Some paths to new inequalities

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August 2016

Abstract

We establish some new inequalities from a difficult problem.

The inequality is a familiar topic for all of mathematicians. In this paper, we refer to some exploitation and creation of a difficult problem.

Problem 1 *Given positive real numbers x, y, z such that $x^2 + y^2 + z^2 = 1$. Prove that*

$$\frac{x}{1 - x^2} + \frac{y}{1 - y^2} + \frac{z}{1 - z^2} \geq \frac{3\sqrt{3}}{2}.$$

The following are some solutions.

Solution 1 (See [1])

From the hypothesis, we have $0 < x, y, z < 1$. Applying Cauchy 's inequality to three positive numbers $2x^2, 1 - x^2, 1 - x^2$, we have

$$\frac{2x^2 + (1 - x^2) + (1 - x^2)}{3} \geq \sqrt[3]{2x^2(1 - x^2)^2}.$$

It follows

$$\frac{2}{3} \geq \sqrt[3]{2x^2(1 - x^2)^2} \Rightarrow x(1 - x^2) \leq \frac{2}{3\sqrt{3}}.$$

It is equivalent to $\frac{1}{x(1 - x^2)} \geq \frac{3\sqrt{3}}{2}$.

Thus,

$$\frac{x}{1 - x^2} \geq \frac{3\sqrt{3}}{2}x^2 \ (1)$$

Similarly, we have

$$\frac{y}{1 - y^2} \geq \frac{3\sqrt{3}}{2}y^2 \ (2)$$
$$\frac{z}{1 - z^2} \geq \frac{3\sqrt{3}}{2}z^2 \ (3)$$

Adding the inequalities (1), (2), (3) termwise, we have

$$\frac{x}{1 - x^2} + \frac{y}{1 - y^2} + \frac{z}{1 - z^2} \geq \frac{3\sqrt{3}}{2}(x^2 + y^2 + z^2) = \frac{3\sqrt{3}}{2}.$$

The equality happens if and only if $x = y = z = \frac{1}{\sqrt{3}}$ (they satisfy the condition $x^2 + y^2 + z^2 = 1$).

Solution 2

We need to prove $\frac{x}{1 - x^2} \geq \frac{3\sqrt{3}}{2}x^2$. It is equivalent to

$$\begin{aligned}
&3\sqrt{3}x^3 - 3\sqrt{3}x + 2 \geq 0\\
&\Leftrightarrow (\sqrt{3}x)^3 - 3(\sqrt{3}x) + 2 \geq 0\\
&\Leftrightarrow (\sqrt{3}x - 1)^2(\sqrt{3}x + 2) \geq 0
\end{aligned}$$

This inequality holds true for all $x \in (0\ ;\ 1)$. Thus, we have $\frac{x}{1 - x^2} \geq \frac{3\sqrt{3}}{2}x^2$. Similarly, we also have $\frac{y}{1 - y^2} \geq \frac{3\sqrt{3}}{2}y^2$, $\frac{z}{1 - z^2} \geq \frac{3\sqrt{3}}{2}z^2$.

Adding these inequalities termwise, we have

$$\frac{x}{1 - x^2} + \frac{y}{1 - y^2} + \frac{z}{1 - z^2} \geq \frac{3\sqrt{3}}{2}(x^2 + y^2 + z^2) = \frac{3\sqrt{3}}{2}.$$

The equality happens if and only if $x = y = z = \frac{1}{\sqrt{3}}$ (they satisfy the condition $x^2 + y^2 + z^2 = 1$).

Solution 3 (See [2], p.154-155)

We need to prove

$$\frac{x}{1-x^2} + \frac{y}{1 - y^2} + \frac{z}{1 - z^2} \geq \frac{3\sqrt{3}}{2}.$$

It is equivalent to

$$\frac{x^2}{x(1 - x^2)} + \frac{y^2}{y(1 - y^2)} + \frac{z^2}{z(1 - z^2)} \geq \frac{3\sqrt{3}}{2}.$$

Function $f(t) = t(1 - t^2) = -t^3 + t$ is defined on the interval $(0;1)$. Since $f'(t) = -3t^2 + 1$, we have the variation chart as follows

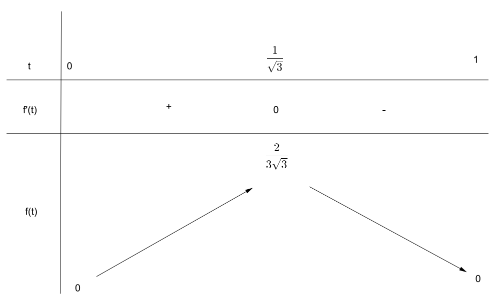

Figure 1: The variation chart

From this result, we follow that if $0 < x < 1$, $0 < y < 1$, $0 < z < 1$, then

$0 < x(1 - x^2) \le \frac{2}{3\sqrt{3}}$; $0 < y(1 - y^2) \le \frac{2}{3\sqrt{3}}$; $0 < z(1 - z^2) \le \frac{2}{3\sqrt{3}}$.

Hence,

$$\frac{x^2}{x(1 - x^2)} + \frac{y^2}{y(1 - y^2)} + \frac{z^2}{z(1 - z^2)} \ge \frac{3\sqrt{3}}{2}(x^2 + y^2 + z^2) = \frac{3\sqrt{3}}{2}.$$

The equality happens if and only if $x = y = z = \frac{1}{\sqrt{3}}$ (they satisfy the condition $x^2 + y^2 + z^2 = 1$).
We generalize problem 1 to the following one

Problem 2 *Given positive real numbers $x,\ y,\ z$ and α such that $\sqrt{\alpha}\left(x^2 + y^2 + z^2\right) = 1$. Prove that*

$$\frac{x}{1 - \alpha x^2} + \frac{y}{1 - \alpha y^2} + \frac{z}{1 - \alpha z^2} \ge \frac{3\sqrt{3}}{2}.$$

If $\alpha = 1$ then problem 2 becomes to problem 1.
From the hypothesis $0 < x, y, z < 1$. Applying Cauchy 's inequality to three positive numbers $2\alpha x^2,\ 1 - \alpha x^2,\ 1 - \alpha x^2$, we have

$$\frac{2\alpha x^2 + (1 - \alpha x^2) + (1 - \alpha x^2)}{3} \ge \sqrt[3]{2\alpha x^2(1 - \alpha x^2)^2}.$$

It follows

$$\frac{2}{3} \ge \sqrt[3]{2\alpha x^2(1 - \alpha x^2)^2} \Rightarrow x(1 - \alpha x^2) \le \frac{2}{3\sqrt{3\alpha}}.$$

It is equivalent to $\frac{1}{x(1-\alpha x^2)} \ge \frac{3\sqrt{3\alpha}}{2}$.
Thus,

$$\frac{x}{1 - \alpha x^2} \ge \frac{3\sqrt{3\alpha}}{2}x^2 \ (1)$$

Similarly, we also have

$$\frac{y}{1 - \alpha y^2} \ge \frac{3\sqrt{3\alpha}}{2}y^2 \ (2)$$
$$\frac{z}{1 - \alpha z^2} \ge \frac{3\sqrt{3\alpha}}{2}z^2 \ (3)$$

Adding the inequalities (1), (2), (3), termwise, we obtain

$$\frac{x}{1 - \alpha x^2} + \frac{y}{1 - \alpha y^2} + \frac{z}{1 - \alpha z^2} \ge \frac{3\sqrt{3\alpha}}{2}(x^2 + y^2 + z^2) = \frac{3\sqrt{3}}{2}.$$

The equality is for the readers.
We continue generalize problem 2 to the following one

Problem 3 *Given positive real numbers $x,\ y,\ z$ and $\alpha,\ \beta,\ \gamma$ such that $\sqrt{\alpha}x^2 + \sqrt{\beta}y^2 + \sqrt{\gamma}z^2 = 1$. Prove that*

$$\frac{x}{1 - \alpha x^2} + \frac{y}{1 - \beta y^2} + \frac{z}{1 - \gamma z^2} \ge \frac{3\sqrt{3}}{2}.$$

If $\alpha = \beta = \gamma$ then problem 3 becomes problem 2.
Applying Cauchy 's inequality to three positive real numbers $2\alpha x^2,\ 1 - \alpha x^2,\ 1 - \alpha x^2$, we have

$$\frac{2\alpha x^2 + (1 - \alpha x^2) + (1 - \alpha x^2)}{3} \geq \sqrt[3]{2\alpha x^2(1 - \alpha x^2)^2}.$$

It follows

$$\frac{2}{3} \geq \sqrt[3]{2\alpha x^2(1 - \alpha x^2)^2} \Rightarrow x(1 - \alpha x^2) \leq \frac{2}{3\sqrt{3\alpha}}.$$

It is equivalent to

$$\frac{1}{x(1 - \alpha x^2)} \geq \frac{3\sqrt{3\alpha}}{2}.$$

Thus,

$$\frac{x}{1 - \alpha x^2} \geq \frac{3\sqrt{3\alpha}}{2}x^2 \ (1)$$

Similarly, we also have

$$\frac{y}{1 - \beta y^2} \geq \frac{3\sqrt{3\beta}}{2}y^2 \ (2)$$
$$\frac{z}{1 - \gamma z^2} \geq \frac{3\sqrt{3\gamma}}{2}z^2 \ (3)$$

Adding these inequalities (1), (2), (3), termwise, we obtain

$$\frac{x}{1 - \alpha x^2} + \frac{y}{1 - \beta y^2} + \frac{z}{1 - \gamma z^2} \geq \frac{3\sqrt{3}}{2}(\sqrt{\alpha}x^2 + \sqrt{\beta}y^2 + \sqrt{\gamma}z^2) = \frac{3\sqrt{3}}{2}.$$

The equality is for the readers.
We generalize problem 3 to the following one

Problem 4 *Given positive real numbers* $x,\ y,\ z$ *and* $\alpha,\ \beta,\ \gamma$ *such that* $\sqrt[2k]{\alpha}x^2 + \sqrt[2k]{\beta}y^2 + \sqrt[2k]{\gamma}z^2 = 1$ *(*k *is a positive integer). Prove that*

$$\frac{x}{1 - \alpha x^{2k}} + \frac{y}{1 - \beta y^{2k}} + \frac{z}{1 - \gamma z^{2k}} \geq \frac{(2k+1)\sqrt[2k]{2k + 1}}{2k}.$$

If $k = 1$ then problem 4 becomes to problem 3.
Applying Cauchy' inequality, we have

$$\frac{2k}{2k + 1} = \frac{1 - \alpha x^{2k} + 1 - \alpha x^{2k} + ... + 1 - \alpha x^{2k} + 2k\alpha x^{2k}}{2k + 1}$$
$$\geq \sqrt[2k+1]{2k\alpha x^{2k}(1 - \alpha x^{2k})^{2k}}.$$

Thus,

$$\left(\frac{2k}{2k + 1}\right)^{2k+1} \geq 2k\alpha x^{2k}(1 - \alpha x^{2k})^{2k} \Leftrightarrow \frac{(2k)^{2k}}{(2k + 1)^{2k+1}.\alpha} \geq (x(1 - \alpha x^{2k}))^{2k}.$$

Thus,

$$x(1 - \alpha x^{2k}) \leq \frac{2k}{(2k+1)\sqrt[2k]{(2k+1).\alpha}}$$

This relation is equivalent to

$$\frac{1}{x(1 - \alpha x^{2k})} \geq \frac{(2k+1)\sqrt[2k]{(2k+1).\ \alpha}}{2k}$$

Thus,

$$\frac{x}{1 - \alpha x^{2k}} \geq \frac{(2k+1)\sqrt[2k]{(2k+1).\ \alpha}}{2k}.x^2$$

Similarly, we also have

$$\frac{y}{1 - \beta y^{2k}} \geq \frac{(2k+1)\sqrt[2k]{(2k+1).\beta}}{2k}.y^2$$

and

$$\frac{z}{1 - \gamma z^{2k}} \geq \frac{(2k+1)\sqrt[2k]{(2k+1).\gamma}}{2k}.z^2$$

Thus,

$$\frac{x}{1-\alpha x^{2k}} + \frac{y}{1-\beta y^{2k}} + \frac{z}{1-\gamma z^{2k}} \geq \frac{(2k+1)\sqrt[2k]{2k+1}}{2k}\left(\sqrt[2k]{\alpha}x^2 + \sqrt[2k]{\beta}y^2 + \sqrt[2k]{\gamma}z^2\right).$$

We follow

$$\frac{x}{1 - \alpha x^{2k}} + \frac{y}{1 - \beta y^{2k}} + \frac{z}{1 - \gamma z^{2k}} \geq \frac{(2k+1)\sqrt[2k]{2k+1}}{2k}.$$

The equality is for the readers.
We generalize problem 1 to the problem as follows:

Problem 5 *Given positive real numbers* $x,\ y,\ z$ *and* $\alpha,\ \beta,\ \gamma$ *such that* $\sqrt[n]{\alpha}x^n + \sqrt[n]{\beta}y^n + \sqrt[n]{\gamma}z^n = 1$ *(*n *is a positive integer (*$n \geq 2$*)). Prove that*

$$\frac{x^{n-1}}{1 - \alpha x^n} + \frac{y^{n-1}}{1 - \beta y^n} + \frac{z^{n-1}}{1 - \gamma z^n} \geq \frac{(n+1)\sqrt[n]{n+1}}{n}.$$

Applying Cauchy 's inequality, we have

$$\frac{n}{n+1} = \frac{n\alpha x^n + (1 - \alpha x^n) + \ldots + (1 - \alpha x^n)}{n+1} \geq \sqrt[n+1]{n\alpha x^n (1 - \alpha x^n)^n}.$$

Thus,

$$\left(\frac{n}{n+1}\right)^{n+1} \geq n\alpha x^n(1 - \alpha x^n)^n \Leftrightarrow \frac{n^n}{(n+1)^{n+1}.\alpha} \geq (x(1 - \alpha x^n))^n.$$

This relation is equivalent to

$$\frac{n}{(n+1)\sqrt[n]{(n+1)\alpha}} \geq x(1 - \alpha x^n) \Leftrightarrow \frac{x^{n-1}}{1 - \alpha x^n} \geq \frac{(n+1)\sqrt[n]{(n+1)\alpha}}{n}x^n.$$

Similarly, we also have

$$\frac{y^{n-1}}{1 - \beta y^n} \geq \frac{(n+1)\sqrt[n]{(n+1)\beta}}{n}y^n;\ \frac{z^{n-1}}{1 - \gamma z^n} \geq \frac{(n+1)\sqrt[n]{(n+1)\gamma}}{n}z^n.$$

Since $\sqrt[n]{\alpha}x^n + \sqrt[n]{\beta}y^n + \sqrt[n]{\gamma}z^n = 1$, we have

$$\frac{x^{n-1}}{1 - \alpha x^n} + \frac{y^{n-1}}{1 - \beta y^n} + \frac{z^{n-1}}{1 - \gamma z^n} \geq \frac{(n+1)\sqrt[n]{n+1}}{n}.$$

The equality is for the readers.
We just have had some generalizations of problem 1. We now go to its similar problem.
Consider the Cartesian coordinate system as the bellow figure.

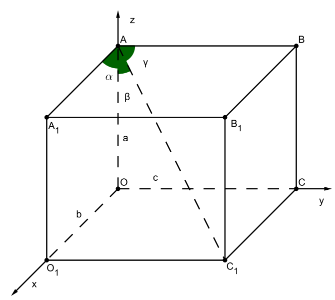

Figure 2: The Cartesian coordinate system

Let $OA = a$, $OC = c$, $OO_1 = b$. Since OCC_1O_1 is a rectangular in the plane (Oxy), we have:

$$b^2 + c^2 = O_1C^2 = OC_1^2.$$

Thus, $a^2 + b^2 + c^2 = a^2 + OC_1^2 = AC_1^2$.
Hence

$$\frac{a^2}{AC_1^2} + \frac{b^2}{AC_1^2} + \frac{c^2}{AC_1^2} = 1.$$

Letting $\frac{A_1A}{AC_1} = \frac{b}{AC_1} = \cos\alpha$; $\frac{OA}{AC_1} = \frac{a}{AC_1} = \cos\beta$, $\frac{AB}{AC_1} = \frac{c}{AC_1} = \cos\gamma$, we have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and $\cos\alpha,\ \cos\beta,\ \cos\gamma > 0$.
Thus, we can state problem 1 as the following equivalent problem

Problem 6 *Given a rectangular prism $OABCO_1A_1B_1C_1$. Let $\widehat{A_1AC_1} = \alpha$, $\widehat{OAC_1} = \beta$, $\widehat{BAC_1} = \gamma$. Prove that*

$$\frac{\cot an\alpha}{\sin\alpha} + \frac{\cot an\beta}{\sin\beta} + \frac{\cot an\gamma}{\sin\gamma} \geq \frac{3\sqrt{3}}{2}.$$

Clearly, problem 5 is problem 1; however it is much more difficult.
We have some discoveries around a problem. Some different solutions, similar and generalized problems make us interesting. Do you have any comments on this paper? Please share with us!
The last are some exercises that have close relationships with problem 1.

Problem 7 *Given positive real numbers* $x_1, x_2, ..., x_n$ $(n \in N^*)$, $n \geq 3$ *such that* $x_1^2 + x_2^2 + .. + x_n^2 = 1$ *then*

$$\frac{x_1}{1 - x_1^{2k}} + \frac{x_2}{1 - x_2^{2k}} + ... + \frac{x_n}{1 - x_n^{2k}} \geq \frac{2k + 1}{2k} \sqrt[2k]{2k + 1} \ (k \in N^*).$$

Problem 8 *Given positive real numbers* $x_1, x_2, ..., x_n$ $(n \in N^*)$, $n \geq 3$ *such that* $x_1^2 + x_2^2 + .. + x_n^2 = 1$ *then*

$$\frac{x_1^{n-1}}{1 - x_1^n} + \frac{x_2^{n-1}}{1 - x_2^n} + ... + \frac{x_{n+1}^{n-1}}{1 - x_{n+1}^n} \geq \frac{n+1}{n} \sqrt[n]{n + 1}.$$

Problem 9 *Given positive real numbers* x, y, z *such that* $x + y + z = 1$. *Prove that*

$$\frac{1 + \sqrt{x}}{y + z} + \frac{1 + \sqrt{y}}{z + x} + \frac{1 + \sqrt{z}}{x + y} \geq \frac{9 + 3\sqrt{3}}{2}.$$

Problem 10 *Given positive real numbers* x, y, z. *Prove that*

$$\frac{\sqrt{x + y + z} + \sqrt{x}}{y + z} + \frac{\sqrt{x + y + z} + \sqrt{y}}{z + x} + \frac{\sqrt{x + y + z} + \sqrt{z}}{x + y} \geq \frac{9 + 3\sqrt{3}}{2\sqrt{x + y + z}}.$$

References

[1] Phan Huy Khai (2006), *Special subject of improvement of unders secondary school students good at mathematics, The maximum and minimum values of function*, The Vietnam Education Publishing House.

[2] Phan Duc Chinh, Dang Khai (1993), *Instructions on solving the mathematical problems of the university, college and technical secondary school entrance examination (volume 1)*, The Vietnam Education Publishing House.

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