COMMENTED PROBLEM

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ABSTRACT. In this article it is solved and commented a problem published in GMB 5/2016 with three distinct lines of solving belongs to 9; 10; 12 - standard classes

In GM5/2016 was published the following problem:
„Le a, b, c be real and strictly positive numbers sucht that $a + b + c = 2 \leq abc$. Prove that $abc \geq 8$." (Author Daniel Sitaru)

Proof. Although published for the 9th grade problems - it can be given an elementary proof, that don't require knowledge above the 7th grade. We start from hypothesis:

$$(a + b + c) + 2 \leq abc$$

$$2(a + b + c) + 2 \leq abc + (a + b + c)$$

$$2(a + b + c) + 3 \leq abc + (a + b + c) + 1$$

$$(ab + bc + ca) + 2(a + b + c) + 3 \leq abc + (ab + bc + ca) + (a + b + c) + 1$$

$$(a + 1)(b + 1) + (b + 1)(c + 1) + (c + 1)(a + 1) \leq (a + 1)(b + 1)(c + 1)$$

$$\frac{1}{c+1} + \frac{1}{a+1} + \frac{1}{b+1} \leq 1 \tag{0.1}$$

Reducing the hypothesis to this inequality suggest using the following substitutions:

$$x = \frac{1}{a+1}; y = \frac{1}{b+1}; z = \frac{1}{c+1} \text{ de unde:}$$

$$a = \frac{1-x}{x}; b = \frac{1-y}{y}; c = \frac{1-z}{z}$$

Processed hypothesis 0.1 becomes:
$x + y + z \leq 1$ or $1 - x \geq y + z; 1 - y \geq x + z; 1 - z \geq x + y$.
In this conditions:

$$abc = \frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \geq \frac{(y+z)(z+x)(x+y)}{xyz} \geq \overbrace{\geq}^{\text{AM-GM}}$$

$$\geq \frac{2\sqrt{yz} \cdot 2\sqrt{zx} \cdot 2\sqrt{xy}}{xyz} = \frac{8xyz}{x+yz} = 8$$

A solution for the 10th level grade, where we use the means inequality belongs to Marian Cucoaneş

$$abc \geq 2 + a + b + c \geq 4\sqrt[4]{2abc}$$

$$(abc)^3 \geq 4^4 \cdot 2 = 2^9 \Rightarrow abc \geq 8$$

Key words and phrases. substitutions, inequalities, Viète.

A solution for the 12th level grade, belonging to Leonard Giugiuc uses Viète relationships. Denoting: $a+b+c=3s$ and $abc=p$, from means inequality we obtain $p \leq s^3 \Rightarrow 3s+2 \leq s^3 \Rightarrow (s-2)(s+1)^2 \geq 0 \Rightarrow s \geq 2$.
From hypothesis $3s+2 \leq p$ wherefrom:

$$8 = 3 \cdot 2 + 2 \leq 3s + 2 \leq p \Rightarrow p \geq 8 \Rightarrow abc \geq 8$$

The condition $a+b+c+2 \leq abc$ implies (like the author's solution) the relationship:

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1$$

observed indian mathematician Abhay Chandra. Following this idea, by applying means inequality he obtains:

$$\frac{2}{\sqrt{(a+1)(b+1)}} \leq \frac{1}{1+a} + \frac{1}{1+b} \leq 1 - \frac{1}{1+c} = \frac{c}{c+1}$$
$$\frac{2}{\sqrt{(b+1)(c+1)}} \leq \frac{1}{1+b} + \frac{1}{1+c} \leq 1 - \frac{1}{1+a} \leq \frac{a}{a+1}$$
$$\frac{2}{\sqrt{(c+1)(a+1)}} \leq \frac{1}{1+c} + \frac{1}{1+a} \leq 1 - \frac{1}{1+b} \leq \frac{b}{b+1}$$

By multiplying this relationships we obtain:

$$\frac{8}{(a+1)(b+1)(c+1)} \leq \frac{abc}{(a+1)(b+1)(c+1)}$$

wherefrom: $abc \geq 8$.
The methods indicated may be helpful in solving many problems of this type. □

References

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