On some triangle inequalities

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Abstract. In this article we will use two known triangle inequalities to give some other results.

1 Introduction

The following two inequalities have been very popular.

$$\frac{x^2+y^2+z^2}{2} \geq yz.\cos A + zx.\cos B + xy.\cos C \tag{1}$$

$$\frac{x^2+y^2+z^2}{2} \geq yz.\sin\frac{A}{2} + zx.\sin\frac{B}{2} + xy.\sin\frac{C}{2} \tag{2}$$

where A, B, C are three angles of a triangle and x, y, z are any real numbers. We can rewrite these two inequalities as

$$x^2+y^2+z^2 \geq yz.\frac{b^2+c^2-a^2}{bc} + zx.\frac{c^2+a^2-b^2}{ca} + xy.\frac{a^2+b^2-c^2}{ab} \tag{3}$$

$$\frac{x^2+y^2+z^2}{2} \geq yz.\sqrt{\frac{(s-b)(s-c)}{bc}} + zx.\sqrt{\frac{(s-c)(s-a)}{ca}} + xy.\sqrt{\frac{(s-a)(s-b)}{ab}} \tag{4}$$

There are two popular proofs for (1) as follows

Using properties of vector. Let I be the incenter of triangle ABC, and let X, Y, Z be respectively feets of perpendicular lines from I to sides BC, CA, AB. By full expanding the following self-evident inequality

$$(x\overrightarrow{IX} + y\overrightarrow{IY} + z\overrightarrow{IZ})^2 \geq 0$$

we get the desired inequality. The equality occurs if and only if

$$x\overrightarrow{IX} + y\overrightarrow{IY} + z\overrightarrow{IZ} = \vec{0}.$$

On the other hand, according to the porcupine theorem, we have

$$a\overrightarrow{IX} + b\overrightarrow{IY} + c\overrightarrow{IZ} = \vec{0}.$$

Thus the necessary and sufficient conditions such that the equality occurs as

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

□

Using algebraic techniques. The inequality (1) is equivalent to

$$2yz\cos A + 2zx\cos B - 2xy\cos(A+B) \le x^2 + y^2 + z^2,$$

$$2yz\cos A + 2zx\cos B - 2xy\cos A\cos B \le x^2 + y^2 + z^2 - 2xy\sin A\sin B,$$

$$2yz\cos A + 2zx\cos B - 2xy\cos A\cos B + y^2\sin^2 A + x^2\sin^2 B \le x^2 + y^2 + z^2 + (y\sin A - x\sin B)^2,$$

$$2yz\cos A + 2zx\cos B - 2xy\cos A\cos B \le x^2\cos^2 B + y^2\cos^2 A + z^2 + (y\sin A - x\sin B)^2,$$

$$2z(y\cos A + x\cos B) \le (y\cos A + x\cos B)^2 + z^2 + (y\sin A - x\sin B)^2,$$

$$(y\cos A + x\cos B - z)^2 + (y\sin A - x\sin B)^2 \ge 0$$

Which is obviously true. The equility occurs iff

$$\begin{cases} \frac{x}{\sin A} = \frac{y}{\sin B}, \\ z = y\cos A + x\cos B. \end{cases}$$

To do more clearly, we set $\dfrac{x}{\sin A} = \dfrac{y}{\sin B} = k$. Then $x = k.\sin A, y = k.\sin B$ and $z = k(\sin A\cos B + \sin B\cos A) = k.\sin C$. So, the conditions above are equivalent to

$$\frac{x}{\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C}.$$

This is the necessary and sufficient conditions to the equality happens. □

Applying (1) for a triangle which has three angles $\frac{B+C}{2}, \frac{C+A}{2}, \frac{A+B}{2}$ we obtain (2).

Remark 1. From the second proof we observe that for any angles α, β, γ (they are not necessary three angles of a triangle) such that $\alpha + \beta + \gamma = \pi$, the inequality below is also true (for all real numbers x, y, z)

$$x^2 + y^2 + z^2 \ge 2yz\cos\alpha + 2zx\cos\beta + 2xy\cos\gamma. \quad (5)$$

2 Some results

In all problems below, we use known notations of triangle ABC and note that S denotes its area.

In (3) we replace (x, y, z) by $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ to yield

$$a + b + c \ge \frac{b^2 + c^2 - a^2}{\sqrt{bc}} + \frac{c^2 + a^2 - b^2}{\sqrt{ca}} + \frac{a^2 + b^2 - c^2}{\sqrt{ab}}.$$

This inequality has equivalent forms as

$$\frac{a^2}{\sqrt{bc}} + \frac{b^2}{\sqrt{ca}} + \frac{c^2}{\sqrt{ab}} + a + b + c \ge \sqrt{\frac{a^3}{b}} + \sqrt{\frac{b^3}{c}} + \sqrt{\frac{c^3}{a}} + \sqrt{\frac{b^3}{a}} + \sqrt{\frac{c^3}{b}} + \sqrt{\frac{a^3}{c}},$$

$$\frac{\cot A}{\sqrt{bc}} + \frac{\cot B}{\sqrt{ca}} + \frac{\cot C}{\sqrt{ab}} \le \frac{1}{2r}.$$

If we substitue (a^2, b^2, c^2) for (x, y, z), we get

$$a^4 + b^4 + c^4 \ge bc(b^2 + c^2 - a^2) + ca(c^2 + a^2 - b^2) + ab(a^2 + b^2 - c^2)$$

or

$$a^4 + b^4 + c^4 + abc(a + b + c) \ge ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2).$$

This is Schur's inequality of fourth degree that is well-known.

We repalace (x, y, z) in (1) by $(\frac{1}{\sqrt{s-a}}, \frac{1}{\sqrt{s-b}}, \frac{1}{\sqrt{s-c}})$ (and for the acute triangle ABC) we have

$$\begin{aligned}\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} &\geq \sum_{cyc}\frac{2}{\sqrt{(s-b)(s-c)}}\cos A\\ &\geq \frac{4}{a}\cos A+\frac{4}{b}\cos B+\frac{4}{c}\cos C.\end{aligned}$$

This can be rewritten by other forms as follows

$$\frac{r_a+r_b+r_c}{4S} \geq \frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}.$$

When (x, y, z) is replaced by $(\sqrt{r_a}, \sqrt{r_b}, \sqrt{r_c})$ (the triangle ABC is also acute) we obtain

$$\begin{aligned}\frac{r_a+r_b+r_c}{2} &\geq \sqrt{r_b r_c}\cos A+\sqrt{r_c r_a}\cos B+\sqrt{r_a r_b}\cos C\\ &\geq h_a\cos A+h_b\cos B+h_c\cos C\end{aligned}$$

Which is the other form of one of the results above.

To be continue, we replace again (x, y, z) in (1) by $(\sqrt{\frac{r_b r_c}{r_a}}, \sqrt{\frac{r_c r_a}{r_b}}, \sqrt{\frac{r_a r_b}{r_c}})$ to get

$$\frac{r_b r_c}{r_a}+\frac{r_c r_a}{r_b}+\frac{r_a r_b}{r_c} \geq 2(r_a\cos A+r_b\cos B+r_c\cos C).$$

In (1),(2) we replace respectively (x, y, z) by $(\frac{1}{\sqrt{h_a}}, \frac{1}{\sqrt{h_b}}, \frac{1}{\sqrt{h_c}})$ and $(\frac{1}{\sqrt{r_a}}, \frac{1}{\sqrt{r_b}}, \frac{1}{\sqrt{r_c}})$ and using

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$$

we obtain the following results

$$\frac{\cos A}{\sqrt{h_b h_c}}+\frac{\cos B}{\sqrt{h_c h_a}}+\frac{\cos C}{\sqrt{h_a h_b}} \leq \frac{1}{2r},$$

$$\frac{\sin\frac{A}{2}}{\sqrt{r_b r_c}}+\frac{\sin\frac{B}{2}}{\sqrt{r_c r_a}}+\frac{\sin\frac{C}{2}}{\sqrt{r_a r_b}} \leq \frac{1}{2r}.$$

Chosing $x=\frac{1}{s-a}, y=\frac{1}{s-b}, z=\frac{1}{s-c}$ and substitue it into (2) gives

$$\frac{\sin\frac{A}{2}}{(s-b)(s-c)}+\frac{\sin\frac{B}{2}}{(s-c)(s-a)}+\frac{\sin\frac{C}{2}}{(s-a)(s-b)} \leq \frac{1}{2}\left(\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}\right),$$

which is equivalent to

$$\begin{aligned}&(s-a)\sin\frac{A}{2}+(s-b)\sin\frac{B}{2}+(s-c)\sin\frac{C}{2} \leq\\ &\leq \frac{(s-a)(s-b)(s-c)}{2}\left(\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}\right)\end{aligned}$$

On the other hand, it's easy to show that

$$(s-a)(s-b)(s-c) \leq \frac{abc}{8}$$

We infer that

$$(s-a)\sin\frac{A}{2}+(s-b)\sin\frac{B}{2}+(s-c)\sin\frac{C}{2} \leq \frac{abc}{16}\left(\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}\right).$$

From (3) chosing $x = \frac{a}{b+c}, y = \frac{b}{c+a}, z = \frac{c}{a+b}$ yields

$$\left(\frac{a}{b+c}\right)^2 + \left(\frac{b}{c+a}\right)^2 + \left(\frac{c}{a+b}\right)^2 \geq \frac{b^2+c^2-a^2}{(b+a)(c+a)} + \frac{c^2+a^2-b^2}{(c+b)(a+b)} + \frac{a^2+b^2-c^2}{(a+c)(b+c)}.$$

This inequality has equivalent forms as follows

$$\frac{\cot A}{(a+b)(a+c)} + \frac{\cot B}{(b+c)(b+a)} + \frac{\cot C}{(c+a)(c+b)} \leq \frac{1}{4S}\left(\left(\frac{a}{b+c}\right)^2 + \left(\frac{b}{c+a}\right)^2 + \left(\frac{c}{a+b}\right)^2\right).$$

Chosing $x = \sqrt{\frac{a}{b+c}}, y = \sqrt{\frac{b}{c+a}}, z = \sqrt{\frac{c}{a+b}}$ and replace it into (4) we get

$$\sqrt{\frac{(s-b)(s-c)}{(a+b)(a+c)}} + \sqrt{\frac{(s-c)(s-a)}{(b+c)(b+a)}} + \sqrt{\frac{(s-a)(s-b)}{(c+a)(c+b)}} \leq \frac{1}{2}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

In (4) we replace (x, y, z) by $(\sqrt{s-a}, \sqrt{s-b}, \sqrt{s-c})$ then

$$\frac{(s-b)(s-c)}{\sqrt{bc}} + \frac{(s-c)(s-a)}{\sqrt{ca}} + \frac{(s-a)(s-b)}{\sqrt{ab}} \leq \frac{s}{2}.$$

This inequality has equivalent forms as follows

$$\sqrt{bc}\sin^2\frac{A}{2} + \sqrt{ca}\sin^2\frac{B}{2} + \sqrt{ab}\sin^2\frac{C}{2} \leq \frac{a+b+c}{4}.$$

Chosing again $x = \frac{1}{\sqrt{(s-b)(s-c)}}, y = \frac{1}{\sqrt{(s-c)(s-a)}}, z = \frac{1}{\sqrt{(s-a)(s-b)}}$ and then respectively substitue it into (1), (4) and note that

$$\frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} + \frac{1}{(s-a)(s-b)} = \frac{1}{r^2}$$

we have the following results

$$\frac{\cos A}{(s-a)\sqrt{(s-b)(s-c)}} + \frac{\cos B}{(s-b)\sqrt{(s-c)(s-a)}} + \frac{\cos C}{(s-c)\sqrt{(s-a)(s-b)}} \leq \frac{1}{2r^2},$$

$$\frac{1}{(s-a)\sqrt{bc}} + \frac{1}{(s-b)\sqrt{ca}} + \frac{1}{(s-c)\sqrt{ab}} \leq \frac{1}{2r^2}.$$

Now we chose $x = \sqrt{\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}}, y = \sqrt{\frac{\sin\frac{C}{2}\sin\frac{A}{2}}{\sin\frac{B}{2}}}, z = \sqrt{\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{C}{2}}}$ and substitue it into (2) to get

$$\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}} + \frac{\sin\frac{C}{2}\sin\frac{A}{2}}{\sin\frac{B}{2}} + \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{C}{2}} \geq 2\sin^2\frac{A}{2} + 2\sin^2\frac{B}{2} + 2\sin^2\frac{C}{2}.$$

When replace (x, y, z) by $\left(\sqrt{\frac{s-a}{a}}, \sqrt{\frac{s-b}{b}}, \sqrt{\frac{s-c}{c}}\right)$ into (4) we have

$$\frac{1}{2}\left(\frac{s-a}{a} + \frac{s-b}{b} + \frac{s-c}{c}\right) \geq \frac{(s-b)(s-c)}{bc} + \frac{(s-c)(s-a)}{ca} + \frac{(s-a)(s-b)}{ab}$$

This is equivalent to

$$\frac{s-a}{a} + \frac{s-b}{b} + \frac{s-c}{c} \geq 2\sin^2\frac{A}{2} + 2\sin^2\frac{B}{2} + 2\sin^2\frac{C}{2}$$

or

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+2(\cos A+\cos B+\cos C)\geq 12,$$

or

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{2r}{R}\geq 10.$$

The next, we also replace (x,y,z) by $(\sqrt{bc},\sqrt{ca},\sqrt{ab})$ into (4) then obtain

$$a\sqrt{(s-b)(s-c)}+b\sqrt{(s-c)(s-a)}+c\sqrt{(s-a)(s-b)}\leq\frac{ab+bc+ca}{2}.$$

In (4) we chose $x=\frac{a}{s-a},y=\frac{b}{s-b},z=\frac{c}{s-c}$ then

$$\left(\frac{a}{s-a}\right)^2+\left(\frac{b}{s-b}\right)^2+\left(\frac{c}{s-c}\right)^2\geq\frac{2}{\sin\frac{A}{2}}+\frac{2}{\sin\frac{B}{2}}+\frac{2}{\sin\frac{C}{2}}\geq 12.$$

Now we replace again (x,y,z) by $(\sin A',\sin B',\sin C')$, where A,B,C are three angles of any triangle, into (1), we find that

$$\sum_{cyc}\sin B'\sin C'\cos A\leq\frac{1}{2}(\sin^2 A'+\sin^2 B'+\sin^2 C')$$

Dividing both sides of this inequality by $\sin A'\sin B'\sin C'$, we get

$$\frac{\cos A}{\sin A'}+\frac{\cos B}{\sin B'}+\frac{\cos C}{\sin C'}\leq\frac{1}{2}\left(\frac{\sin A'}{\sin B'\sin C'}+\frac{\sin B'}{\sin C'\sin A'}+\frac{\sin C'}{\sin A'\sin B'}\right)$$

Note that

$$\begin{aligned}\frac{\sin A'}{\sin B'\sin C'}=\frac{\sin(B'+C')}{\sin B'\sin C'}&=\frac{\sin B'\cos C'+\cos B'\sin C')}{\sin B'\sin C'}\\&=\cot B'+\cot C'\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\sin B'}{\sin C'\sin A'}&=\cot C'+\cot A'\\\frac{\sin C'}{\sin A'\sin B'}&=\cot A'+\cot B'\end{aligned}$$

Thus, we have a result: For any two triangles ABC and $A'B'C'$, the following inequality holds

$$\cot A+\cot B+\cot C\geq\frac{\cos A'}{\sin A}+\frac{\cos B'}{\sin B}+\frac{\cos C'}{\sin C}.$$

If we replace again (x,y,z) by (a',b',c') into (1) and using formula $a'^2+b'^2+c'^2=4S'(\cot A'+\cot B'+\cot C')$ then to obtain

$$2S'(\cot A'+\cot B'+\cot C')\geq b'c'\cos A+c'a'\cos B+a'b'\cos C.$$

This is equivalent to

$$\cot A'+\cot B'+\cot C'\geq\frac{\cos A}{\sin A'}+\frac{\cos B}{\sin B'}+\frac{\cos C}{\sin C'}.$$

Thus, we get again the above result.

We consider any point P in triangle ABC and let PX, PY, PZ be internal bisectors of $\angle BPC, \angle CPA, \angle APB$, respectively We set $\angle BPC = 2\alpha, \angle CPA = 2\beta, \angle APB = 2\gamma$. Using the known formulas about the length of bisectors in a triangle, we have

$$PX = \frac{2PB.PC}{PB+PC}\cos\alpha, PY = \frac{2PC.PA}{PC+PA}\cos\beta, PZ = \frac{2PA.PB}{PA+PB}\cos\gamma$$

or

$$2\cos\alpha = PX\left(\frac{1}{PB}+\frac{1}{PC}\right), 2\cos\beta = PY\left(\frac{1}{PC}+\frac{1}{PA}\right), 2\cos\gamma = PZ\left(\frac{1}{PA}+\frac{1}{PB}\right).$$

Applying (5) for α, β, γ which are determined above, we have

$$PX\left(\frac{1}{PB}+\frac{1}{PC}\right)yz + PY\left(\frac{1}{PC}+\frac{1}{PA}\right)zx + PZ\left(\frac{1}{PA}+\frac{1}{PB}\right)xy \le x^2+y^2+z^2.$$

We continue take $x = \sqrt{PA}, y = \sqrt{PB}, z = \sqrt{PC}$ then to get

$$PX\left(\frac{\sqrt{PC}}{\sqrt{PB}}+\frac{\sqrt{PB}}{\sqrt{PC}}\right) + PY\left(\frac{\sqrt{PA}}{\sqrt{PC}}+\frac{\sqrt{PC}}{\sqrt{PA}}\right) + PZ\left(\frac{\sqrt{PB}}{\sqrt{PA}}+\frac{\sqrt{PA}}{\sqrt{PB}}\right) \le PA+PB+PC.$$

which implies that

$$PA+PB+PC \ge 2(PX+PY+PZ) \ge 2(PP_a+PP_b+PP_c)$$

where P_a, P_b, P_c are feet of perpendicular lines from P to the sides BC, CA, AB, respectively. We have just received a result which is stronger than Erdos-Mordell inequality.

We have known that if a, b, c are side-lengths of a triangle then there exist positive real numbers u, v, w such that

$$a = v+w, b = w+u, c = u+v$$

which is called Ravi's substitutions. Using this substitutions, (3) can be written as

$$\begin{aligned} x^2+y^2+z^2 &\ge \sum_{cyc} yz\cdot\frac{(w+u)^2+(u+v)^2-(v+w)^2}{(w+u)(u+v)} \\ &= \sum_{cyc} yz\cdot\frac{2(u^2+uv+uw-vw)}{(w+u)(u+v)} \\ &= \sum_{cyc} yz\cdot\frac{2(u+v)(u+w)-4vw}{(u+v)(u+w)} \\ &= \sum_{cyc} yz\left(2-\frac{4vw}{(u+v)(u+w)}\right) \end{aligned}$$

It follows that

$$\begin{aligned} yz\frac{vw}{(u+v)(u+w)} + zx\frac{wu}{(v+w)(v+u)} + xy\frac{uv}{(w+u)(w+v)} &\ge \frac{2(xy+yz+zx)-(x^2+y^2+z^2)}{4} \\ &= \frac{(x+y+z)^2-2(x^2+y^2+z^2)}{4} \end{aligned}$$

Note that the equality occurs iff

$$\frac{x}{v+w} = \frac{y}{w+u} = \frac{z}{u+v}$$

We choose $x = y = z$ to get the known result

$$\frac{vw}{(u+v)(u+w)} + \frac{wu}{(v+w)(v+u)} + \frac{uv}{(w+u)(w+v)} \geq \frac{3}{4}$$

When choosing $x = 1, y = \frac{1}{2}, z = \frac{1}{3}$ then we obtain

$$\frac{vw}{(u+v)(u+w)} + \frac{2wu}{(v+w)(v+u)} + \frac{3uv}{(w+u)(w+v)} > \frac{23}{24}$$

(because the equality does not occur). Applying (3) for a triangle which has three side-lengths m_a, m_b, m_c we get

$$yz\frac{5a^2 - b^2 - c^2}{m_b m_c} + zx\frac{5b^2 - c^2 - a^2}{m_c m_a} + xy\frac{5c^2 - a^2 - b^2}{m_a m_b} \leq 4(x^2 + y^2 + z^2)$$

A simple consequent of this result as

$$\frac{5a^2 - b^2 - c^2}{m_b m_c} + \frac{5b^2 - c^2 - a^2}{m_c m_a} + \frac{5c^2 - a^2 - b^2}{m_a m_b} \leq 12.$$

In (3) we replace (x, y, z) by (xa, yb, zc) to give

$$x^2a^2 + y^2b^2 + z^2c^2 \geq yz(b^2 + c^2 - a^2) + zx(c^2 + a^2 - b^2) + xy(a^2 + b^2 - c^2)$$

which is equivalent to

$$a^2(x^2 + 2yz) + b^2(y^2 + 2zx) + c^2(z^2 + 2xy) \geq (a^2 + b^2 + c^2)(xy + yz + zx).$$

The equality holds for $x = y = z$. In (4) we replace (x, y, z) by $(x\sqrt{a}, y\sqrt{b}, z\sqrt{c})$ to yield

$$yz\sqrt{(s-b)(s-c)} + zx\sqrt{(s-c)(s-a)} + xy\sqrt{(s-a)(s-b)} \leq \frac{x^2a + y^2b + z^2c}{2}.$$

From here, choosing again $(x, y, z) = (\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$ we have

$$\frac{\sqrt{(s-b)(s-c)}}{bc} + \frac{\sqrt{(s-c)(s-a)}}{ca} + \frac{\sqrt{(s-a)(s-b)}}{ab} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

By similar ways, you can also establish new results for yourself. To finish this article, we will give some problems that can be solved by using elementary inequality above.

3 Proposed problems

Excersise 1 (S.Klamkin)**.** Let ABC be a triangle and let x, y, z be any real numbers. Prove that

$$x^2 + y^2 + z^2 \geq 2(-1)^{n+1}(yz\cos nA + zx\cos nB + xy\cos nC)$$

where n is a natural number. The equality occurs iff

$$\frac{x}{\sin nA} = \frac{y}{\sin nB} = \frac{z}{\sin nC}.$$

Hint: The desired inequality is equivalent to

$$(x + (-1)^n(y\cos nC + z\cos nB))^2 + (y\sin nC - z\sin nB)^2 \geq 0.$$

Excersise 2. Let ABC be a triangle and let n be a natural number. Prove that

(a) $\cos(2nA) + \cos(2nB) + \cos(2nC) \geq -3/2$.

(b) $\cos(2n+1)A + \cos(2n+1)B + \cos(2n+1)C \leq 3/2$.

Excersise 3. Prove that for all any triangle ABC, then

$$\sqrt{3}\cos A + 2\cos B + 2\sqrt{3}\cos C \leq 4.$$

Excersise 4. Let ABC be a triangle and let x, y, z be any real numbers. Prove that

$$\left(\frac{ax+by+cz}{4S}\right)^2 \geq \frac{xy}{ab} + \frac{yz}{bc} + \frac{zx}{ca}.$$

Excersise 5. Prove that for all any triangle ABC, then

$$\left(\frac{a^2+b^2+c^2}{4S}\right)^2 \geq \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

Excersise 6 (IMO Shortlist, 1995)**.** Given positive real numbers a, b, c. Find all triples (x, y, z) of real numbers satisfying the following system of equation

$$\begin{cases} x+y+z = a+b+c, \\ 4xyz - (a^2x + b^2y + c^2z) = abc. \end{cases}$$

Excersise 7 (China TST, 2007)**.** Let x, y, z be positive real numbers such that

$$x + y + z + \sqrt{xyz} = 4.$$

Prove that

$$\sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}} \geq x + y + z.$$

Tài liệu

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