The development and creation of a nice inequality problem

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Abstract

We create and develop an inequality problem of a Vietnamese mathematical textbook.

There are a lot of exploitation of a problem such as finding out many solutions, finding out similar and generalized problems of this one. These make us interesting. We refer to these things through a nice inequality problem.

Problem 1 (Problem 9, page 110, Vietnamese Advanced Algebraic textbook 10^{th}, (2016)) *Prove that, if $a \geq 0$ and $b \geq 0$ then*

$$\frac{a+b}{2} \times \frac{a^2+b^2}{2} \leq \frac{a^3+b^3}{2}.$$

Solution 1

$$\begin{aligned}
&\frac{a+b}{2} \times \frac{a^2+b^2}{2} \leq \frac{a^3+b^3}{2} \\
&\Leftrightarrow (a+b)(a^2+b^2) \leq 2(a^3+b^3) \\
&\Leftrightarrow a^3+ab^2+a^2b+b^3 \leq 2a^3+2b^3 \\
&\Leftrightarrow ab^2+a^2b \leq a^3+b^3 \\
&\Leftrightarrow a^2(a-b)+b^2(b-a) \geq 0 \\
&\Leftrightarrow (a-b)(a^2-b^2) \geq 0 \\
&\Leftrightarrow (a-b)^2(a+b) \geq 0.
\end{aligned}$$

This thing holds true. Hence, we have

$$\frac{a+b}{2} \times \frac{a^2+b^2}{2} \leq \frac{a^3+b^3}{2}.$$

Solution 2
We have the identity

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

If $a = b = c = 0$ then $\frac{a+b}{2} \times \frac{a^2+b^2}{2} \leq \frac{a^3+b^3}{2}$.
If $a + b > 0$ then

$$
\begin{aligned}
&\tfrac{a+b}{2} \times \tfrac{a^2+b^2}{2} \leq \tfrac{a^3+b^3}{2}\\
\Leftrightarrow\ &\tfrac{a+b}{2} \times \tfrac{a^2+b^2}{2} \leq \tfrac{(a+b)(a^2-ab+b^2)}{2}\\
\Leftrightarrow\ &\tfrac{a^2+b^2}{2} \leq a^2 - ab + b^2\\
\Leftrightarrow\ &a^2 - 2ab + b^2 \geq 0\\
\Leftrightarrow\ &(a-b)^2 \geq 0.
\end{aligned}
$$

Clearly, this holds true. We have $\frac{a+b}{2} \times \frac{a^2+b^2}{2} \leq \frac{a^3+b^3}{2}$.

Solution 3

Without loss of generality, suppose that $a \leq b$. Since $a \geq 0$ and $b \geq 0$ we have $a^2 \leq b^2$.

Applying Chebyshev 's inequality to two increasing sequences $a \leq b$ and $a^2 \leq b^2$, we have

$$(a+b)(a^2+b^2) \leq 2(a^3+b^3).$$

Thus,

$$\tfrac{a+b}{2} \times \tfrac{a^2+b^2}{2} \leq \tfrac{a^3+b^3}{2}.$$

From problem 1, we follow the problem

Problem 2 *Prove that, if $a > b$ then*

$$\tfrac{a-b}{2} \times \tfrac{a^2+b^2}{2} \leq \tfrac{a^3-b^3}{2}.$$

This problem also have many different solutions. Indeed

$$
\begin{aligned}
&\tfrac{a-b}{2} \times \tfrac{a^2+b^2}{2} \leq \tfrac{a^3-b^3}{2}\\
\Leftrightarrow\ &(a-b)(a^2+b^2) \leq 2(a^3-b^3)\\
\Leftrightarrow\ &a^3 + ab^2 - ba^2 - b^3 \leq 2a^3 - 2b^3\\
\Leftrightarrow\ &a^3 + a^2b - b^3 - b^2a \geq 0\\
\Leftrightarrow\ &a^2(a+b) - b^2(a+b) \geq 0\\
\Leftrightarrow\ &(a+b)(a^2-b^2) \geq 0\\
\Leftrightarrow\ &(a+b)^2(a-b) \geq 0.
\end{aligned}
$$

Since $a \geq b$, we always have $(a+b)^2(a-b) \geq 0$. Thus,

$$\tfrac{a-b}{2} \times \tfrac{a^2+b^2}{2} \leq \tfrac{a^3-b^3}{2}.$$

Solution 2

Applying the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, we have

If $a = b$ then $\frac{a-b}{2} \times \frac{a^2+b^2}{2} \leq \frac{a^3-b^3}{2}$.

If $a > b$ then

$$
\begin{aligned}
&\tfrac{a-b}{2} \times \tfrac{a^2+b^2}{2} \leq \tfrac{a^3-b^3}{2}\\
\Leftrightarrow\ &\tfrac{a-b}{2} \times \tfrac{a^2+b^2}{2} \leq \tfrac{(a-b)(a^2+ab+b^2)}{2}\\
\Leftrightarrow\ &a^2 + b^2 \leq 2(a^2+ab+b^2)\\
\Leftrightarrow\ &(a+b)^2 \geq 0.
\end{aligned}
$$

This is obviously. Thus, $\frac{a-b}{2} \times \frac{a^2+b^2}{2} \leq \frac{a^3-b^3}{2}$.
We generalize problem 1 to the following one

Problem 3 *Prove that, if* $a \geq 0$ *and* $b \geq 0$ *and* $m, n \in N$ *then*

$$\frac{a^m+b^m}{2} \times \frac{a^n+b^n}{2} \leq \frac{a^{m+n}+b^{m+n}}{2}.$$

This problem is quite interesting. We have the following solution

$$\begin{aligned}
&\frac{a^m+b^m}{2} \times \frac{a^n+b^n}{2} \leq \frac{a^{m+n}+b^{m+n}}{2}\\
\Leftrightarrow\ & (a^m+b^m)(a^n+b^n) \leq 2(a^{m+n}+b^{m+n})\\
\Leftrightarrow\ & a^{m+n}+a^mb^n+b^ma^n+b^{m+n} \leq 2a^{m+n}+2b^{m+n}\\
\Leftrightarrow\ & a^{m+n}-a^mb^n+b^{m+n}-b^ma^n \geq 0\\
\Leftrightarrow\ & a^m(a^n-b^n)+b^m(b^n-a^n) \geq 0\\
\Leftrightarrow\ & (a^m-b^m)(a^n-b^n) \geq 0.
\end{aligned}$$

Because the roles of a and b are the same, without loss of the generality suppose that $a \geq b$, we have $a^m \geq b^m$ and $a^n \geq b^n$. Hence,

$$(a^m-b^m)(a^n-b^n) \geq 0.$$

Thus,

$$\frac{a^m+b^m}{2} \times \frac{a^n+b^n}{2} \leq \frac{a^{m+n}+b^{m+n}}{2}.$$

The similar problem of problem 3 is as follows

Problem 4 *Prove that, if* $a \geq b \geq 0$ *and* $m, n \in N$ *then*

$$\frac{a^m-b^m}{2} \times \frac{a^n+b^n}{2} \leq \frac{a^{m+n}-b^{m+n}}{2}.$$

Indeed, we have

$$\begin{aligned}
&\frac{a^m-b^m}{2} \times \frac{a^n+b^n}{2} \leq \frac{a^{m+n}-b^{m+n}}{2}\\
\Leftrightarrow\ & (a^m-b^m)(a^n+b^n) \leq 2(a^{m+n}-b^{m+n})\\
\Leftrightarrow\ & a^{m+n}+a^mb^n-a^nb^m-b^{m+n} \leq 2a^{m+n}-2b^{m+n}\\
\Leftrightarrow\ & a^{m+n}-a^mb^n-b^{m+n}+b^ma^n \geq 0\\
\Leftrightarrow\ & a^n(a^m+b^m)-b^n(a^m+b^m) \geq 0\\
\Leftrightarrow\ & (a^n-b^n)(a^m+b^m) \geq 0.
\end{aligned}$$

Because $a \geq b \geq 0$, $(a^n-b^n)(a^m+b^m) \geq 0$. Thus,

$$\frac{a^m-b^m}{2} \times \frac{a^n+b^n}{2} \leq \frac{a^{m+n}-b^{m+n}}{2}.$$

We have some generalizations of two real numbers a and b. How about three real numbers? We have the generalized problem as follows

Problem 5 *Prove that, if* $a \geq 0$*,* $b \geq 0$ *and* $c \geq 0$ *then*

$$\frac{a+b+c}{3} \times \frac{a^2+b^2+c^2}{3} \leq \frac{a^3+b^3+c^3}{3}$$

Applying Chebyshev 's inequality to two increasing sequences $a \leq b \leq c$ and $a^2 \leq b^2 \leq c^2$, we have

$$(a + b + c)(a^2 + b^2 + c^2) \leq 3(a^3 + b^3 + c^3).$$

Thus, we always have

$$\frac{a + b + c}{3} \times \frac{a^2 + b^2 + c^2}{3} \leq \frac{a^3 + b^3 + c^3}{3}$$

We have some discoveries around a nice inequality problem. The different solutions, the similar and generalized problems make us interesting. Do you have any comments on this paper? Please share with us! The last are some exercises.

Problem 6 *Prove that, if* $a \geq 0$, $b \geq 0$, $c \geq 0$ *and* $m, n \in N$ *then*

$$\frac{a^m + b^m + c^m}{3} \times \frac{a^n + b^n + c^n}{3} \leq \frac{a^{m+n} + b^{m+n} + c^{m+n}}{3}.$$

Problem 7 *Prove that, if* $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ *then*

$$\frac{a + b + c + d}{4} \times \frac{a^2 + b^2 + c^2 + d^2}{4} \leq \frac{a^3 + b^3 + c^3 + d^3}{4}$$

References

[1] Doan Quynh, Nguyen Huy Doan, Nguyen Xuan Liem, Dang Hung Thang, Tran Van Vuong (2016), *Advanced algebra* 11^{th}, The Vietnam Educational Publishing House.

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