The hidden beauties of an inequality problem of a Vietnamese mathematical textbook

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Abstract

We create an inequality problem of a Vietnamese mathematical textbook.

There are a lot of simple problems however they have *charming beauties*. The *beauties* are illustrated by many different solutions, similar and generalized problems. We refer to these things through the following problem of a Vietnamese mathematical textbook.

Problem 1 (Problem 20, page 112, Advanced Algebra textbook 11^{th}, (2016)) *Given real numbers x, y such that $x^2 + y^2 = 1$. Prove that*

$$|x + y| \leq \sqrt{2}\ ;$$

Solution 1
Applying Cauchy 's inequality to two non-negative real numbers, we have

$$\frac{\frac{x^2}{x^2 + y^2} + \frac{1}{2}}{2} \geq \frac{|x|}{\sqrt{2}\sqrt{x^2 + y^2}}\ ;$$
$$\frac{\frac{y^2}{x^2 + y^2} + \frac{1}{2}}{2} \geq \frac{|y|}{\sqrt{2}\sqrt{x^2 + y^2}}.$$

Thus,

$$\frac{\frac{x^2}{x^2 + y^2} + \frac{1}{2} + \frac{y^2}{x^2 + y^2} + \frac{1}{2}}{2} \geq \frac{1}{\sqrt{2}} \cdot \frac{|x| + |y|}{\sqrt{x^2 + y^2}}.$$

Since $x^2 + y^2 = 1$ we have $|x| + |y| \leq \sqrt{2}$.
Applying the triangle inequality $|x| + |y| \geq |x + y|$, we follow $|x + y| \leq \sqrt{2}$.
The equality happens if and only if $x = y = \frac{1}{\sqrt{2}}$ or $x = y = -\frac{1}{\sqrt{2}}$.
Solution 2
Applying Cauchy-Bouniakowski-Schwarz 's inequality, we have

$$(x + y)^2 \leq (x^2 + y^2)(1^2 + 1^2) = 2.$$

Thus, $|x + y| \leq \sqrt{2}$.
The equality happens if and only if $x = y = \frac{1}{\sqrt{2}}$ or $x = y = -\frac{1}{\sqrt{2}}$.
Solution 3
Let $M = x + y$.
The domain of M includes all of values of M that make the system of equations

$$\begin{cases} x^2 + y^2 = 1 \\ M = x + y \end{cases}$$

has roots.
Substituting $y = M - x$ for the equation $x^2 + y^2 = 1$ to find the domain of M, we have $x^2 + (M - x)^2 - 1 = 0$.
Thus, $2x^2 - 2Mx + M^2 - 1 = 0$.
The condition that the equation has solution is $\Delta' \geq 0$. It follows $M^2 - 2(M^2 - 1) \geq 0$
Hence, $M^2 - 2 \leq 0$.
Thus, $|x + y| \leq \sqrt{2}$.
The equality happens if and only if $x = y = \frac{1}{\sqrt{2}}$ or $x = y = -\frac{1}{\sqrt{2}}$.
Solution 4
Because $x^2 + y^2 = 1$, we let $x = \sin\alpha$; $y = \cos\alpha$.
We have $|x + y| = |\sin\alpha + \cos\alpha| = \sqrt{2}|\sin(\alpha + \frac{\pi}{4})| \leq \sqrt{2}$
The equality happens if and only if $|\sin(\alpha + \frac{\pi}{4})| = 1$. It means $x = y = \frac{1}{\sqrt{2}}$ or $x = y = -\frac{1}{\sqrt{2}}$.
Solution 5
Consider $\overrightarrow{U} = (1 ; 1)$; $\overrightarrow{V} = (x ; y)$.
We have

$$|\overrightarrow{U} . \overrightarrow{V}| = |x + y| \leq |\overrightarrow{U}| . |\overrightarrow{V}| = \sqrt{1^2 + 1^2} . \sqrt{x^2 + y^2} = \sqrt{2}.$$

The equality happens if and only if $x = y = \frac{1}{\sqrt{2}}$ or $x = y = -\frac{1}{\sqrt{2}}$.
Solution 6

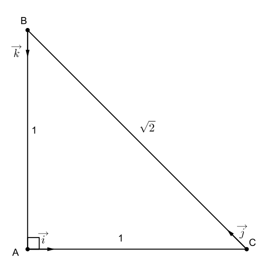

Figure 1: The vectorial method 1

Let ABC be an isosceles right triangle that all of two lengths of two legs AB and AC equal to 1 and the length of the hypotenuse equals to $\sqrt{2}$.
Let $\overrightarrow{i}$, $\overrightarrow{j}$, $\overrightarrow{k}$ be unit vectors on the sides AC, CB, BA, respectively.
We have

$$0 \leq (|x|\overrightarrow{i} + \overrightarrow{j} + |y|\overrightarrow{k})^2.$$

Hence, $0 \leq x^2 + 1 + y^2 - 2|x|\cos C - 2|y|\cos B$.
Thus, $x^2 + y^2 + 1 \geq 2|x| . \frac{1}{\sqrt{2}} + 2|y| . \frac{1}{\sqrt{2}}$.
Thus, $2 \geq \sqrt{2} . (|x| + |y|) \geq \sqrt{2} . |x + y|$.
It means $|x + y| \leq \sqrt{2}$.
The equality happens if and only if $x = y = \frac{1}{\sqrt{2}}$ or $x = y = -\frac{1}{\sqrt{2}}$.
Solution 7

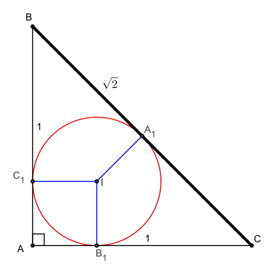

Figure 2: The vectorial method 2

Let ABC be an isosceles right triangle that all of two lengths of two legs AB and AC equal to 1 and the length of the hypotenuse equals to $\sqrt{2}$.
Without loss of generality, suppose that the incircle radius equals to 1. Denote by I the incircle center and A_1, B_1, C_1 the tangent points as the above figure 2.
We have

$$0 \leq (\overrightarrow{IA_1} + |y|\overrightarrow{IB_1} + |x|\overrightarrow{IC_1})^2.$$

Hence,

$$0 \leq 1 + x^2 + y^2 + 2|y|\cos(\pi - C) + 2|x|\cos(\pi - B).$$

Thus,

$$x^2 + y^2 + 1 \geq 2|y|\cos C + 2|x|\cos B = 2|x|.\frac{1}{\sqrt{2}} + 2|y|.\frac{1}{\sqrt{2}} = \sqrt{2}(|x| + |y|).$$

It means

$$2 \geq \sqrt{2}(|x| + |y|).$$

Applying the triangle inequality $|x| + |y| \geq |x + y|$, we follows $\sqrt{2} \geq |x + y|$. The equality happens if and only if $x = y = \frac{1}{\sqrt{2}}$ or $x = y = -\frac{1}{\sqrt{2}}$.

Solution 8

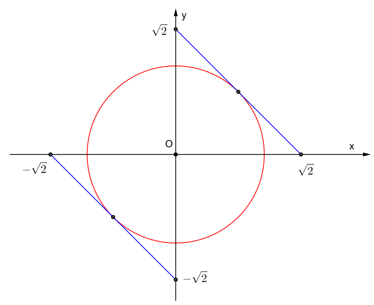

Figure 3: The analytic method 1

The system of equations

$$\begin{cases} x^2 + y^2 = 1 \\ M = x + y \end{cases}$$

has roots if and only if the line d: $x + y - M = 0$ meets the circle (C) : $x^2 + y^2 = 1$. Find all of values of M such that the line d meets the circle (C).

The required values of M satisfy that the line d lies between two tangent lines parallel to d.

d becomes to the tangent line of (C) if and only if $1 \,.\, 1 + 1 \,.\, 1 = M^2$. Then, $M = -\sqrt{2}$ or $M = \sqrt{2}$. These are the minimum and maximum values of M. The equality happens if and only if $x = y = \frac{1}{\sqrt{2}}$ or $x = y = -\frac{1}{\sqrt{2}}$.

Solution 9

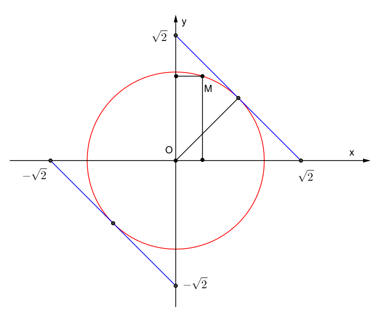

Figure 4: The analytic method 2

We see that lines $x + y - \sqrt{2}$ and $x + y + \sqrt{2}$ are tangent to the circle $(C): x^2 + y^2 = 1$.
Because these lines are tangent to the circle (C), the point lies on the circle (different from two tangent points) then it does not lie on the lines. In other words, if $x^2 + y^2 = 1$ then $|x + y| \leq \sqrt{2}$.
The equality happens if and only if $x = y = \frac{1}{\sqrt{2}}$ or $x = y = -\frac{1}{\sqrt{2}}$.
We now generalize problem 1 to the following one:

Problem 2 *Given real numbers x, y, z such that $x^2 + y^2 + z^2 = 1$. Prove that*

$$|x + y + z| \leq \sqrt{3}.$$

Similarly to problem 1, this problem has many solutions.
Solution 1
Applying Cauchy's inequality to three non-negative real numbers, we have

$$\frac{\frac{x^2}{x^2 + y^2 + z^2} + \frac{1}{3}}{2} \geq \frac{|x|}{\sqrt{3}\sqrt{x^2 + y^2 + z^2}};$$
$$\frac{\frac{y^2}{x^2 + y^2 + z^2} + \frac{1}{3}}{2} \geq \frac{|y|}{\sqrt{3}\sqrt{x^2 + y^2 + z^2}};$$
$$\frac{\frac{z^2}{x^2 + y^2 + z^2} + \frac{1}{3}}{2} \geq \frac{|y|}{\sqrt{3}\sqrt{x^2 + y^2 + z^2}}$$

Thus, $\frac{\frac{x^2}{x^2 + y^2 + z^2} + \frac{1}{3} + \frac{y^2}{x^2 + y^2 + z^2} + \frac{1}{3} + \frac{z^2}{x^2 + y^2 + z^2} + \frac{1}{3}}{2} \geq \frac{1}{\sqrt{3}} \cdot \frac{|x| + |y| + |z|}{\sqrt{x^2 + y^2 + z^2}}$.
Since $x^2 + y^2 + z^2 = 1$, we have $|x| + |y| + |z| \leq \sqrt{3}$.
Applying the triangle inequality $|x| + |y| + |z| \geq |x + y + z|$, we follows $|x + y + z| \leq \sqrt{3}$.
The equality happens if and only if $x = y = z = \frac{1}{\sqrt{3}}$ or $x = y = z = -\frac{1}{\sqrt{3}}$.
Solution 2
Applying Cauchy-Bouniakowski-Schwarz 's inequality, we have

$$(x + y + z)^2 \leq (x^2 + y^2 + z^2)(1^2 + 1^2 + 1^2) = 3.$$

Thus, $|x + y + z| \leq \sqrt{3}$.
The equality happens if and only if $x = y = z = \frac{1}{\sqrt{3}}$ or $x = y = z = -\frac{1}{\sqrt{3}}$.
Solution 3
Consider the vectors $\overrightarrow{U} = (1;\ 1;\ 1)$, $\overrightarrow{V} = (x;\ y;\ z)$.
We have

$$\left|\overrightarrow{U} \,.\, \overrightarrow{V}\right| = |x + y + z| \leq \left|\overrightarrow{U}\right| . \left|\overrightarrow{V}\right| =$$
$$\sqrt{1^2 + 1^2 + 1^2} \,.\, \sqrt{x^2 + y^2 + z^2} = \sqrt{3}.$$

The equality happens if and only if $x = y = z = \frac{1}{\sqrt{3}}$ or $x = y = z = -\frac{1}{\sqrt{3}}$.
We have some discoveries around a textbook problem. There are a lot of different methods in solving this one that are algebraic methods, vectorial methods and geometric methods. Do you have any comments on the paper? Please share with us!
The last are some exercises which are generalized problems of problem 1.

Problem 3 *Given real numbers x and y such that*
a) $x^2 + y^2 = \frac{c^2}{a^2 + b^2}$. *Prove that* $|ax \pm by| \leq |c|$.
b) $|ax \pm by| = |c|$. *Prove that* $x^2 + y^2 \geq \frac{c^2}{a^2 + b^2}$

Problem 4 *Given real numbers x and y such that*
a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a, b > 0)$. *Prove that* $\left|\frac{x}{a} + \frac{y}{b}\right| \leq \sqrt{2}$.
b) $\left|\frac{x}{a} \pm \frac{y}{b}\right| = |c|$. *Prove that* $\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq \frac{c^2}{2}$.

References

[1] Doan Quynh, Nguyen Huy Doan, Nguyen Xuan Liem, Dang Hung Thang, Tran Van Vuong (2016), *Advanced algebra* 11^{th}, The Vietnam Educational Publishing House.

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