APPLICATIONS OF FERMAT'S AND LAGRANGE'S THEOREMS

DANIEL SITARU

Abstract. In the following we will show the way in which we can use Fermat's and Lagrange's theorems to solve problems whose solution, at first sight, does not seems to use these results.

Problem 0.1. *Let be* $a_i \in (0, \infty) \setminus \{1\}, i \in \overline{1, n}$ *sucht that, for any* $x \in \mathbb{R}$,

$$a_1^x + a_2^x + \ldots + a_n^x \geq n.$$

Prove that $a_1 a_2 \ldots a_n = 1$.

Proof. Lets consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = a_1^x + a_2^x + \ldots + a_n^x.$$

The hypothesis says that $f(x) \geq f(0), \forall x \in \mathbb{R}$, so $x = 0$ is a point of minimum local for this function. According to Fermat's theorem, it follows that $f'(0) = 0$. As $f'(x) = a_1^x \ln a_1 + a_2^x \ln a_2 + \ldots + a_n^x \ln a_n$, we obtain $\ln(a_1 a_2 \ldots a_n) = 0$, namely $a_1 a_2 \ldots a_n = 1$.

Remark 0.1. From means inequality it follows that the reciprocal affirmation from problem 1 is, also, true: if $a_1, a_2, \ldots, a_n > 0$ and $a_1 a_2 \ldots a_n = 1$, then

$$a_1^x + a_2^x + \ldots + a_n^x \geq n \sqrt[n]{a_1^x a_2^x \ldots a_n^x} = n, \forall x \in \mathbb{R}.$$

□

Problem 0.2. *Let be* $a_i, b_i \in (0, \infty) \setminus \{1\}, i \in \overline{1, n}$ *such that, for any* $x \in \mathbb{R}$,

$$a_1 b_1^x + a_2 b_2^x + \ldots + a_n b_n^x \geq a_1 + a_2 + \ldots + a_n.$$

Prove that $b_1^{a_1} b_2^{a_2} \ldots b_n^{a_n} = 1$.

Proof. We define the function $f : \mathbb{R} \to \mathbb{R}$ through

$$f(x) = a_1 b_1^x + a_2 b_2^x + \ldots a_n b_n^x \tag{0.1}$$

We can notice that $f(0) = a_1 + a_2 + \ldots + a_n$ and that $f(x) \geq f(0), \forall x \in \mathbb{R}$. It follows that $x = 0$ is a point of minimum local for f. From Fermat's theorem it follows that $f'(0) = 0$. Then $f'(x) = a_1 b_1^x \ln b_1 + a_2 b_2^x \ln b_2 + \ldots + a_n b_n^x \ln b_n$, so $f'(0) = a_1 \ln b_1 + a_2 \ln b_2 + \ldots + a_n \ln b_n = \ln(b_1^{a_1} b_2^{a_2} \ldots b_n^{a_n})$, wherefrom $b_1^{a_1} b_2^{a_2} \ldots b_n^{a_n} = 1$.

Remark 0.2. And the reciprocal of the affirmation form problem 2 is true: if $a_i, b_i \in (0, \infty) \setminus \{1\}, i \in \overline{1, n}$ and $b_1^{a_1} b_2^{a_2} \ldots b_n^{a_n} = 1$, then, because the function defined by 0.1 is convex (it has the second derivative positive) and $f'(0) = 0$ it follows that it is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$, so

$$f(x) \geq f(0) = a_1 + a_2 + \ldots + a_n, \forall x \in \mathbb{R}.$$

□

Problem 0.3. *Find the numbers* $a, b, c \in (0, \infty)$ *which have the property* $a^x + b^x + c^x \geq x^a + x^b + x^c, \forall x \in (0, \infty)$.

Proof. We consider the function $f : (0, \infty) \to \mathbb{R}, f(x) = e \ln x - x$. Its derivative is $f'(x) = \frac{e}{x} - 1$ and it is positive on $(0, e]$ and negative on $[e, \infty)$, so the function have in point $x = e$ a global maximum. It follows $f(x) \leq f(e) = 0$, namely $x^e \leq e^x, \forall x > 0$, with equality only if $x = e$.
On the other hand, from hypothesis it follows $a^e + b^e + c^e \geq e^a + e^b + e^c$, which is, according to the ones mention above, it is possible only if $a = b = c = e$.
We notice that these values check the given property. □

Problem 0.4. *Solve the following equation* $3^x + 6^x = 4^x + 5^x$.

Proof. We consider x fixed, solution of the equation and the function $f : (0, \infty) \to \mathbb{R}, f(t) = t^x$, with the derivative $f'(t) = xt^{x-1}$. Lagrange's theorem applied to the function on the intervals $[3, 4]$ and $[5, 6]$ guarantees the existence of some numbers $c_1 \in (3, 4)$ and $c_2 \in (5, 6)$ such that $f(4) - f(3) = xc_1^{x-1}$, respectively $f(6) - f(5) = xc_2^{x-1}$.
Because the equation is equivalent with $f(4) - f(3) = f(6) - f(5)$, it follows $x = 0$ and $c_1^{x-1} = c_2^{x-1}$, namely $x = 1$. We notice that both numbers are solutions of the initial equation.

Remark 0.3. We consider the function $g : \mathbb{R} \to \mathbb{R}, g(x) = 6^x - 5^x - 4^x + 3^x$. It is continuous and it is canceled only in 0 and 1, so it has a constant sign on the intervals $(-\infty, 0), (0, 1)$ and $(1, \infty)$. From

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} 6^x \left(1 - \left(\frac{5}{6}\right)^x - \left(\frac{4}{6}\right)^x + \left(\frac{3}{6}\right)^x\right) = +\infty$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 3^x \left(\left(\frac{6}{3}\right)^x - \left(\frac{5}{3}\right)^x - \left(\frac{4}{3}\right)^x + 1\right) = +\infty$$

it follows that the function is positive on $(-\infty, 0) \cup (1, \infty)$. Then, because $g'(0) \neq 0$, 0 is not a point of extremely local, so the function is negative on $(0, 1)$. We deduce:

$$6^x + 3^x > 4^x + 5^x, \forall x \in (-\infty, 0) \cup (1, \infty)$$

and

$$6^x + 3^x < 4^x + 5^x, \forall x \in (0, 1)$$

□

References

[1] Gh. Gussi, O. Stănăşilă, T. Stoica, *Mathematics.* manual for the XI th grade, EDP Publishing House, Bucharest, 1991
[2] M. Ganga, *Elements of Mathematical Analysis* Math Press Publishing House, Ploieşti, 1991
[3] D. Sitaru, Claudia Nănuţi, *Mathematics for contests* ECKO - Print Publishing House, Drobeta Turnu-Severin, 2012
[4] Daniel Sitaru, *Math Phenomenon.* Paralela 45, Publishing House, Piteşti, Romania, 2016.

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.
E-mail address: `dansitaru63@yahoo.com`