The creation of a sequential limited problem

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Abstract

We create a sequential limited problem. The limited problem is very important in mathematics. That 's reason why we need more exploitation of this problem.

1 Introduction

There are a lot of different ways of creation mathematics such as finding out many solutions, finding out similar and generalized problems of a problem. These make us interesting. We refer to these things through a nice sequential limited problem of a Vietnamese textbook.

Problem 1 (Problem 58, p. 178, Vietnamese Advanced Algebraic and analytic textbook 11^{th}, (2016)) *Find the limit of the sequence (u_n) such that*

$$u_n = \frac{1}{1\ .\ 2} + \frac{1}{2\ .\ 3} + ... + \frac{1}{n(n+1)}.$$

We will go to prove $u_n = 1 - \frac{1}{n+1}$. There are many solutions to prove this thing.
Solution 1
For each of positive integers, we have

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

From this, we have

$$u_n = \frac{1}{1\ .\ 2} + \frac{1}{2\ .\ 3} + ... + \frac{1}{n(n+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + ... + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

Thus $\lim u_n = \lim(1 - \frac{1}{n+1}) = 1$.
Solution 2
We will prove that $\frac{1}{1\ .\ 2} + \frac{1}{2\ .\ 3} + ... + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$ (1).
. With $n = 1$, we have $\frac{1}{1\ .\ 2} = 1 - \frac{1}{2}$. Thus, (1) will hold true when n = 1.
. Suppose that (1) holds true from $n = 1$ to $n = k$, it means,

$$\frac{1}{1\ .\ 2} + \frac{1}{2\ .\ 3} + ... + \frac{1}{k(k+1)} = 1 - \frac{1}{k+1}$$

we will prove that it will hold true when $n = k + 1$, it means,

$$\frac{1}{1\ .\ 2} + \frac{1}{2\ .\ 3} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k+2}.$$

Indeed, since the inductive hypothesis, we have

$$\frac{1}{1\ .\ 2} + \frac{1}{2\ .\ 3} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{(k+1)(k+2) - (k+2) + 1}{(k+1)(k+2)} = \frac{(k+1)(k+2) - (k+1)}{(k+1)(k+2)} = \frac{k+1}{k+2} = 1 - \frac{1}{k+2}.$$

Thus, (1) holds true for all of positive integer n's.
Thus, $\lim u_n = \lim(1 - \frac{1}{n+1}) = 1$.
we extend problem 1 to the problem as follows

Problem 2 *Find the limit of the sequence (u_n) such that*

$$u_n = \frac{1}{1\ .\ 2\ .\ 3} + \frac{1}{2\ .\ 3\ .\ 4} + ... + \frac{1}{n(n+1)(n+2)}.$$

This problem also has many solutions.
Solution 1

$$\frac{1}{1\ .\ 2\ .\ 3} = \frac{1}{2}\left(\frac{1}{1\ .\ 2} - \frac{1}{2\ .\ 3}\right),\ \frac{1}{2\ .\ 3\ .\ 4} = \frac{1}{2}\left(\frac{1}{2\ .\ 3} - \frac{1}{3\ .\ 4}\right),\ ...,$$
$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right).$$

From this, we have

$$u_n = \frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{1\,.\,2} - \frac{1}{(n+1)(n+2)}\right)$$
$$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

Thus, $\lim u_n = \lim\left(\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right) = \frac{1}{4}.$

Solution 2

We will prove

$$\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)} \quad (2)$$

. With $n = 1$, we have

$$\frac{1}{1\,.\,2\,.\,3} = \frac{1}{6} = \frac{1}{4} - \frac{1}{12} = \frac{1}{4} - \frac{1}{2.(1+1)(1+2)}.$$

Thus, (1) will hold true when $n = 1$.

. Suppose that (1) holds true from $n = 1$ to $n = k$, it means,

$$\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \dots + \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)},$$

we will prove that it will hold true when $n = k + 1$, it means,

$$\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \dots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$
$$= \frac{1}{4} - \frac{1}{2(k+2)(k+3)}$$

Indeed, since the inductive hypothesis, we have

$$\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \dots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$
$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$
$$= \frac{(k+1)(k+2)(k+3) - 2(k+3) + 4}{4(k+1)(k+2)(k+3)} = \frac{(k+1)(k+2)(k+3) - 2(k+1)}{4(k+1)(k+2)(k+3)}$$
$$= \frac{1}{4} - \frac{1}{2(k+2)(k+3)}.$$

Thus, (2) will hold true when $n = 1$.

Thus, $\lim u_n = \lim\left(\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right) = \frac{1}{4}.$

We now find out the similar problem of problem 1, we have

Problem 3 *Find the limit of the sequence* (u_n) *such that*

$$u_n = \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+\dots+n}.$$

We will go to prove $S = 1 + 2 + \dots + n = \frac{n(n+1)}{2}.$

Indeed,

$$\begin{aligned} S &= 1 + \quad 2 + \dots + n\,; \\ S &= n + (n-1) + \dots + 1. \end{aligned}$$

Thus, $2S = n(n+1)$. It means $1 + 2 + \dots + n = \frac{n(n+1)}{2}.$

Thus,

$$u_n = \frac{2}{1\,.\,2} + \frac{2}{2\,.\,3} + \dots + \frac{2}{n(n+1)}.$$

By the problem 1, we have

$$\lim u_n = \lim 2\,.\,(1 - \frac{1}{n+1}) = 2.$$

We continue to exploit problem 1 by remarking that $n^2 < n(n+1) < (n+1)^2$ so $n < \sqrt{n(n+1)} < n+1$. Hence $[\sqrt{n(n+1)}] = n$. It follows

$$[\sqrt{1\,.\,2}] + [\sqrt{2\,.\,3}] + \dots + [\sqrt{n\,.\,(n+1)}] = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

We combine algebraic method with arithmetical method to obtain the similar problem

Problem 4 *Find the limit of the sequence* (u_n) *such that*

$$u_n = \frac{1}{[\sqrt{1\,.\,2}]} + \frac{1}{[\sqrt{1\,.\,2}] + [\sqrt{2\,.\,3}]} + \frac{1}{[\sqrt{1\,.\,2}] + [\sqrt{2\,.\,3}] + [\sqrt{3\,.\,4}]} + \dots$$
$$+ \frac{1}{[\sqrt{1\,.\,2}] + [\sqrt{2\,.\,3}] + \dots + [\sqrt{n\,.\,(n+1)}]}.$$

Thus, $u_n = \frac{2}{1 . 2} + \frac{2}{2 . 3} + ... + \frac{2}{n(n+1)}$.
Since problem 1, we easily calculus $\lim u_n = \lim 2 . (1 - \frac{1}{n+1}) = 2$.
We continue to notice that the sum equals to $\frac{n(n+1)}{2}$. We go to the following problem

Problem 5 *Find the limit of the sequence* (u_n) *such that*

$$u_n = \frac{1}{\sqrt{1^3}} + \frac{1}{\sqrt{1^3 + 2^3}} + \frac{1}{\sqrt{1^3 + 2^3 + 3^3}} + ... + \frac{1}{\sqrt{1^3 + 2^3 + ... + n^3}}.$$

We need to prove

$$\sqrt{1^3 + 2^3 + 3^3 + ... + n^3} = \frac{n(n+1)}{2}.$$

In order to prove this equality, we need to prove the equivalent formula

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

We first calculus $1^2 + 2^2 + ... + n^2$.
Indeed, we have the identity $(n+1)^3 = n^3 + 3n^2 + 3n + 1$. It is equivalent to

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1.$$

From this, we have

$$\begin{aligned} 2^3 - 1^3 &= 3 . 1^2 + 3 . 1 + 1 \\ 3^3 - 2^3 &= 3 . 2^2 + 3 . 2 + 1 \\ &... \\ (n+1)^3 - n^3 &= 3 . n^2 + 3 . n + 1 \end{aligned}$$

Adding the results termwise, we have

$$(n+1)^3 - 1 = 3 . (1^2 + 2^2 + 3^2 + ... + n^2) + 3 . \frac{n(n+1)}{2} + n.$$

The result is equivalent to the result

$$2(n^3 + 3n^2 + 3n) = 6 . (1^2 + 2^2 + ... + n^2) + 3 . (n^2 + n) + 2n.$$

Thus, $1^2 + 2^2 + ... + n^2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$.
Next, we calculus the sum $1^3 + 2^3 + 3^3 + ... + n^3$.
We have the identity

$$(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1.$$

Thus,

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1.$$

This equality holds true for all of positive integer $n, n = 1, 2, 3, ...$:

$$\begin{aligned} 2^4 - 1^4 &= 4 . 1^3 + 6 . 1^2 + 4 . 1 + 1 \\ 3^4 - 2^4 &= 4 . 2^3 + 6 . 2^2 + 4 . 2 + 1 \\ &... \\ (n+1)^4 - n^4 &= 4 . n^3 + 6 . n^2 + 4 . n + 1 \end{aligned}$$

Thus,

$$(n+1)^4 - 1 = 4 . (1^3 + 2^3 + ... + n^3) + 6 . (1^2 + 2^2 + ... + n^2) + 4 . (1 + 2 + ... + n) + n.$$

As we known that $1 + 2 + ... + n = \frac{n(n+1)}{2}$ and $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, we have

$$\begin{aligned} 4 . (1^3 + 2^3 + ... + n^3) &= (n+1)^4 - (n+1) - 2n(n+1) - n(n+1)(2n+1) \\ &= (n+1)[n^3 + 3n^2 + 3n - 2n - n(2n+1)] \\ &= (n+1)n[n^2 + 3n + 1 - (2n+1)]. \end{aligned}$$

Thus, $1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$. It means that $\sqrt{1^3 + 2^3 + 3^3 + ... + n^3} = \frac{n(n+1)}{2}$.
Thus,

$$u_n = \frac{2}{1 \cdot 2} + \frac{2}{2 \cdot 3} + ... + \frac{2}{n(n+1)}.$$

Since problem 1, we easily to calculus that $\lim u_n = \lim 2 \cdot (1 - \frac{1}{n+1}) = 2$.
A different problem is similar to problem 1 as follows

Problem 6 *Find the limit of the sequence* (u_n) *such that*

$$u_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{1} + \sqrt{1+3}} + ... + \frac{1}{\sqrt{1} + \sqrt{1+3} + .. \sqrt{1+3+...+(2n-1)}}$$

We go to prove that

$$\sqrt{1} + \sqrt{1+3} + \sqrt{1+3+...+(2n-1)} = 1 + 2 + 3 + ... + n.$$

This equality is equivalent to $1 + 3 + ... + (2n - 1) = n^2$.
Indeed, we have

$$\begin{array}{rcl} S & = & 1 + 3 + ... + (2n-1) \\ S & = & (2n-1) + (2n-3) + ... + 1 \end{array}$$

Thus, $2S = 2n + 2n + + 2n = 2n^2$. Thus $S = n^2$. Hence $1 + 3 + ... + (2n - 1) = n^2$.
From this, $\sqrt{1 + 3 + ... + (2n - 1)} = n$. It means that

$$\sqrt{1} + \sqrt{1+3} + \sqrt{1+3+...+(2n-1)} = 1 + 2 + 3 + ... + n.$$

Thus,

$$u_n = \frac{2}{1 \cdot 2} + \frac{2}{2 \cdot 3} + ... + \frac{2}{n(n+1)}.$$

Since problem 1, we have $\lim u_n = \lim 2 \cdot (1 - \frac{1}{n+1}) = 2$.
We have some exploitation of a problem. All of different solutions, similar problems and generalized problems make us interesting. Do you have any comments on this paper! Please share with us!
The last are some exercises

Problem 7 *Find the limit of the sequence* (u_n) *such that*

$$u_n = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + ... + \frac{1}{n(n+1)(n+2)(n+3)}.$$

Problem 8 *Find the limit of the sequence* (u_n) *such that*

$$u_n = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + ... + \frac{1}{n(n+1)(n+2)(n+3)(n+4)}.$$

References

[1] Doan Quynh, Nguyen Huy Doan, Nguyen Xuan Liem, Dang Hung Thang, Tran Van Vuong (2016), *Advanced algebra* 11^{th}, The Vietnam Educational Publishing House.

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