Around A Nice Inequality

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Abstract. On the Mathematical Reflections Journal, at the article "An Unexpectedly Useful Inequality" of the author Pham Huu Đuc and the article "On Some Geometric Inequalities" of the author Tran Quang Hung, there are many problem is established based on a nice inequality. In this paper, we are continue to give some problems from that original inequality and its generalizations.

1 The original inequality and its generalizations

The first we will recall the original inequality and its proof

Problem 1. For all non-negative real number a, b, c, x, y, z, the following inequality holds

$$(b+c)x + (c+a)y + (a+b)z \geq 2\sqrt{(ab+bc+ca)(xy+yz+zx)}.$$

Proof. Applying Cauchy-Schwarz inequality twice we obtain

$$\begin{aligned}
&(b+c)x + (c+a)y + (a+b)z \\
&= (a+b+c)(x+y+z) - (ax+by+cz) \\
&= \sqrt{[(a^2+b^2+c^2)+2(ab+bc+ca)][(x^2+y^2+z^2)+2(xy+yz+zx)]} - (ax+by+cz) \\
&\geq \sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} + 2\sqrt{(ab+bc+ca)(xy+yz+zx)} - (ax+by+cz) \\
&\geq 2\sqrt{(ab+bc+ca)(xy+yz+zx)}.
\end{aligned}$$

The equality occurs if and only if $\dfrac{a}{x} = \dfrac{b}{y} = \dfrac{c}{z}$. □

The above inequality has a generalization as follows

Problem 2 (Generalization)**.** If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are non-negative real numbers and let $S = b_1 + b_2 + \cdots + b_n$, then

$$a_1(S-b_1) + a_2(S-b_2) + \cdots + a_n(S-b_n) \geq 2\sqrt{\Big(\sum_{1\leq i<j\leq n} a_i a_j\Big)\Big(\sum_{1\leq i<j\leq n} b_i b_j\Big)}.$$

Proof. Using Cauchy-Schwarz inequality yields

$$\begin{aligned}(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n) &= \sqrt{(a_1+a_2+\cdots+a_n)^2(b_1+b_2+\cdots+b_n)^2}\\ &= \sqrt{\left[(a_1^2+\cdots+a_n^2)+2\sum_{1\le i<j\le n}a_ia_j\right]\left[(b_1^2+\cdots+b_n^2)+2\sum_{1\le i<j\le n}b_ib_j\right]}\\ &\ge \sqrt{(a_1^2+\cdots+a_n^2)((b_1^2+\cdots+b_n^2)}+2\sqrt{(\sum_{1\le i<j\le n}a_ia_j)(\sum_{1\le i<j\le n}b_ib_j)}\\ &\ge a_1b_1+a_2b_2+\cdots+a_nb_n+2\sqrt{(\sum_{1\le i<j\le n}a_ia_j)(\sum_{1\le i<j\le n}b_ib_j)}\end{aligned}$$

It follows that

$$a_1(S-b_1)+a_2(S-b_2)+\cdots+a_n(S-b_n)\ge 2\sqrt{(\sum_{1\le i<j\le n}a_ia_j)(\sum_{1\le i<j\le n}b_ib_j)}.$$

The proof is complete. The equality occurs when

$$a_1=\lambda b_1, a_2=\lambda b_2,\ldots,a_n=\lambda b_n.$$

□

The case $n=4$, we have a rather beautiful result as follows

Problem 3. For all non-negative real numbers a,b,c,d,x,y,z,t the following inequality holds

$$\begin{aligned}&(b+c+d)x+(c+d+a)y+(d+a+b)z+(a+b+c)t\\ &\ge 2\sqrt{(ab+ac+ad+bc+bd+cd)(xy+xz+xt+yz+yt+zt)}.\end{aligned}$$

Especially when we take $(x,y,z,t)=(4,3,2,1)$ then to get a rather interesting problem below

Problem 4. For all non-negative real numbers a,b,c,d the following inequality holds

$$(6a+7b+8c+9d)^2\ge 140(ab+ac+ad+bc+bd+cd).$$

Remark 1. This problem is even true for all arbitrary real numbers a,b,c,d. Indeed, it is equivalent to

$$\frac{7}{3}(3a-4b)^2+11(a-2c)^2+4(a-4d)^2+\frac{7}{3}(2b-3c)^2+\frac{7}{3}(b-3d)^2+(c-2d)^2\ge 0$$

which is obviously true. The equality happens iff

$$\frac{a}{4}=\frac{b}{3}=\frac{c}{2}=d.$$

Applying the problem 3 and using the result

$$\frac{ab}{(b+c)(c+d)}+\frac{bc}{(c+d)(d+a)}+\frac{cd}{(d+a)(a+b)}+\frac{da}{(a+b)(b+c)}+\frac{ac}{(b+c)(d+a)}+\frac{bd}{(c+d)(a+b)}\ge 1$$

(note that the equality occurs when $a=c=0,b=d$ or $a=c,b=d=0$) we obtain

$$(y+z+t)\frac{a}{b+c}+(z+t+x)\frac{b}{c+d}+(t+x+y)\frac{c}{d+a}+(x+y+z)\frac{d}{a+b}\ge 2\sqrt{xy+yz+zt+tx+xz+yt}.$$

We have another extension of the original problem using Holder's inequality as follows

Problem 5. For all non-negative real numbers $a, b, c, x, y, z, u, v, w$ the following inequality holds

$$ax(v+w)+by(w+u)+cz(u+v)+[a(y+z)+b(z+x)+c(x+y)](u+v+w) \geq 3\sqrt[3]{ABC}$$

where

$$A=(a+b)(b+c)(c+a), B=(x+y)(y+z)(z+x), C=(u+v)(v+w)(w+u).$$

Solution. We have

$$\begin{aligned}
LHS &= (a+b+c)(x+y+z)(u+v+w)-(axu+byv+czw)\\
&= \sqrt[3]{(a+b+c)^3(x+y+z)^3(u+v+w)^3}-(axu+byv+czw)\\
&= \sqrt[3]{(a^3+b^3+c^3+3(a+b)(b+c)(c+a))(x^3+y^3+z^3+3(x+y)(y+z)(z+x))}\\
&\times \sqrt[3]{(u^3+v^3+w^3+3(u+v)(v+w)(w+u))}-(axu+byv+czw)\\
&= \sqrt[3]{(a^3+b^3+c^3+3A)(x^3+y^3+z^3+3B)(u^3+v^3+w^3+3C)}-(axu+byv+czw)
\end{aligned}$$

Applying Holder's inequality twice gives us

$$\begin{aligned}
LHS &\geq \sqrt[3]{(a^3+b^3+c^3)(x^3+y^3+z^3)(u^3+v^3+w^3)}+3\sqrt[3]{ABC}-(axu+byv+czw)\\
&\geq 3\sqrt[3]{ABC}
\end{aligned}$$

as desired. □

2 Some applications

Now we will use the original inequality to give some new problems. In problems below we will use the known normal notations about triangle ABC. For example a, b, c are side-lengths; h_a, h_b, h_c are the lengths of altitudes; r_a, r_b, r_c are exradii correspond to sides BC, CA, AB; S, s, R, and r are respectively the area, semi-perimeter, circumradius and inradius of triangle ABC.

At problem 1, if we replace (x, y, z) by $(\tan\frac{A'}{2}, \tan\frac{B'}{2}, \tan\frac{C'}{2})$ and using the known identity

$$\tan\frac{A'}{2}\tan\frac{B'}{2}+\tan\frac{B'}{2}\tan\frac{C'}{2}+\tan\frac{C'}{2}\tan\frac{A'}{2}=1$$

then to get the following problem

Problem 6. For two any triangles ABC and $A'B'C'$ the following inequality holds

$$(b+c)\tan\frac{A'}{2}+(c+a)\tan\frac{B'}{2}+(a+b)\tan\frac{C'}{2} \geq 2\sqrt{ab+bc+ca}.$$

From this problem we continue to substitute (a, b, c) by (r_a, r_b, r_c) and note that identity $r_ar_b+r_br_c+r_cr_a=s^2$, we have

Problem 7. If ABC and $A'B'C'$ are two triangles then

$$(r_b+r_c)\tan\frac{A'}{2}+(r_c+r_a)\tan\frac{B'}{2}+(r_a+r_b)\tan\frac{C'}{2} \geq a+b+c.$$

From the problem 7, we replace again (r_a, r_b, r_c) by $(\sqrt{r_a}, \sqrt{r_b}, \sqrt{r_c})$ and using the relations

$$\sqrt{r_br_c} \geq h_a, \sqrt{r_cr_a} \geq h_b, \sqrt{r_ar_b} \geq h_c$$

then to obtain

Problem 8. If ABC and $A'B'C'$ are two triangles then

$$(\sqrt{r_b}+\sqrt{r_c})\tan\frac{A'}{2}+(\sqrt{r_c}+\sqrt{r_a})\tan\frac{B'}{2}+(\sqrt{r_a}+\sqrt{r_b})\tan\frac{C'}{2}\geq 2\sqrt{h_a+h_b+h_c}\geq 6\sqrt{r}.$$

At problem 6, substituting (a,b,c) by $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ and using the identity

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}$$

we get that

Problem 9. For two any triangles ABC and $A'B'C'$ the following inequality holds

(a) $\left(\frac{1}{b}+\frac{1}{c}\right)\tan\frac{A'}{2}+\left(\frac{1}{c}+\frac{1}{a}\right)\tan\frac{B'}{2}+\left(\frac{1}{a}+\frac{1}{b}\right)\tan\frac{C'}{2}\geq\sqrt{\frac{2}{Rr}}.$

(b) $\left(\frac{1}{b}+\frac{1}{c}\right)\frac{1}{a'}+\left(\frac{1}{c}+\frac{1}{a}\right)\frac{1}{b'}+\left(\frac{1}{a}+\frac{1}{b}\right)\frac{1}{c'}\geq\frac{1}{\sqrt{RR'rr'}}.$

Also from the problem 6, replacing (a,b,c) by $(s-a,s-b,s-c)$ and using the identity

$$(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)=r(4R+r)$$

we obtain

Problem 10. For two any triangles ABC and $A'B'C'$ the following inequality holds

$$a\cdot\tan\frac{A'}{2}+b\cdot\tan\frac{B'}{2}+c\cdot\tan\frac{C'}{2}\geq 2\sqrt{r(4R+r)}.$$

By similar way we also establish two results below

Problem 11. Prove that in any triangle ABC,

$$PA+PB+PC\geq 2\sqrt{r(4R+r)}\geq 6r.$$

Problem 12. For any triangle ABC and all positive real numbers x,y,z the following inequality holds

$$a\cdot\frac{x}{y+z}+b\cdot\frac{y}{z+x}+c\cdot\frac{z}{x+y}\geq\sqrt{3r(4R+r)}\geq 3\sqrt{3}r.$$

We start again from problem 1, substituting (x,y,z) by $\left(\frac{1}{s-a},\frac{1}{s-b},\frac{1}{s-c}\right)$ and using the known identity

$$\frac{1}{(s-a)(s-b)}+\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}=\frac{1}{r^2}$$

gives us

Problem 13. For two any triangles ABC and $A'B'C'$ we have

(a) $\frac{b+c}{s'-a'}+\frac{c+a}{s'-b'}+\frac{a+b}{s'-c'}\geq\frac{2}{r'}\sqrt{ab+bc+ca}.$

(b) $\frac{r_b+r_c}{s'-a'}+\frac{r_c+r_a}{s'-b'}+\frac{r_a+r_b}{s'-c'}\geq\frac{a+b+c}{r'}.$

Similarly as above we also have

Problem 14. For any triangle ABC and an abitrary point in its plane, the following inequality holds

$$\frac{PA}{(s-b)(s-c)}+\frac{PB}{(s-c)(s-a)}+\frac{PC}{(s-a)(s-b)}\geq\frac{2}{r}.$$

At problem 6, substituting (a,b,c) by $\left(\sqrt{\frac{s-a}{a}},\sqrt{\frac{s-b}{b}},\sqrt{\frac{s-c}{c}}\right)$ and using the known formulas

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}},\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$$

we get that

Problem 15. For two any triangles ABC and $A'B'C'$ the following inequality holds

$$\left(\sqrt{\frac{s-b}{b}}+\sqrt{\frac{s-c}{c}}\right)\tan\frac{A'}{2}+\left(\sqrt{\frac{s-c}{c}}+\sqrt{\frac{s-a}{a}}\right)\tan\frac{B'}{2}+\left(\sqrt{\frac{s-a}{a}}+\sqrt{\frac{s-b}{b}}\right)\tan\frac{C'}{2}$$
$$\geq 2\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}.$$

At problem 1, replacing (x,y,z) by $\left(\frac{x}{y+z},\frac{y}{z+x},\frac{z}{x+y}\right)$ and (a,b,c) by $\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ and $\left(\frac{PA}{a},\frac{PB}{b},\frac{PC}{c}\right)$, respectively, combining with the known result

$$\frac{xy}{(z+x)(z+y)}+\frac{yz}{(x+y)(x+z)}+\frac{zx}{(y+z)(y+x)}\geq\frac{3}{4} \tag{1}$$

leads us to

Problem 16. Given a triangle ABC and any point P in its plane. The following inequalities hold for all positive real numbers x,y,z

(a) $\frac{x}{y+z}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{y}{z+x}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{z}{x+y}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq\sqrt{3}.$

(b) $\frac{x}{y+z}\left(\frac{PB}{b}+\frac{PC}{c}\right)+\frac{y}{z+x}\left(\frac{PC}{c}+\frac{PA}{a}\right)+\frac{z}{x+y}\left(\frac{PA}{a}+\frac{PB}{b}\right)\geq\sqrt{3}.$

Moreover, by problem 1 we can easily solve the problem below

Problem 17. Prove that for all positive real numbers a,b,c,x,y,z

$$\frac{x(b+c)}{y+z}+\frac{y(c+a)}{z+x}+\frac{z(a+b)}{x+y}\geq\frac{3(ab+bc+ca)}{a+b+c}.$$

Solution. Indeed, according to the result of problem 1 and inequality (1) we have

$$\frac{x(b+c)}{y+z}+\frac{y(c+a)}{z+x}+\frac{z(a+b)}{x+y}\geq\sqrt{3(ab+bc+ca)}.$$

Therefore it suffices to show that

$$\sqrt{3(ab+bc+ca)}\geq\frac{3(ab+bc+ca)}{a+b+c}$$

This inequality is equivalent to

$$(a+b+c)^2\geq 3(ab+bc+ca)$$

which is the well-known result and we are done. □

We continue to profit by problem 1, substituting (x,y,z) by $(\frac{x}{y+z},\frac{y}{z+x},\frac{z}{x+y})$ and (a,b,c) by $\left(\frac{1}{s-a},\frac{1}{s-b},\frac{1}{s-c}\right)$ we obtain

Problem 18. For any triangle ABC and all positive real numbers x,y,z then

$$\frac{xa}{(y+z)(s-b)(s-c)}+\frac{yb}{(z+x)(s-c)(s-a)}+\frac{zc}{(x+y)(s-a)(s-b)}\geq\frac{\sqrt{3}}{r}.$$

The above result can be rewritten as

$$\frac{xa(p-a)}{y+z}+\frac{yb(p-b)}{z+x}+\frac{zc(p-c)}{x+y}\geq\sqrt{3}S$$

which is equivalent to

$$\frac{x}{y+z}\cdot\frac{p-a}{h_a}+\frac{y}{z+x}\cdot\frac{p-b}{h_b}+\frac{z}{x+y}\cdot\frac{p-c}{h_c}\geq\frac{\sqrt{3}}{2}.$$

In problem 1 we replace (a,b,c) by $(\cot A,\cot B,\cot C)$ and (x,y,z) by $(\cot A',\cot B',\cot C')$ then to get that

Problem 19. If ABC and $A'B'C'$ are two acute triangles then

$$(\cot B+\cot C)\cot A'+(\cot C+\cot A)\cot B'+(\cot A+\cot B)\cot C'\geq 2.$$

Remark 2. Using the formula $\cot A=\frac{b^2+c^2-a^2}{4S}$ we can rewrite the above inequality as follows

$$(b^2+c^2-a^2)a'^2+(c^2+a^2-b^2)b'^2+(a^2+b^2-c^2)c'^2\geq 16SS'$$

which is Neuberg-Pedoe inequality - a well-known result.

Also, if we substitute (a,b,c) by (a^3,b^3,c^3) and (x,y,z) by $(\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b})$ then to obtain

$$\sum_{\text{cyclic}}(b^3+c^3)\frac{1}{b+c}\geq 2\sqrt{(a^3b^3+b^3c^3+c^3a^3)\left[\frac{1}{(a+b)(b+c)}+\frac{1}{(b+c)(c+a)}+\frac{1}{(c+a)(a+b)}\right]},$$

or

$$\sum_{\text{cyclic}}(b^2-bc+c^2)\geq 2\sqrt{\frac{2(a+b+c)(a^3b^3+b^3c^3+c^3a^3)}{(a+b)(b+c)(c+a)}},$$

or

$$2(a^2+b^2+c^2)\geq ab+bc+ca+2\sqrt{\frac{2(a+b+c)(a^3b^3+b^3c^3+c^3a^3)}{(a+b)(b+c)(c+a)}}.$$

So we have just got a problem as folows

Problem 20. For all positive real numbers a,b,c the following inequality holds

$$a^2+b^2+c^2\geq\frac{1}{2}(ab+bc+ca)+\sqrt{\frac{2(a+b+c)(a^3b^3+b^3c^3+c^3a^3)}{(a+b)(b+c)(c+a)}}.$$

By similar way as the above problem but we substitute (x,y,z) by $\left(\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\right)$ then to obtain

Problem 21. The following inequality holds for all positive real numbers a, b, c

$$ab(a+b)+bc(b+c)+ca(c+a) \geq 3abc+\sqrt{3(a^3b^3+b^3c^3+c^3a^3)}.$$

Also at the original problem, we replace (a, b, c) by $(\frac{PA}{a}, \frac{PB}{b}, \frac{PC}{c})$ and (x, y, z) by $(\frac{P'A'}{a'}, \frac{P'B'}{b'}, \frac{P'C'}{c'})$ and combine with Hayashi's inequality, then to have a problem below

Problem 22. Consider two triangles ABC and $A'B'C'$. P and P' are two any points. The following inequality holds

$$\left(\frac{PB}{b}+\frac{PC}{c}\right)\frac{P'A'}{a'}+\left(\frac{PC}{c}+\frac{PA}{a}\right)\frac{P'B'}{b'}+\left(\frac{PA}{a}+\frac{PB}{b}\right)\frac{P'C'}{c'} \geq 2.$$

Basing on the original problem you can find many other interesting inequalities by yourself. To finish this paper, we will give some problem for your practice

3 Proposed problems

Excersise 1. Prove that for two any triangles ABC and $A'B'C'$ the following inequality holds

$$(b+c)a'+(c+a)b'+(a+b)c' \geq 8\sqrt{3SS'}.$$

Excersise 2. Prove that for any triangle ABC and all positive real numbers x, y, z the following inequality holds

$$\frac{x}{y+z}\cdot\frac{\sin A}{\sin B\sin C}+\frac{y}{z+x}\cdot\frac{\sin B}{\sin C\sin A}+\frac{z}{x+y}\cdot\frac{\sin C}{\sin A\sin B} \geq \sqrt{3}.$$

Excersise 3. Prove that for two any triangles ABC and $A'B'C'$

$$\frac{\sqrt{r_b}+\sqrt{r_c}}{s'-a'}+\frac{\sqrt{r_c}+\sqrt{r_a}}{s'-b'}+\frac{\sqrt{r_a}+\sqrt{r_b}}{s'-c'} \geq \frac{2}{r'}\sqrt{h_a+h_b+h_c} \geq \frac{6r}{r'}.$$

Excersise 4. Prove that for two any triangles ABC and $A'B'C'$

$$(b+c)\tan\frac{A'}{2}+(c+a)\tan\frac{B'}{2}+(a+b)\tan\frac{C'}{2} \geq 12r.$$

Excersise 5 (Titu Andreescu, Gabriel Dospinescu)**.** Let a, b, c, x, y, z be positive real numbers such that $xy+yz+zx=3$. Prove that

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y) \geq 3.$$

Excersise 6. Let ABC be a triangle and let P be any point in the plane. Prove that

$$\left(\frac{1}{b}+\frac{1}{c}\right)\frac{PA}{a}+\left(\frac{1}{c}+\frac{1}{a}\right)\frac{PB}{b}+\left(\frac{1}{a}+\frac{1}{b}\right)\frac{PC}{c} \geq \sqrt{\frac{2}{Rr}}.$$

Excersise 7. Let a, b, c be three side-lengths of a triangle with the area S and let x, y, z be positive real numbers. Prove that

$$\frac{x(b+c)}{y+z}+\frac{y(c+a)}{z+x}+\frac{z(a+b)}{x+y} \geq \sqrt{6\sqrt{3}S+\frac{1}{2}(a^2+b^2+c^2)}.$$

Excersise 8. Prove that for two any triangles ABC and $A'B'C'$ we have

$$\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\tan\frac{A'}{2}+\left(\frac{1}{h_c}+\frac{1}{h_a}\right)\tan\frac{B'}{2}+\left(\frac{1}{h_a}+\frac{1}{h_b}\right)\tan\frac{C'}{2}\geq\frac{12}{a+b+c}.$$

Excersise 9. Let a,b,c,x,y,z be positive real numbers such that $xy+yz+zx+2xyz=1$. Prove that

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y)\geq\frac{3}{2}.$$

Excersise 10. Given a triangle ABC and M is any point in its plane. Let x,y,z be positive real numbers such that $xy+yz+zx+2xyz=1$. Prove that

$$(y+z)\frac{MA}{a}+(z+x)\frac{MB}{b}+(x+y)\frac{MC}{c}\geq\sqrt{3}.$$

Excersise 11. Prove that for two any triangles ABC and $A'B'C'$ we have

(a) $\left(\frac{m_b}{b}+\frac{m_c}{c}\right)\tan\frac{A'}{2}+\left(\frac{m_c}{c}+\frac{m_a}{a}\right)\tan\frac{B'}{2}+\left(\frac{m_a}{a}+\frac{m_b}{b}\right)\tan\frac{C'}{2}\geq 3.$

(b) $\left(\frac{b}{m_b}+\frac{c}{m_c}\right)\tan\frac{A'}{2}+\left(\frac{c}{m_c}+\frac{a}{m_a}\right)\tan\frac{B'}{2}+\left(\frac{a}{m_a}+\frac{b}{m_b}\right)\tan\frac{C'}{2}\geq 4.$

Excersise 12 (Nguyen Viet Hung)**.** Let ABC be a triangle and let M be any point in its plane. Prove that

(a) $\left(\frac{m_b}{b}+\frac{m_c}{c}\right)\frac{MA}{a}+\left(\frac{m_c}{c}+\frac{m_a}{a}\right)\frac{MB}{b}+\left(\frac{m_a}{a}+\frac{m_b}{b}\right)\frac{MC}{c}\geq 3.$

(b) $\left(\frac{b}{m_b}+\frac{c}{m_c}\right)\frac{MA}{a}+\left(\frac{c}{m_c}+\frac{a}{m_a}\right)\frac{MB}{b}+\left(\frac{a}{m_a}+\frac{b}{m_b}\right)\frac{MC}{c}\geq 4.$

(Vietnamese Mathematics and Young Magazine)

Excersise 13 (Nguyen Viet Hung)**.** Let ABC be a triangle and let P be any point in its plane. Prove that

$$(b+c)PA+(c+a)PB+(a+b)PC\geq 2\sqrt{abc(a+b+c)}.$$

(Romanian Mathematical Magazine)

Excersise 14. Let ABC be a triangle and let P be any point in its plane. Prove that

$$\left(\frac{r_b}{b}+\frac{r_c}{c}\right)\frac{PA}{a}+\left(\frac{r_c}{c}+\frac{r_a}{a}\right)\frac{PB}{b}+\left(\frac{r_a}{a}+\frac{r_b}{b}\right)\frac{PC}{c}\geq 2\sqrt{3}.$$

Excersise 15. Prove that in any triangle ABC,

$$\frac{ar_a}{m_a}+\frac{br_b}{m_b}+\frac{cr_c}{m_c}\geq 2\sqrt{3r(4R+r)}.$$

Excersise 16. Prove that the following inequality holds for all positive real numbers a,b,c

$$\sum_{cyc}(a^2+bc)(b^2-bc+c^2)\geq 2\sqrt{(a^2+b^2+c^2)(a^3b^3+b^3c^3+c^3a^3)}.$$

Hint: Using the identity

$$\frac{a^2+bc}{b+c}\cdot\frac{b^2+ca}{c+a}+\frac{b^2+ca}{c+a}\cdot\frac{c^2+ab}{a+b}+\frac{c^2+ab}{a+b}\cdot\frac{a^2+bc}{b+c}=a^2+b^2+c^2.$$

Tài liệu

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[3] *Mathematics and Young Magazine*, Education Publishing House of Vietnam.