A new proof of Cauchy-Bouniakowski-Schwarz's inequality

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Abstract

We present a new proof of Cauchy-Bouniakowski-Schwarz's inequality.

Bouniakowski's inequality is a famous one. It is stated as follows:

Theorem 1 *Given*

$$x_1, x_2, ..., x_n; y_1, y_2, ..., y_n \in R.$$

Prove that:

$$(x_1^2 + x_2^2 + ... + x_n^2)(y_1^2 + y_2^2 + ... + y_n^2) \geq (x_1y_1 + x_2y_2 + ... + x_ny_n)^2.$$

We give a new proof as follows:
Case 1. If

$$x_1^2 + x_2^2 + ... + x_n^2 = 0$$

or

$$y_1^2 + y_2^2 + ... + y_n^2 = 0$$

we have Q. E. D.
Case 2. If

$$x_1^2 + x_2^2 + ... + x_n^2 \neq 0$$

and

$$y_1^2 + y_2^2 + ... + y_n^2 \neq 0$$

Let

$$R_x^2 = x_1^2 + x_2^2 + ... + x_n^2; R_y^2 = y_1^2 + y_2^2 + ... + y_n^2. \ (1)$$

Then

$$\begin{cases} x_1 &= R_x \sin\alpha_1 \sin\alpha_2 ... \sin\alpha_{n-2} \sin\alpha_{n-1} \\ x_2 &= R_x \sin\alpha_1 \sin\alpha_2 ... \sin\alpha_{n-2} \cos\alpha_{n-1} \\ x_3 &= R_x \sin\alpha_1 \sin\alpha_2 ... \cos\alpha_{n-2} \\ ... \\ x_n &= R_x \cos\alpha_1 \end{cases}$$

and

$$\begin{cases} y_1 = R_y \sin\beta_1 \sin\beta_2 ... \sin\beta_{n-2} \sin\beta_{n-1} \\ y_2 = R_y \sin\beta_1 \sin\beta_2 ... \sin\beta_{n-2} \cos\beta_{n-1} \\ y_3 = R_y \sin\beta_1 \sin\beta_2 ... \cos\beta_{n-2} \\ ... \\ y_n = R_y \cos\beta_1 \end{cases}$$

We have

$$x_1 y_1 = R_x R_y \prod_{k=1}^{n-2} \sin\alpha_k \sin\beta_k \sin\alpha_{n-1} \sin\beta_{n-1}; \; x_2 y_2 = R_x R_y \prod_{k=1}^{n-2} \sin\alpha_k \sin\beta_k \cos\alpha_{n-1} \cos\beta_{n-1}.$$

Thus,

$$\begin{aligned} x_1 y_1 + x_2 y_2 \leq |x_1 y_1 + x_2 y_2| &= |R_x R_y| |\prod_{k=1}^{n-2} \sin\alpha_k \sin\beta_k|.|\cos(\alpha_{n-1} - \beta_{n-1})| \\ &\leq |R_x R_y| |\prod_{k=1}^{n-2} \sin\alpha_k \sin\beta_k|. \end{aligned}$$

From this relation, we have:

$$\begin{aligned} x_1 y_1 + x_2 y_2 + x_3 y_3 &\leq |x_1 y_1 + x_2 y_2 + x_3 y_3| \leq |R_x R_y| |\prod_{k=1}^{n-3} \sin\alpha_k \sin\beta_k|.|\cos(\alpha_{n-2} - \beta_{n-2})| \\ &\leq |R_x R_y| |\prod_{k=1}^{n-3} \sin\alpha_k \sin\beta_k| \\ ... \\ x_1 y_1 + x_2 y_2 + ... + x_n y_n &\leq |x_1 y_1 + x_2 y_2 + ... + x_n y_n| \leq |R_x R_y| \; (2) \end{aligned}$$

From (1) and (2), we have

$$(x_1^2 + x_2^2 + ... + x_n^2)(y_1^2 + y_2^2 + ... + y_n^2) \geq (x_1 y_1 + x_2 y_2 + ... + x_n y_n)^2.(Q.E.D)$$

The quality happens if and only if

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = ... = \frac{x_n}{y_n}.$$

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