A TYPE OF USEFUL SUBSTITUTIONS IN TRIANGLE GEOMETRY

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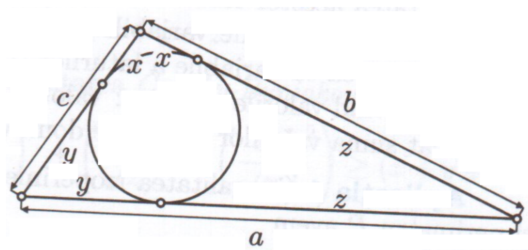
ABSTRACT. In the following lesson we will present a substitution type which could simplify proving some triangle properties, named, in some papers, Ravi's substitutions.

Let x, y, z be real strictly positive numbers . We denote $a = y + z, b = z + x$, $c = x + y$. Then $a + b > c > 0$ and analogs; in this way a, b, c are the sides of a triangle.
Reciprocal, if a, b, c are the sides of a triangle then the system

$$x + y = c, x + z = b, y + z = a$$

has a unique solution

$$x = \frac{b + c - a}{2} > 0, y = \frac{a + c - b}{2} > 0, z = \frac{a + b - c}{2} > 0.$$



These relationships have a geometric interpretation: x, y, z represents the determined segments on the triangle's sides by the points of contact of inscribed circle to the sides.
By adding the relationships and using the standard notation $p =$ triangle's semi-perimeter we obtain $a+b+c = 2(x+y+z)$ and then $2p = 2(x+y+z), p = x+y+z$. Then we deduce that $a = p - x, b = p - y, c = p - z$, wherefrom

$$x = p - a, y = p - b, z = p - c.$$

Aria S of the triangle becomes

$$S = \sqrt{p(p-a)(p-b)(p-c)} = \sqrt{xyz(x+y+z)}.$$

The radius of the inscribed circle is

$$r = \frac{S}{p} = \frac{\sqrt{xyz(x+y+z)}}{x+y+z} = \sqrt{\frac{xyz}{x+y+z}}.$$

The radius of triangle's circumscribed circle is

$$R = \frac{abc}{4S} = \frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}}.$$

Some fundamental formulas for the angles are

$$\sin\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}}=\sqrt{\frac{yz}{(x+z)(x+y)}},$$

$$\cos\frac{A}{2}=\sqrt{\frac{p(p-a)}{bc}}=\sqrt{\frac{x(x+y+z)}{(x+z)(x+y)}},$$

$$\tan\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}=\sqrt{\frac{yz}{x(x+y+z)}}$$

and the analogs.

The lengths of the bisectors and the heights of the triangle are given by

$$l_a=\frac{2}{b+c}\sqrt{bcp(p-a)}=\frac{2}{2x+y+z}\sqrt{x(x+z)(x+y)(x+y+z)},$$

$$h_a=\frac{2S}{a}=\frac{2\sqrt{xyz(x+y+z)}}{y+z}$$

and the analogs.

The utility of these formulas resides from the fact that they express elements of the triangle in function of the independent positive arbitrary variables x, y, z, while using as triangle's sides's variables means the occurrence of some restrictions on their values: the values of each variable must be smaller then the sum of the values of the other two variables.

Application 0.1. *(iso-perimetric inequality). In any triangle having the area S and the perimeter P we have*

$$36S\le\sqrt{3}P^2,$$

with equality just for the equilateral triangles (in other words, among all the triangles with perimeter P, the one with the biggest area is obtained when the sides are equal).

Proof. $P=2p$ and the inequality can be written $9S\le\sqrt{3}p^2$. With Ravi's substitutions the inequality becomes $9\sqrt{xyz(x+y+z)}\le\sqrt{3}(x+y+z)^2$, which is equivalent with $27xyz\le(x+y+z)^3$. But the last inequality is equivalent with means inequality: $3\sqrt[3]{xyz}\le x+y+z$, so it's true.
The equality is obtained just in the case $x=y=z$. □

Application 0.2. *(Euler's inequality). In any triangle we have*

$$R\ge 2r.$$

Proof. We write the inequality

$$\frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}}\ge 2\sqrt{\frac{xyz}{x+y+z}}$$

or

$$(x+y)(z+x)(y+z)\ge 8xyz. \tag{0.1}$$

On the other hand, $x+y\ge 2\sqrt{xy}$ and the analogs. By multiplying we obtain 0.1. □

Application 0.3. *(Mitrinovič inequality). In any triangle we have*

$$\frac{p}{r}\ge 3\sqrt{3}$$

Proof. We write the inequality

$$\frac{p}{r} = \frac{p^2}{S} = \frac{(x+y+z)^2}{\sqrt{xyz(x+y+z)}} \geq 3\sqrt{3},$$

or $(x+y+z)^2 \geq 3\sqrt{3xyz(x+y+z)}$, namely $(x+y+z)^3 \geq 27xyz$, which is the means inequality. □

Application 0.4. *(IMO 1983, problem 6 - this problem can be found also in the collection of problems by C. Coşniţă and F. Turtoiu). Let a, b, c be the sides of a triangle. Prove that*

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Proof. After replacements and calculus, the inequality can be written equivalent

$$x^3z + y^3x + z^3y \geq x^2yz + xy^2z + xyz^2,$$

or

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z.$$

The last relationship follows from Cauchy-Buniakovski-Schwarz's inequality:

$$\left((\sqrt{y})^2 + (\sqrt{z})^2 + (\sqrt{x})^2\right)\left(\left(\frac{x}{\sqrt{y}}\right)^2 + \left(\frac{y}{\sqrt{z}}\right)^2 + \left(\frac{z}{\sqrt{x}}\right)^2\right) \geq (x+y+z)^2,$$

dividing with $x + y + z > 0$. □

Exercises 0.1. *Prove that if a, b, c are the sides of a triangle, then*

$$a^3 + b^3 + c^3 + 3abc \geq 2ab^2 + 2bc^2 + 2ca^2,$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

Does these inequalities hold for any real positive numbers a, b, c?

References

[1] L. Giugiuc, *The Dual Ravi's Theorem*, RMT 2/2015.
[2] Samer Seraj (editor), *Geometric Inequalities Marathon*, compilation on www.artofproblemsolving.com.
[3] I. Tomescu, *The XXIV th International Mathematical Olympiad*, G.M. nr. 1/1984.
[4] C. Coşniţă, F. Turoiu, *Algebra Problems*, Tehnica Publishing House, 1972.
[5] D. Sitaru, *Affine and Euclidian Geometry Problems*, Ecko - Print Publishing House, Drobeta Turnu-Severin, 2013.
[6] D. Sitaru, Claudia Nănuţi, *Mathematics for contests*, Ecko - Print Publishing House, Drobeta Turnu-Severin, 2012
[7] Daniel Sitaru, *Math Phenomenon*, Paralela 45 Publishing House, Piteşti, Romania, 2016.

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