The methods of using Torricelli point to prove inequalities

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Abstract

We use Torricelli point to prove inequalities.

There is an important point in geometry which is the Torricelli point. This point is applied in solving a lot of real problems. In this paper, we refer to an another aspect of the Torricelli point which is applying it to prove inequality problems.

Problem 1 *Given real numbers x, y, z. Prove that*

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

Denote by $[O\alpha)$, $[O\beta)$, $[O\gamma)$ three rays having the same origin O such that the angle between two arbitrary rays equals 120^0. Let A, B, C be points on $[O\alpha)$, $[O\beta)$, $[O\gamma)$, respectively, such that

$$OA = |x|, \; OB = |y|, \; OC = |z|.$$

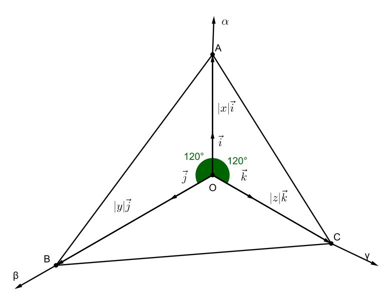

Figure 1: The method 1 of using Torricelli point

Then O is the Torricelli point of triangle ABC. We have

$$\left(|x|\overrightarrow{i} + |y|\overrightarrow{j} + |z|\overrightarrow{k}\right)^2 \geq 0 \Leftrightarrow x^2 + y^2 + z^2 - |xy| - |yz| - |zx| \geq 0.$$

Thus, $x^2 + y^2 + z^2 \geq |xy| + |yz| + |zx|$ (1)
On the other hand, we also have

$$|xy| + |yz| + |zx| \geq xy + yz + zx \text{ (2)}$$

Since (1) and (2), we have

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

We now use the Torricelli point to solve a different problem as follows:

Problem 2 (Murray S. Klamkin 's inequality) *Given positive real numbers x, y, z. Prove that*

$$\sqrt{x^2 + xy + y^2} + \sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2} \geq 3\sqrt{xy + yz + zx}.$$

This problem can be proved by using intermediate inequalities as follows:

$$\sqrt{x^2 + xy + y^2} \geq \frac{\sqrt{3}}{2}(x + y),$$
$$\sqrt{3}(x + y + z) \geq 3\sqrt{xy + yz + zx}.$$

Here is the proof of using the Torricelli point.

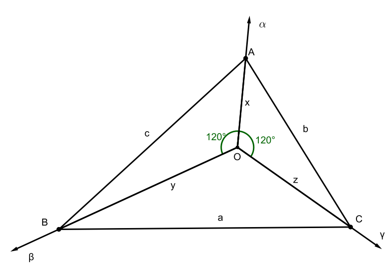

Figure 2: The method 2 of using Torricelli point

Denote by $[O\alpha)$, $[O\beta)$, $[O\gamma)$ three rays having the same origin O such that the angle between two arbitrary rays equals 120^0. Let A, B, C be points on $[O\alpha)$, $[O\beta)$, $[O\gamma)$, respectively, such that

$$OA = x, \; OB = y, \; OC = z.$$

Then O is the Torricelli point of triangle ABC.
Let S be the area of triangle ABC.
Applying Cosine rule formula

$$\begin{array}{l} a^2 = b^2 + c^2 - 2bc\cos A; \\ b^2 = c^2 + a^2 - 2ca\cos B; \\ c^2 = a^2 + b^2 - 2ab\cos C \end{array}$$

to triangles OAB, OBC, OCA, we obtain:

$$a = BC = \sqrt{y^2 + yz + z^2};\ b = CA = \sqrt{z^2 + zx + x^2};\ c = AB = \sqrt{x^2 + xy + y^2}.$$

We have

$$\frac{\sqrt{3}}{4}(xy + yz + zx) = \frac{1}{2}(xy\sin 120^0 + yz\sin 120^0 + zx\sin 120^0) = S.$$

Thus, the required inequality is equivalent to:

$$a + b + c \geq 3\sqrt{\tfrac{4S}{\sqrt{3}}} \Leftrightarrow p^2 \geq 3\sqrt{3}S\ (p = \tfrac{a+b+c}{2})\ (3)$$

We have

$$\begin{array}{l} p = (p - a) + (p - b) + (p - c) \geq 3\sqrt[3]{(p - a)(p - b)(p - c)} \\ \Leftrightarrow p^3 \geq 27(p - a)(p - b)(p - c) \\ \Leftrightarrow p^4 \geq 27S^2 \Leftrightarrow p^2 \geq 3\sqrt{3}S. \end{array}$$

Thus, (3) holds true. We have Q. E. D.
We continue to discover a different method of using the Torricelli point as follows:

Problem 3 *Find the minimum of the following expression:*

$$S = \sqrt{(x - 1)^2 + (y + 1)^2} + \sqrt{(x + 1)^2 + (y - 1)^2} + \sqrt{(x + 2)^2 + (y + 2)^2}.$$

Letting $A(-1;\ 1)$, $B(-2;\ -2)$, $C(1;\ -1)$ and $M(x;\ y)$, we have:

$$\begin{array}{l} MA = \sqrt{(x + 1)^2 + (y - 1)^2};\ MB = \sqrt{(x + 2)^2 + (y + 2)^2}; \\ MC = \sqrt{(x - 1)^2 + (y + 1)^2}. \end{array}$$

We also have $AC = \sqrt{8}$, $BA = BC = \sqrt{10}$. Applying Cosine rule formula

$$\begin{array}{l} a^2 = b^2 + c^2 - 2bc\cos A; \\ b^2 = c^2 + a^2 - 2ca\cos B; \\ c^2 = a^2 + b^2 - 2ab\cos C, \end{array}$$

we easily prove triangle ABC is acute and isosceles.
Problem 3 becomes to the following geometric problem

Problem 4 *Given an acute triangle ABC with three sides* $AC = \sqrt{8}$, $BC = BA = \sqrt{10}$. *Find the point M lying on the plane containing the triangle ABC such that the sum of* $MA + MB + MC$ *is minimum.*

According to the well-known result of geometry, point M is Torricelli point. Because triangle ABC is acute and isosceles, M lies on the median BO of triangle ABC and $\widehat{AMB} = \widehat{BMC} = \widehat{CMA} = 120^0$.

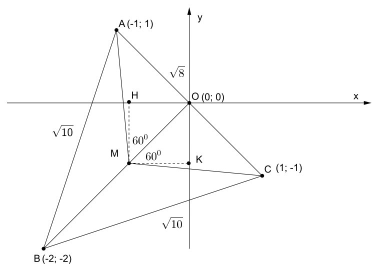

Figure 3: The method 3 of using Torricelli point

Then, triangle AMB is isosceles satisfying the relation $\widehat{AMO} = 60^0$.
We have:

$$\frac{AO}{MO} = \frac{\sqrt{2}}{MO} = \tan 60^0 = \sqrt{3} \Rightarrow MO = \sqrt{\frac{2}{3}}.$$

Through point M draw lines MH, MK perpendicular to axes Ox, Oy that meet Ox, Oy at H and K, respectively. we have

$$\frac{OH}{MO} = OH\sqrt{\frac{3}{2}} = \sin 45^0 = \frac{1}{\sqrt{2}} = \frac{OK}{MO}.$$

Thus, $OH = OK = \frac{1}{\sqrt{3}}$.
Back to problem 3, we see that the minimum value of S happens if $x = y = -\frac{1}{\sqrt{3}}$.
Then, we have

$$S = 2\sqrt{\left(\frac{-1}{\sqrt{3}} - 1\right)^2 + \left(\frac{-1}{\sqrt{3}} + 1\right)^2} + \sqrt{2\left(\frac{-1}{\sqrt{3}} + 2\right)^2}$$
$$S = 2\sqrt{\frac{1}{3} + 1 + \frac{2}{\sqrt{3}} + \frac{1}{3} + 1 - \frac{2}{\sqrt{3}}} + \left(2 - \frac{1}{\sqrt{3}}\right)\sqrt{2}$$
$$S = 4\sqrt{\frac{2}{3}} + 2\sqrt{2} - \sqrt{\frac{2}{3}}$$
$$S = \sqrt{2}(2 + \sqrt{3}).$$

Clearly, The solution is very nice.
We have some discoveries around the Torricelli point. The methods of solving

three problems are different ones on using the Torricelli point to prove inequalities. Do you have any comments on this paper? Please share with us!
The last are some exercises of using the Torricelli point.

Problem 5 *Given positive real numbers x, y, z. Prove that*

$$\sqrt{x^2 + xy + y^2} \,.\, \sqrt{y^2 + yz + z^2} + \sqrt{y^2 + yz + z^2} \,.\, \sqrt{z^2 + zx + x^2} + \sqrt{z^2 + zx + x^2} \,.\, \sqrt{x^2 + xy + y^2} \geq (x + y + z)^2.$$

Problem 6 *Given positive real numbers x, y, z such that $x + y + z = 1$. Prove that*

$$\sqrt{x^2 + xy + y^2} \,.\, \sqrt{y^2 + yz + z^2} + \sqrt{y^2 + yz + z^2} \,.\, \sqrt{z^2 + zx + x^2} + \sqrt{z^2 + zx + x^2} \,.\, \sqrt{x^2 + xy + y^2} \geq 1.$$

References

[1] Le Quang Nam (2000), *Finding for learning mathemtics*, The National University Publishing House, Ho Chi Minh city.

[2] *Inequality 40 (Murray Klamkin)*, available at https://gbas2010.wordpress.com/2010/03/07/inequality-40murray-klamkin/

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