About An Identity With The Condition

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Abstract. In this paper, we will mention a nice and useful identity which is very known, then we give its generalization and apply them to solve some problems.

1 Introduction

We will start with the following simple problem

Problem 1. Let a, b, c be real numbers such that $abc = 1$. Prove that

(a) $\dfrac{1}{1+a+ab} + \dfrac{1}{1+b+bc} + \dfrac{1}{1+c+ca} = 1.$

(b) $\dfrac{a}{1+a+ab} + \dfrac{b}{1+b+bc} + \dfrac{c}{1+c+ca} = 1.$

We will give two solutions for this problem as follows

Solution 1. (a) We have

$$\begin{aligned}\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} &= \frac{c}{c+ca+abc} + \frac{a}{a+ab+abc} + \frac{1}{1+c+ca} \\ &= \frac{c}{c+ca+1} + \frac{a}{a+ab+1} + \frac{1}{1+c+ca} \\ &= \frac{c}{1+c+ca} + \frac{ca}{ca+1+c} + \frac{1}{1+c+ca} \\ &= 1.\end{aligned}$$

(b) Also similarly as above, we have

$$\begin{aligned}\frac{a}{1+a+ab} + \frac{b}{1+b+bc} + \frac{c}{1+c+ca} &= \frac{ca}{c+ca+1} + \frac{ab}{a+ab+1} + \frac{c}{1+c+ca} \\ &= \frac{ca}{c+ca+1} + \frac{1}{ca+1+c} + \frac{c}{1+c+ca} \\ &= 1.\end{aligned}$$

The proofs are completed. □

Solution 2. From the given condition $abc = 1$, it follows that there exist the numbers x, y, z such that

$$a = \frac{y}{x}, b = \frac{z}{y}, c = \frac{x}{z}.$$

Then

(a)

$$\begin{aligned}\frac{1}{1+a+ab}+\frac{1}{1+b+bc}+\frac{1}{1+c+ca}&=\frac{1}{1+\dfrac{y}{x}+\frac{z}{x}}+\frac{1}{1+\dfrac{z}{y}+\dfrac{x}{y}}+\frac{1}{1+\dfrac{x}{z}+\frac{y}{z}}\\&=\frac{x}{x+y+z}+\frac{y}{x+y+z}+\frac{z}{x+y+z}\\&=1.\end{aligned}$$

(b)

$$\begin{aligned}\frac{a}{1+a+ab}+\frac{b}{1+b+bc}+\frac{c}{1+c+ca}&=\frac{\frac{y}{x}}{1+\frac{y}{x}+\dfrac{z}{x}}+\frac{\frac{z}{y}}{1+\frac{z}{y}+\frac{x}{y}}+\frac{\frac{x}{z}}{1+\frac{x}{z}+\frac{y}{z}}\\&=\frac{y}{x+y+z}+\frac{z}{x+y+z}+\frac{x}{x+y+z}\\&=1.\end{aligned}$$

We are done. □

Both solutions above are very simple, but the second one may help us to extend the original result from three variables to four variables as follows

Problem 2. Let a, b, c, d be real numbers such that $abcd = 1$. Prove that

(a) $\dfrac{1}{1+a+ab+abc}+\dfrac{1}{1+b+bc+bcd}+\dfrac{1}{1+c+cd+cda}+\dfrac{1}{1+d+da+dab}=1.$

(b) $\dfrac{a}{1+a+ab+abc}+\dfrac{b}{1+b+bc+bcd}+\dfrac{c}{1+c+cd+cda}+\dfrac{d}{1+d+da+dab}=1.$

Solution. Also similarly the second solution of the firs problem, since $abcd = 1$, there are real numbers x, y, z, t such that

$$a=\frac{y}{x}, b=\frac{z}{y}, c=\frac{t}{z}, d=\frac{x}{t}.$$

□

Furthermore, we can extend for n variables

Problem 3. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

(a) $\frac{1}{1+a_1+a_1a_2+\cdots+a_1a_2\cdots a_{n-1}}+\frac{1}{1+a_2+a_2a_3+\cdots+a_2a_3\cdots a_n}+\cdots+\frac{1}{1+a_n+a_na_1+\cdots+a_na_1\cdots a_{n-2}}=1.$

(b) $\frac{a_1}{1+a_1+a_1a_2+\cdots+a_1a_2\cdots a_{n-1}}+\frac{a_2}{1+a_2+a_2a_3+\cdots+a_2a_3\cdots a_n}+\cdots+\frac{a_n}{1+a_n+a_na_1+\cdots+a_na_1\cdots a_{n-2}}=1.$

Solution. From $a_1a_2\cdots a_n = 1$, it follows that there are real numbers $x_1, x_2, \ldots, x_n$ such that

$$a_1=\frac{x_2}{x_1}, a_2=\frac{x_3}{x_2}, \ldots, a_n=\frac{x_1}{x_n}$$

Then

(a)

$$\frac{1}{1+a_1+a_1a_2+\cdots+a_1a_2\cdots a_{n-1}} = \frac{1}{1+\frac{x_2}{x_1}+\frac{x_3}{x_1}+\cdots+\frac{x_n}{x_1}} = \frac{x_1}{x_1+x_2+\cdots+x_n}$$
$$\frac{1}{1+a_2+a_2a_3+\cdots+a_2a_3\cdots a_n} = \frac{1}{1+\frac{x_3}{x_2}+\frac{x_4}{x_2}+\cdots+\frac{x_1}{x_2}} = \frac{x_2}{x_1+x_2+\cdots+x_n}$$
$$\cdots \quad \cdots \quad \cdots$$
$$\frac{1}{1+a_n+a_na_1+\cdots+a_na_1\cdots a_{n-2}} = \frac{1}{1+\frac{x_1}{x_n}+\frac{x_2}{x_n}+\cdots+\frac{x_{n-1}}{x_n}} = \frac{x_n}{x_1+x_2+\cdots+x_n}$$

Summing up these relations we obtain the desired result.

(b) Similar to the part (a).

□

2 Applications

Now we will use the above results to prove the problems below

Problem 4. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a) $\dfrac{1}{3+2a^2+b^2}+\dfrac{1}{3+2b^2+c^2}+\dfrac{1}{3+2c^2+a^2} \le \dfrac{1}{2}$.

(b) $\dfrac{a}{3+2a^2+b^2}+\dfrac{b}{3+2b^2+c^2}+\dfrac{c}{3+2c^2+a^2} \le \dfrac{1}{2}$.

Solution. (a) Using the AM-GM inequality we get

$$\frac{1}{3+2a^2+b^2} = \frac{1}{2+(1+a^2)+(a^2+b^2)} \le \frac{1}{2+2a+2ab}$$

Similarly

$$\frac{1}{3+2b^2+c^2} \le \frac{1}{2+2b+2bc}$$
$$\frac{1}{3+2c^2+a^2} \le \frac{1}{2+2c+2ca}$$

Adding up these relations and using the result of the part (a) of problem 1, we obtain the desired inequality.

(b) We do similarly as above and using the part (b) of problem 1.

□

Problem 5. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a) $\dfrac{1}{(a+2)^3+(a+b+1)^3+27}+\dfrac{1}{(b+2)^3+(b+c+1)^3+27}+\dfrac{1}{(c+2)^3+(c+a+1)^3+27} \le \dfrac{1}{27}$.

(b) $\dfrac{a}{(a+2)^3+(a+b+1)^3+27}+\dfrac{b}{(b+2)^3+(b+c+1)^3+27}+\dfrac{c}{(c+2)^3+(c+a+1)^3+27} \le \dfrac{1}{27}$.

Solution. (a) Applying AM-GM inequality we obtain

$$\frac{1}{(a+2)^3+(a+b+1)^3+27} \le \frac{1}{27a+27ab+27}$$

Similarly

$$\frac{1}{(b+2)^3+(b+c+1)^3+27} \le \frac{1}{27b+27bc+27}$$

$$\frac{1}{(c+2)^3+(c+a+1)^3+27} \le \frac{1}{27c+27ca+27}$$

Summing up these inequalities and using the part (a) of problem 1, the conclusion follows.

(b) We do similarly as above and using the part (b) of problem 1.

□

Problem 6. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

(a) $\dfrac{1}{6+3a^3+2b^3+c^3}+\dfrac{1}{6+3b^3+2c^3+d^3}+\dfrac{1}{6+3c^3+2d^3+a^3}+\dfrac{1}{6+3d^3+2a^3+b^3} \le \dfrac{1}{3}$.

(b) $\dfrac{a}{6+3a^3+2b^3+c^3}+\dfrac{b}{6+3b^3+2c^3+d^3}+\dfrac{c}{6+3c^3+2d^3+a^3}+\dfrac{d}{6+3d^3+2a^3+b^3} \le \dfrac{1}{3}$.

Solution. (a) Applying AM-GM inequality yields

$$\begin{aligned}\frac{1}{6+3a^3+2b^3+c^3} &= \frac{1}{3+(1+1+a^3)+(1+a^3+b^3)+(a^3+b^3+c^3)}\\ &\le \frac{1}{3+3a+3ab+3abc}\end{aligned}$$

Similarly

$$\frac{1}{6+3b^3+2c^3+d^3} \le \frac{1}{3+3b+3bc+3bcd},$$

$$\frac{1}{6+3c^3+2d^3+a^3} \le \frac{1}{3+3c+3cd+3cda},$$

$$\frac{1}{6+3d^3+2a^3+b^3} \le \frac{1}{3+3d+3da+3dab}.$$

Adding up these relations and using the part (a) of problem 2, we get the desired result.

(b) We do similarly as above and note that the part (b) of problem 2.

□

Problem 7. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a) $\dfrac{1}{(1+a+ab)^2}+\dfrac{1}{(1+b+bc)^2}+\dfrac{1}{(1+c+ca)^2} \ge \dfrac{1}{3}$.

(b) $\dfrac{a}{(1+a+ab)^2}+\dfrac{b}{(1+b+bc)^2}+\dfrac{c}{(1+c+ca)^2} \ge \dfrac{1}{a+b+c}$.

Solution. (a) By Cauchy-Schwarz ineuality we have

$$\begin{aligned}\frac{1}{(1+a+ab)^2}+\frac{1}{(1+b+bc)^2}+\frac{1}{(1+c+ca)^2} &\ge \frac{1}{3}\left(\frac{1}{1+a+ab}+\frac{1}{1+b+bc}+\frac{1}{1+c+ca}\right)^2\\ &= \frac{1}{3}.\end{aligned}$$

(b) We also use Cauchy-Schwarz inequality to obtain

$$\begin{aligned}\frac{a}{(1+a+ab)^2}+\frac{b}{(1+b+bc)^2}+\frac{c}{(1+c+ca)^2} &= \frac{(\frac{a}{1+a+ab})^2}{a}+\frac{(\frac{b}{1+b+bc})^2}{b}+\frac{(\frac{c}{1+c+ca})^2}{c}\\ &\geq \frac{(\frac{1}{1+a+ab}+\frac{1}{1+b+bc}+\frac{1}{1+c+ca})^2}{a+b+c}\\ &= \frac{1}{a+b+c}\end{aligned}$$

and we are done.

□

Problem 8. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a) $\dfrac{a}{(1+a+ab)^3}+\dfrac{b}{(1+b+bc)^3}+\dfrac{c}{(1+c+ca)^3} \geq \dfrac{1}{(a+b+c)^2}$.

(b) $\dfrac{a^2}{(1+a+ab)^3}+\dfrac{b^2}{(1+b+bc)^3}+\dfrac{c^2}{(1+c+ca)^3} \geq \dfrac{1}{3(a+b+c)}$.

Solution. (a) Applying Holder's inequality and using the part (a) of problem 1, we obtain

$$\left[\sum_{cyc}\frac{a}{(1+a+ab)^3}\right](a+b+c)(a+b+c) \geq \left(\sum_{cyc}\frac{a}{1+a+ab}\right)^3 = 1$$

It follows that

$$\sum_{cyc}\frac{a}{(1+a+ab)^3} \geq \frac{1}{(a+b+c)^2}.$$

(b) Also according to Holder's inequality and note that the part (a) of problem 1, we have

$$\left[\sum_{cyc}\frac{a^2}{(1+a+ab)^3}\right](a+b+c)(1+1+1) \geq \left(\sum_{cyc}\frac{a}{1+a+ab}\right)^3 = 1$$

Therefore

$$\sum_{cyc}\frac{a^2}{(1+a+ab)^3} \geq \frac{1}{3(a+b+c)}.$$

The conclusion follows.

□

Problem 9. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

(a) $\dfrac{1}{1+a_1+a_1a_2}+\dfrac{1}{1+a_2+a_2a_3}+\cdots+\dfrac{1}{1+a_n+a_na_1} > 1$.

(Russia, 2004)

(b) $\dfrac{a_1}{1+a_1+a_1a_2}+\dfrac{a_2}{1+a_2+a_2a_3}+\cdots+\dfrac{a_n}{1+a_n+a_na_1} > 1$.

Solution. (a) Since $a_1, a_2, \ldots, a_n > 0$ so

$$\begin{aligned}\frac{1}{1+a_1+a_1a_2} &> \frac{1}{1+a_1+a_1a_2+\cdots+a_1a_2\cdots a_{n-1}}\\ \frac{1}{1+a_2+a_2a_3} &> \frac{1}{1+a_2+a_2a_3+\cdots+a_2a_3\cdots a_n}\\ \cdots \quad & \cdots \quad \cdots\\ \frac{1}{1+a_n+a_na_1} &> \frac{1}{1+a_n+a_na_1+\cdots+a_na_1\cdots a_{n-2}}\end{aligned}$$

Summing up these inequalities and using the part (a) of problem 3, we get immediately the required inequality.

(b) We do similarly as above and using the part (b) of problem (3).

□

Problem 10. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a) $\dfrac{1}{a+ab+2} + \dfrac{1}{b+bc+2} + \dfrac{1}{c+ca+2} \le \dfrac{3}{4}.$

(Mathematical Reflections)

(b) $\dfrac{a}{a+ab+2} + \dfrac{b}{b+bc+2} + \dfrac{c}{c+ca+2} \le \dfrac{9+a+b+c}{16}.$

Solution. (a) Using Cauchy-Schwarz inequality gives us

$$\frac{16}{a+ab+2} = \frac{(3+1)^2}{(a+ab+1)+1} \le \frac{3^2}{a+ab+1} + \frac{1^2}{1}$$

or

$$\frac{1}{a+ab+2} \le \frac{9}{16(a+ab+1)} + \frac{1}{16}.$$

Similarly

$$\frac{1}{b+bc+2} \le \frac{9}{16(b+bc+1)} + \frac{1}{16},$$
$$\frac{1}{c+ca+2} \le \frac{9}{16(c+ca+1)} + \frac{1}{16}.$$

Adding these relations and using the part (a) of problem 1 we obtain

$$\begin{aligned}\frac{1}{a+ab+2} + \frac{1}{b+bc+2} + \frac{1}{c+ca+2} &\le \frac{9}{19}\left(\frac{1}{a+ab+1} + \frac{1}{b+bc+1} + \frac{1}{c+ca+1}\right) + \frac{3}{16}\\ &= \frac{9}{16} + \frac{3}{16}\\ &= \frac{3}{4}.\end{aligned}$$

(b) We do similarly as above and using the part (b) of problem 1.

□

Problem 11. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a) $\dfrac{a^2}{a+ab+1} + \dfrac{b^2}{b+bc+1} + \dfrac{c^2}{c+ca+1} \ge 1,$

(b) $\dfrac{\sqrt{a}}{a+ab+1} + \dfrac{\sqrt{b}}{b+bc+1} + \dfrac{\sqrt{c}}{c+ca+1} \le 1.$

Solution 1. (a) By AM-GM inequality we have

$$\frac{a^2}{a+ab+1} + \frac{1}{a+ab+1} \ge \frac{2a}{a+ab+1},$$
$$\frac{b^2}{b+bc+1} + \frac{1}{b+bc+1} \ge \frac{2b}{b+bc+1},$$
$$\frac{c^2}{c+ca+1} + \frac{1}{c+ca+1} \ge \frac{2c}{c+ca+1}.$$

Adding up these inequalities and using two familiar identities we obtain the required result.

(b) We also use AM-GM inequality to get

$$\frac{\sqrt{a}}{a+ab+1} \le \frac{1}{2}\left(\frac{a}{a+ab+1} + \frac{1}{a+ab+1}\right),$$

$$\frac{\sqrt{b}}{b+bc+1} \le \frac{1}{2}\left(\frac{b}{b+bc+1} + \frac{1}{b+bc+1}\right),$$

$$\frac{\sqrt{c}}{c+ca+1} \le \frac{1}{2}\left(\frac{c}{c+ca+1} + \frac{1}{c+ca+1}\right).$$

Summing up these relations and note that the familiar identities yields the desired result.

□

LỜI GIẢI 2. We will apply Cauchy-Schwarz inequality be the following way

(a)

$$\left(\frac{a^2}{a+ab+1} + \frac{b^2}{b+bc+1} + \frac{c^2}{c+ca+1}\right)\left(\frac{1}{a+ab+1} + \frac{1}{b+bc+1} + \frac{1}{c+ca+1}\right)$$
$$\ge \left(\frac{a}{a+ab+1} + \frac{b}{b+bc+1} + \frac{c}{c+ca+1}\right)^2 = 1.$$

This combines with two familiar identities above to give us the required inequality.

(b)

$$\left(\frac{\sqrt{a}}{a+ab+1} + \frac{\sqrt{b}}{b+bc+1} + \frac{\sqrt{c}}{c+ca+1}\right)^2$$
$$\le \left(\frac{a}{a+ab+1} + \frac{b}{b+bc+1} + \frac{c}{c+ca+1}\right)\left(\frac{1}{a+ab+1} + \frac{1}{b+bc+1} + \frac{1}{c+ca+1}\right)$$
$$= 1.$$

This combines with two familiar identities above, the desired inequality follows.

□

Remark 1. By the similar way, we can show that more general results as follows: If $a, b, c > 0$ and $abc = 1$ then for all positive integers n,

(a) $\dfrac{a^n}{a+ab+1} + \dfrac{b^n}{b+bc+1} + \dfrac{c^n}{c+ca+1} \ge 1,$

(b) $\dfrac{\sqrt[n]{a}}{a+ab+1} + \dfrac{\sqrt[n]{b}}{b+bc+1} + \dfrac{\sqrt[n]{c}}{c+ca+1} \le 1.$

Problem 12 (Nguyen Viet Hung)**.** Let a, b, c be real numbers such that $abc = 1$. Prove that

$$\frac{a+ab+1}{(a+ab+1)^2+1} + \frac{b+bc+1}{(b+bc+1)^2+1} + \frac{c+ca+1}{(c+ca+1)^2+1} \le \frac{9}{10}.$$

(Mathematical Reflections)

Solution. The desired inequality is equivalent to

$$\frac{\frac{1}{1+a+ab}}{\frac{1}{(1+a+ab)^2}+1}+\frac{\frac{1}{1+b+bc}}{\frac{1}{(1+b+bc)^2}+1}+\frac{\frac{1}{1+c+ca}}{\frac{1}{(1+c+ca)^2}+1}\le\frac{9}{10}$$

We put $\frac{1}{1+b+bc}=x$, $\frac{1}{1+c+ca}=y$, $\frac{1}{1+a+ab}=z$ and since the condition $abc=1$, then

$$x+y+z=1.$$

Our inequality becomes

$$\frac{x}{x^2+1}+\frac{y}{y^2+1}+\frac{z}{z^2+1}\le\frac{9}{10} \tag{1}$$

Among these numbers $x-\frac{1}{3}, y-\frac{1}{3}$, and $z-\frac{1}{3}$, there exist two numbers that both of them are not positive or not negative. Without loss of generality suppose they are $y-\frac{1}{3}$, and $z-\frac{1}{3}$. Then we have

$$\left(y-\frac{1}{3}\right)\left(z-\frac{1}{3}\right)\ge 0$$

or

$$y^2+z^2\le\frac{1}{9}+\left(y+z-\frac{1}{3}\right)^2=\frac{1}{9}+\left(\frac{2}{3}-x\right)^2 \tag{2}$$

(1) is equivalent to

$$\frac{x}{x^2+1}\le\left(\frac{1}{2}-\frac{y}{y^2+1}\right)+\left(\frac{1}{2}-\frac{z}{z^2+1}\right)-\frac{1}{10}$$

or

$$\frac{2x}{x^2+1}+\frac{1}{5}\le\frac{(y-1)^2}{y^2+1}+\frac{(z-1)^2}{z^2+1}$$

Using the Cauchy-Schwarz inequality and remark (2) we get

$$\frac{(y-1)^2}{y^2+1}+\frac{(z-1)^2}{z^2+1}\ge\frac{(y+z-2)^2}{y^2+z^2+2}=\frac{(1+x)^2}{\frac{1}{9}+\left(\frac{2}{3}-x\right)^2+2}=\frac{9(1+x)^2}{23-12x+9x^2}$$

It remains to prove that

$$\frac{9(1+x)^2}{23-12x+9x^2}\ge\frac{1+10x+x^2}{5(1+x^2)}$$

But this is equivalent to

$$(3x-1)^2(2x^2+2x+11)\ge 0,$$

which is obvious true and we are done. The equality occurs iff $x=y=z=\frac{1}{3}$, i.e. $a=b=c=1$. □

To end this article, we invite readers practise the following problems

3 Proposed problems

Excersise 1. Let a,b,c be positive real numbers such that $abc^2=1$. Prove that

(a) $$\frac{1}{1+a+ab+abc}+\frac{1}{1+b+bc+bc^2}+\frac{1}{1+c+c^2+c^2a}+\frac{1}{1+c+ca+cab}=1.$$

(b) $$\frac{a}{1+a+ab+abc}+\frac{b}{1+b+bc+bc^2}+\frac{c}{1+c+c^2+c^2a}+\frac{c}{1+c+ca+cab}=1.$$

Excersise 2. Let a,b,c be positive real numbers such that $abc=1$. Prove that

(a) $\dfrac{1}{6+2a^3+b^3}+\dfrac{1}{6+2b^3+c^3}+\dfrac{1}{6+2c^3+a^3}\le\dfrac{1}{3}.$

(b) $\dfrac{a}{6+2a^3+b^3}+\dfrac{b}{6+2b^3+c^3}+\dfrac{c}{6+2c^3+a^3}\le\dfrac{1}{3}.$

Excersise 3. Prove the following identity holds

$$\frac{1}{\sqrt[8]{2}+\sqrt[4]{2}+\sqrt{2}-2}+\frac{1}{\sqrt[8]{32}+\sqrt[8]{2}+\sqrt[4]{8}+\sqrt[4]{2}-\sqrt{2}}+\frac{1}{\sqrt[8]{128}+\sqrt[8]{32}+\sqrt[8]{8}+\sqrt[8]{2}+\sqrt[4]{2}+2}$$
$$+\frac{1}{\sqrt[8]{128}+\sqrt[8]{32}+2\sqrt[8]{8}+3\sqrt[8]{2}+\sqrt[4]{8}+2\sqrt[4]{2}+\sqrt{2}+4}=1.$$

Excersise 4. Let a,b,c be positive real numbers such that $abc=1$. Prove that

(a) $\dfrac{1}{(a+b)^2+(a+1)^2+4}+\dfrac{1}{(b+c)^2+(b+1)^2+4}+\dfrac{1}{(c+a)^2+(c+1)^2+4}\le\dfrac{1}{4}.$

(b) $\dfrac{a}{(a+b)^2+(a+1)^2+4}+\dfrac{b}{(b+c)^2+(b+1)^2+4}+\dfrac{c}{(c+a)^2+(c+1)^2+4}\le\dfrac{1}{4}.$

Excersise 5. Let a,b,c be positive real numbers such that $abc=1$. Prove that for every positive integers $n\ge 2$,

(a) $\dfrac{1}{3(n-1)+2a^n+b^n}+\dfrac{1}{3(n-1)+2b^n+c^n}+\dfrac{1}{3(n-1)+2c^n+a^n}\le\dfrac{1}{n}.$

(b) $\dfrac{a}{3(n-1)+2a^n+b^n}+\dfrac{b}{3(n-1)+2b^n+c^n}+\dfrac{c}{3(n-1)+2c^n+a^n}\le\dfrac{1}{n}.$

Excersise 6. Let a,b,c be positive real numbers such that $abc=1$. Prove that for every positive integers $n\ge 2$,

(a) $\dfrac{1}{(1+a+ab)^n}+\dfrac{1}{(1+b+bc)^n}+\dfrac{1}{(1+c+ca)^n}\ge\dfrac{1}{3^{n-1}}.$

(b) $\dfrac{1}{(1+a+ab)^n}+\dfrac{1}{(1+b+bc)^n}+\dfrac{1}{(1+c+ca)^n}\ge\dfrac{1}{3^{n-2}(a^2+b^2+c^2)}.$

Excersise 7. Let a,b,c,d be positive real numbers such that $abcd=1$. Prove that

(a) $\dfrac{1}{(1+a+ab+abc)^2}+\dfrac{1}{(1+b+bc+bcd)^2}+\dfrac{1}{(1+c+cd+cda)^2}+\dfrac{1}{(1+d+da+dab)^2}\ge\dfrac{1}{4}.$

(b) $\dfrac{a}{(1+a+ab+abc)^2}+\dfrac{b}{(1+b+bc+bcd)^2}+\dfrac{c}{(1+c+cd+cda)^2}+\dfrac{d}{(1+d+da+dab)^2}\ge\dfrac{1}{a+b+c+d}.$

Excersise 8. Let a,b,c be positive real numbers such that $abc=1$. Prove that

(a) $\dfrac{1}{(a+1)^2+b^2+1}+\dfrac{1}{(b+1)^2+c^2+1}+\dfrac{1}{(c+1)^2+a^2+1}\le\dfrac{1}{2}.$

(Mathematical Reflections)

(b) $\dfrac{a}{(a+1)^2+b^2+1}+\dfrac{b}{(b+1)^2+c^2+1}+\dfrac{c}{(c+1)^2+a^2+1}\le\dfrac{1}{2}.$

Excersise 9. Let a,b,c be positive real numbers such that $abc=1$. Prove that

(a) $\dfrac{1}{a^2+2ab+3}+\dfrac{1}{b^2+2bc+3}+\dfrac{1}{c^2+2ca+3}\le\dfrac{1}{2}.$

(b) $\dfrac{a}{a^2+2ab+3}+\dfrac{b}{b^2+2bc+3}+\dfrac{c}{c^2+2ca+3}\le\dfrac{1}{2}.$

Excersise 10. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a) $\dfrac{1}{a(1+a+ab)^2}+\dfrac{1}{b(1+b+bc)^2}+\dfrac{1}{c(1+c+ca)^2}\ge\dfrac{1}{a+b+c}.$

(b) $\dfrac{a}{b(1+a+ab)^2}+\dfrac{b}{c(1+b+bc)^2}+\dfrac{c}{a(1+c+ca)^2}\ge\dfrac{1}{ab+bc+ca}.$

Excersise 11. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

(a) $\dfrac{1}{2+a+ab+abc}+\dfrac{1}{2+b+bc+bcd}+\dfrac{1}{2+c+cd+cda}+\dfrac{1}{2+d+da+dab}\le\dfrac{4}{5}.$

(b) $\dfrac{a}{2+a+ab+abc}+\dfrac{b}{2+b+bc+bcd}+\dfrac{c}{2+c+cd+cda}+\dfrac{d}{2+d+da+dab}\le\dfrac{16+a+b+c+d}{25}.$

Excersise 12. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{b+bc-2}{(a+ab+1)^2}+\frac{c+ca-2}{(b+bc+1)^2}+\frac{a+ab-2}{(c+ca+1)^2}\ge 0.$$

References

[1] *Mathematics and Young Magazine,* Education Publishing House of Vietnam.

[2] *Mathematical Reflections Journal.*

[3] Dusan Djukic, Vladimir Jankovic, Ivan Matic, Nikola Petrovic, *The IMO Compendium*, Springer, 2004.

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