PROPERTIES OF THE SETS PROVED WITH THE CHARACTERISTIC FUNCTION

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Let be $E \neq \emptyset$ and $A \subseteq E$. We define the characteristic function of the set A by $\varphi_A : E \to \{0, 1\}$,

$$\varphi_A(x) = \begin{cases} 1, x \in A \\ 0, x \neq A \end{cases} .$$

We can easily prove (exercise!) the following properties of the characteristic function for $A \subseteq E$ and $B \subseteq E$:
1. $\varphi_{A\cup B} = \varphi_A + \varphi_B - \varphi_A\varphi_B$.
2. $\varphi_{A\cap B} = \varphi_A\varphi_B$.
3. $\varphi_{A\setminus B} = \varphi_A - \varphi_A\varphi_B$.
4. $\varphi_{A\Delta B} = \varphi_A + \varphi_B - 2\varphi_A\varphi_B$, where $A\Delta B = (A \setminus B) \cup (B \setminus A)$.
5. $\varphi_A^2 = \varphi_A$.
6. $\varphi_\emptyset = 0, \varphi_E = 1$.
7. $\varphi_{C_E A} = 1 - \varphi_A$.
8. $A = B \Leftrightarrow \varphi_A = \varphi_B$ and, more generally, $A \subseteq B \Leftrightarrow \varphi_A \leq \varphi_B$.

For example, for 1. we notice that, if $x \in E$ and it belongs at least to one of A, B sets, then both members have the value in point x equal with 1, and if $x \in E$ and it does not belong to neither A, B sets, then both members have the value in x point equal to 0.
In the following we will solve some problems from the high school manuals and we will propose some applications.

Problem 0.1. *Prove that if A, B, C are three sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.*

Proof. From $A \cap B = A \cap C$ it follows $\varphi_{A\cap B} = \varphi_{A\cap C}$, namely $\varphi_A\varphi_B = \varphi_A\varphi_C$. From $A \cup B = A \cup C$ it follows $\varphi_{A\cup B} = \varphi_{A\cup C}$, so

$$\varphi_A + \varphi_B - \varphi_A\varphi_B = \varphi_A + \varphi_C - \varphi_A\varphi_C,$$

wherefrom $\varphi_B = \varphi_C$, namely $B = C$. □

Problem 0.2. *Prove that if A, B are sets, then the relationships $A \cap B = A$ and $A \cup B = B$ are equivalent.*

Proof. Let X be a set that includes the sets A, B. Considering the characteristic functions in rapport with X, the relationship $A \cap B = A$ is equivalent with
$\varphi_A\varphi_B = \varphi_A$.
$A \cup B = B$ is equivalent with $\varphi_A + \varphi_B - \varphi_A\varphi_B = \varphi_B$, namely $\varphi_A = \varphi_A\varphi_B$. So, $A \cap B = A$ and $A \cup B = B$ are equivalent with the same realtionship. □

Problem 0.3. *Prove that $(A\Delta B)\Delta C = A\Delta(B\Delta C)$*

Proof. $\varphi_{(A\Delta B)\Delta C} = \varphi_{A\Delta B} + \varphi_C - 2\varphi_{A\Delta B\varphi_C} = \varphi_A + \varphi_B + \varphi_C - 2\varphi_A\varphi_B - 2\varphi_A\varphi_C - - 2\varphi_B\varphi_C + 4\varphi_A\varphi_B\varphi_C$.
For $\varphi_{A\Delta(B\Delta C)} = \varphi_{(B\Delta C)\Delta A}$ we obtain the same expresion, because the preceding result is symmetric in A, B, C. So, $\varphi_{(A\Delta B)\Delta C} = \varphi_{A\Delta(B\Delta C)}$, namely
$(A\Delta B)\Delta C = A\Delta(B\Delta C)$. □

Problem 0.4. *Prove that* $A \cap (B\Delta C) = (A \cap B)\Delta(A \cap C)$.

Proof.

$$\varphi_{A\cap(B\Delta C)} = \varphi_A\varphi_{B\Delta C} = \varphi_A(\varphi_B + \varphi_C - 2\varphi_B\varphi_C) =$$
$$= \varphi_A\varphi_B + \varphi_A\varphi_C - 2\varphi_A\varphi_B\varphi_C;$$
$$\varphi(A \cap B)\Delta(A \cap C) = \varphi_{A\cap B} + \varphi_{A\cap C} - 2\varphi_{A\cap B}\varphi_{A\cap C};$$
$$= \varphi_A\varphi_B + \varphi_A\varphi_C - 2\varphi_A\varphi_B\varphi_A\varphi_C =$$
$$= \varphi_A\varphi_B + \varphi_A\varphi_C - 2\varphi_A\varphi_B\varphi_C.$$

From the relationship above we deduce $\varphi_{A\cap(B\Delta C)} = \varphi_{(A\cap B)\Delta(\Delta\cap C)}$, hence the conclusion. □

Problem 0.5. *Let E be a set and the function that associate to each set included in E its complement: $C_E : \mathcal{P}(E) \to \mathcal{P}(E), C_E(X) = E \setminus X, \hat{A}$ for any $x \in \mathcal{P}(E)$. Prove that the application C_E has the following properies (Morgan's laws):*
1. $C_E(X \cup Y) = C_E(X) \cap C_E(Y)$;
2. $C_E(X \cap Y) = C_E(X) \cup C_E(Y)$.

Proof. 1. $\varphi_{C_E(X\cup Y)} = 1 - \varphi_{X\cup Y} = 1 - \varphi_X - \varphi_Y + \varphi_X\varphi_Y$.
$\varphi_{C_E(X)\cap C_E(Y)} = \varphi_{C_E(X)}\varphi_{C_E(Y)} = (1 - \varphi_X)(1 - \varphi_Y) =$
$= 1 - \varphi_X - \varphi_Y + \varphi_X\varphi_Y$, which proves the requirement.
2. $\varphi_{C_E(X\cap Y)} = 1 - \varphi_{X\cap Y} = 1 - \varphi_X\varphi_Y$.
$\varphi_{C_E(X)\cup C_E(Y)} = \varphi_{C_E(X)} + \varphi_{C_E(Y)} - \varphi_{C_E(X)}\varphi_{C_E(Y)} =$
$= 1-\varphi_X+1-\varphi_Y-(1-\varphi_X)(1-\varphi_Y) = 1-\varphi_X\varphi_Y$, which proves the requirement. □

1. Proposed Problems

Problem 1.1. *Let A, B, C be the given sets. Solve the equations in X:*
a. $A\Delta X = B$;
b. $A\Delta X\Delta B = C$.

Problem 1.2. *Let E be a nonempty set and $A \in \mathcal{P}(E)$ a fixed set. Prove that the function $f : \mathcal{P}(E) \to \mathcal{P}(E), f(X) = A\Delta X$ is bijective.*

Problem 1.3. *Let $A_n = \{1, 2, 3, \ldots, n\}, n \in \mathbb{N}^*$, fixed. Prove that $(\mathcal{P}(A_n), \Delta, \cap)$ is a commutative ring with divisors of zero. Find the invertible elements of the ring.*

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