An Application of the Fermat-Torricelli Point

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Abstract: *In this note we will give several proofs of some interesting inequalities concerning the Fermat-Torricelli point of a triangle*

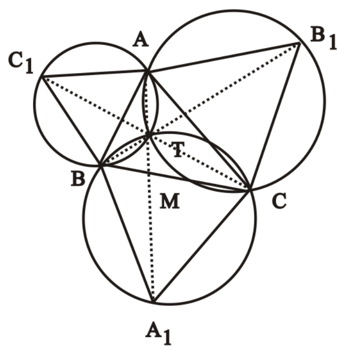
In this paper we will prove some interesting results derived from the Fermat-Torricelli point of a triangle.

At first, we will see some fundamental properties, which are proven in the book [1]

Definition 1. *Let ABC be a triangle. The Fermat point of ABC (also known as its Fermat-Torricelli point) is that point of the plane (ABC) for which the sum $MA + MB + MC$ is minimal, where M is a point of (ABC).*

Theorem 1. (Torricelli) *Let ABC be a triangle with the measure of each angle smaller than $\frac{2\pi}{3}$. Let ABC_1, ACB_1 and BCA_1 be equilateral triangles, with their interiors situated in the exterior of ABC. Then, the circumscribed circles of these triangles have a common point, T.*

Remark. From the proof of the previous theorem it follows that an unique point T of the plane exists such that $\mu\left(\widehat{ATB}\right) = \mu\left(\widehat{ATC}\right) = \mu\left(\widehat{BTC}\right) = \frac{2\pi}{3}$.
T is named the Torricelli point of the triangle ABC. ■



Theorem 2. *Let ABC be a triangle with the measure of each angle smaller than $\frac{2\pi}{3}$, the equilateral triangles ABC_1, ACB_1 and BCA_1 with their interiors situated in the exterior of ABC and the Torricelli point T of ABC. Then:*

(a) *The lines AA_1, BB_1 and CC_1 are concurrent.*

(b) $AT + BT + CT = AA_1 = BB_1 = CC_1$.

Theorem 3. (Fermat) *The sum $MA + MB + MC$, where M is a point of (ABC), is minimal iff M coincides with the Torricelli point T of ABC. Therefore, the Fermat point and the Torricelli point of a triangle coincide.*

Remark. If $A \geq \frac{2\pi}{3}$, then the Fermat-Torricelli point of ABC coincides with A. ■

In what follows, we will see some interesting applications of the Fermat-Torricelli point.

Proposition 1. *Let ABC be a triangle and T its Fermat-Torricelli point. Then,*

$$BT + CT \leq \frac{2}{\sqrt{3}} BC.$$

Proof 1.

Case I: $A < \frac{2\pi}{3}$, $B < \frac{2\pi}{3}$, $C < \frac{2\pi}{3}$.

From Theorem 2 we have $AT + BT + CT = AA_1$.

Let $\{M\} = BC \cap AA_1$.

As $\mu\left(\widehat{BTA_1}\right) = \mu\left(\widehat{CTA_1}\right) = \frac{\pi}{3}$, we obtain $d\left(B, AA_1\right) = \frac{BT\sqrt{3}}{2} \leq BM$ and $d\left(C, AA_1\right) = \frac{CT\sqrt{3}}{2} \leq CM$, so

$$d\left(B, AA_1\right) + d\left(C, AA_1\right) = \frac{BT\sqrt{3}}{2} + \frac{CT\sqrt{3}}{2} \leq BM + CM = BC$$

Therefore

$$BT + CT \leq \frac{2}{\sqrt{3}} BC.$$

The equality holds when $BC \perp AA_1$. In this case, as the triangles BTM and CTM are congruent, it results that $BM = CM$, i.e. AA_1 is the mediator of $[BC]$. In other words, the equality holds iff $AB = AC$.

Case II: $A \geq \frac{2\pi}{3}$.

Then, $cosA \leq -\frac{1}{2}$, $T = A$ and the statement becomes

$$c + b \leq \frac{2}{\sqrt{3}} \cdot a.$$

But $a^2 = b^2 + c^2 - 2bc \cdot cosA \geq b^2 + c^2 + bc \geq \frac{3(b+c)^2}{4}$, and from here we deduce that $a \geq \frac{\sqrt{3}}{2}\left(b + c\right)$. The equality holds iff $b = c$.

Case III: $B \geq \frac{2\pi}{3}$ or $C \geq \frac{2\pi}{3}$.

Then $T = B$ or $T = C$ and the statement becomes $BC \leq \frac{2}{\sqrt{3}} \cdot BC$, which is true.

■

Proof 2. We will use the following result:

Lemma 1 *For all* $x, y, \alpha \in \mathbb{R}$,

$$x^2 - 2xy \cdot \cos\alpha + y^2 \geq (x+y)^2 \cdot \sin^2 \frac{\alpha}{2}.$$

The equality holds iff $x = y$ *or* $\alpha = (2k+1)\pi$, *with* $k \in \mathbb{Z}$.

Proof of Lemma 1: The inequality is equivalent to

$$x^2\left(1 - \sin^2\frac{\alpha}{2}\right) + y^2\left(1 - \sin^2\frac{\alpha}{2}\right) - 2xy\left(\cos\alpha + \sin^2\frac{\alpha}{2}\right) \geq 0 \Leftrightarrow (x-y)^2 \cdot \cos^2\frac{\alpha}{2} \geq 0$$

which is true.
The equality holds iff $x = y$ or $\alpha = (2k+1)\pi$, cu $k \in \mathbb{Z}$.

Back to the proof of Proposition 1:

Case I: $A < \frac{2\pi}{3}$, $B < \frac{2\pi}{3}, C < \frac{2\pi}{3}$.
Then, $\mu\left(\widehat{BTC}\right) = \frac{2\pi}{3}$ and using Lemma 1 we have:

$$a^2 = BT^2 + CT^2 - 2BT \cdot CT \cdot \cos\frac{2\pi}{3} \geq (BT+CT)^2 \cdot \sin^2\frac{\pi}{3}$$

We obtain $BC \geq (BT + CT) \cdot \frac{\sqrt{3}}{2}$.
The equality holds iff $BT = CT$, i.e. iff $AB = AC$.
Case II: $A \geq \frac{2\pi}{3}$. We have $T = A$, therefore $TB + TC = AB + AC$.
Then,

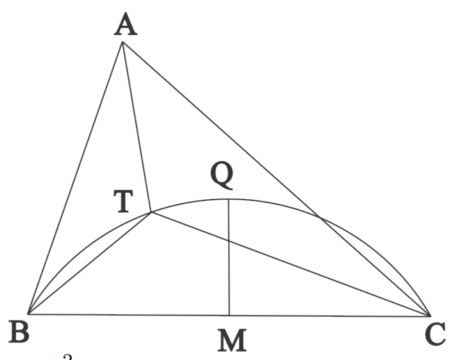
$$\frac{\sqrt{3}}{2} \leq \sin\frac{A}{2} \leq \frac{a}{b+c},$$

with equality iff $b = c$.
We used Lemma 1, for $\alpha = A, x = b, y = c$.
Case III can be solved as in the first proof. ■

Proof 3. (Rachid Moussaoui, Maroc)

Cases II and III can be solved as in the first proof, so we will prove only Case I.

We have $\mu\left(\widehat{BTC}\right) = \frac{2\pi}{3}$.



Then, $BC^2 = BT^2 + CT^2 + BT \cdot CT = (BT + CT)^2 - BT \cdot CT$.

Therefore, the sum $BT + CT$ is maximal iff the product $BT \cdot CT$ is maximal. We have

$BT \cdot CT = \frac{4}{\sqrt{3}} \cdot S_{BTC} \leq \frac{4}{\sqrt{3}} \cdot S_{BQC}$, where Q is the point of the arch BTC for which $QM \perp BC$ and M is the midpoint of $[BC]$. So, $BT \cdot CT \leq \frac{4}{\sqrt{3}} \cdot \frac{BQ^2 \cdot \sin \frac{2\pi}{3}}{2} = \frac{BC^2}{3}$.
Therefore $(BT + CT)^2 \leq BC^2 + BT \cdot CT = \frac{4 \cdot BC^2}{3}$. It follows $BT + CT \leq \frac{2 \cdot BC}{\sqrt{3}}$.
The equality holds iff $T = Q$, i.e. iff $AB = AC$. ■

Proposition 2. *Let ABC be a triangle with $A \leq \frac{2\pi}{3}$ and T its Fermat-Torricelli point. Then,* $3\left(BT^2 + CT^2\right) \geq 2BC^2$.

Proof We will use the following result:

Lemma 2 *If $x, y, \alpha \in \mathbb{R}$, such that $\cos\alpha \leq 0$, then*

$$2\left(x^2 + y^2\right) \cdot \sin^2 \frac{\alpha}{2} \geq x^2 - 2xy \cdot \cos\alpha + y^2$$

The equality holds iff $x = y$ or $\alpha = \frac{(2k+1)\pi}{2}$, with $k \in \mathbb{Z}$.

Proof of Lemma 2: As $2\sin^2 \frac{\alpha}{2} = 1 - \cos\alpha$, the inequality is equivalent with $\left(x^2 + y^2\right) - \left(x^2 + y^2\right) \cdot \cos\alpha \geq x^2 - 2xy \cdot \cos\alpha + y^2 \Leftrightarrow 0 \geq (x - y)^2 \cdot \cos\alpha$, which is true. The equality case is easy to prove.

Back to the proof of Proposition 2:

Case I: $A < \frac{2\pi}{3}, B < \frac{2\pi}{3}, C < \frac{2\pi}{3}$.
Then, $\mu\left(\widehat{BTC}\right) = \frac{2\pi}{3}$ and from Lemma 2 we obtain:

$$a^2 = BT^2 - 2BT \cdot CT \cdot \cos\frac{2\pi}{3} + CT^2 \leq 2\left(BT^2 + CT^2\right) \cdot \sin^2 \frac{\pi}{3},$$

and the conclusion follows.
Case II: $A = \frac{2\pi}{3}$. Then $T = A$, and the inequality reduces to: $3\left(b^2 + c^2\right) \geq 2 \cdot a^2$.
The last inequality is a consequence of Lemma 2, for $x = b$, $y = c, \alpha = A$.
Case III: $B \geq \frac{2\pi}{3}$ or $C \geq \frac{2\pi}{3}$. Then $T = B$ or $T = C$ and the inequality reduces to: $3 \cdot a^2 \geq 2 \cdot a^2$, which is true. ■

Remark. The condition $A \leq \frac{2\pi}{3}$ is an essential one.
Indeed, let ABC be a triangle with $A = \pi - \alpha$, where $\alpha \in \left(0, \frac{\pi}{3}\right)$, and $AB = AC = x > 0$.
Then $T = A$, and the inequality to prove turns into:

$$3\left(AB^2 + AC^2\right) \geq 2BC^2 \Leftrightarrow 6x^2 \geq 4x^2\left(1 + \cos\alpha\right),$$

which is not necessarily true.
(For $\alpha \to 0$ we obtain $6 \geq 8$.) ■

Problem 1 *Prove that for all $x, y, z \in (0, \infty)$,*

$$\frac{x^2 + xy + y^2}{z} + \frac{y^2 + yz + z^2}{x} + \frac{z^2 + zx + x^2}{y} \geq 3\left(x + y + z\right)$$

Proof We will use the following result:

Lemma 3 *Let ABC be a triangle with the measures of all its angles smaller than $\frac{2\pi}{3}$ and T, its Torricelli point. Then,*

$$AT + BT + CT \leq \frac{AB + BC + CA}{\sqrt{3}}$$

Proof of Lemma 3: From the Propozition 1. we obtain:
$BT + CT \leq \frac{2}{\sqrt{3}} BC$, $CT + AT \leq \frac{2}{\sqrt{3}} CA$, and $AT + BT \leq \frac{2}{\sqrt{3}} AB$.
By summing these three inequalities, the conclusion follows .

Back to the proof of the Problem 1:

We choose A, B, C, T such that $AT = x$, $BT = y$, $CT = z$, and $\mu\left(\widehat{BTC}\right) = \mu\left(\widehat{ATC}\right) = \mu\left(\widehat{ATB}\right) = \frac{2\pi}{3}$.
Then T is the Torricelli point of the triangle ABC,
$a^2 = y^2 + yz + z^2$, $b^2 = x^2 + xz + z^2$, $c^2 = x^2 + xy + y^2$ and

$$\frac{x^2 + xy + y^2}{z} + \frac{y^2 + yz + z^2}{x} + \frac{z^2 + zx + x^2}{y} = \frac{c^2}{z} + \frac{a^2}{x} + \frac{b^2}{y} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(a + b + c)^2}{x + y + z} \overset{Lemma3}{\geq} \frac{\left(\sqrt{3}\,(x + y + z)\right)^2}{x + y + z} = 3\,(x + y + z)$$

■

We invite the reader to use this ideas to solve the following:

Problem 2 *Prove that for all $x, y, z \in (0, \infty)$,*

$$\sum_{cyc.} \sqrt{x^2 + y^2} \geq \frac{\sqrt{6}}{3} \cdot \sum_{cyc.} \sqrt{x^2 + xy + y^2}.$$

References

[1] L. Nicolescu, V. Boskoff *Probleme practice de geometrie*, Editura Tehnică, Bucureşti 1990

[2] Olimpiada pe Şcoală (The School Yard Olympiad), Forum interactiv de matematică, https://www.facebook.com/groups/1593739420880226/?fref=ts

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