GPF Inequality

(Generalization for product of functions Inequality)

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When proving inequality [1], I found a lemma for product of functions. Maybe this lemma had been discovered and proved by someone before I found it. So, I will call it by the name: GPF Inequality.

1. Inequalities between $f(x_1)f(x_2)\dots f(x_n)$ and $f(x_1x_2\dots x_n)$, $f(\sqrt[n]{x_1x_2\dots x_n})^n$

Theorem 1: *Let $x_1, x_2, \dots x_n$ be nonnegative real numbers and a positive constant k. Given a function $f(x)$ defined on $[0,\infty)$ satisfied:*
$f(x) \geq 0; f'(x) \geq 0; f''(x)f(x) \geq f'(x)^2$ *and* $kf(x)^n \geq f(x^n)$ *with all* $x \in [0,\infty)$
Then we will have the following true inequality:
$kf(x_1)f(x_2)\dots f(x_n) \geq f(x_1x_2\dots x_n)$

Proof. The inequality is equivalent to:

$$\ln k + \ln f(x_1) + \ln f(x_2) + \dots + \ln f(x_n) \geq \ln f(x_1x_2\dots x_n)$$

Define $g(x) = \ln f(x)$. These conditions: $\frac{f'(x)}{f(x)} \geq 0, f''(x)f(x) \geq f'(x)^2$ give us: $g'(x) \geq 0$ and $g''(x) \geq 0$.
Rewrite the inequality as:

$$\ln k + g(x_1) + g(x_2) + \dots + g(x_n) \geq g(x_1x_2\dots x_n)$$

Since $g'(x) \geq 0, g''(x) \geq 0$, applying Jensen's Inequality and AM-GM, we have:

$$g(x_1^n) + g(x_2^n) + \dots + g(x_n^n) \geq ng\Big(\frac{x_1^n + x_2^n + \dots + x_n^n}{n}\Big) \geq ng(x_1x_2\dots x_n)$$

We need to prove this:

$$n\ln k + ng(x_1) + ng(x_2) + \dots + ng(x_n) \geq g(x_1^n) + g(x_2^n) + \dots + g(x_n^n)$$

We will prove this one: $\ln k + ng(x_n) \geq g(x_n^n)$ with all n.

(1) $\quad \leftrightarrow e^{\ln k + ng(x_n)} \geq e^{g(x_n^n)} \leftrightarrow e^{n\ln f(x_1)} \cdot e^{\ln k} \geq e^{\ln f(x_n^n)}$ or $kf(x_n)^n \geq f(x_n^n)$

But it is true because we have 1 from the condition. □

Theorem 2: *Let $x_1, x_2, \dots x_n \in [m_1, m_2]$ be real numbers and a positive constant k. Given a function $f(x)$ defined on $[m_1, m_2]$ satisfied:*
$f(x) \geq 0; \frac{f'(x)}{f(x)} \leq 0; kf(m_2)^n \geq f(m_1^n)$. *Then we will have the following true inequality:*
$kf(x_1)f(x_2)\dots f(x_n) \geq f(x_1x_2\dots x_n)$.

Proof. The inequality is equivalent to:

(2) $$\ln k + \ln f(x_1) + \ln f(x_2) + \dots + \ln f(x_n) \geq \ln f(x_1x_2\dots + x_n)$$

Define $g(x) = \ln f(x)$. The condition $\frac{f'(x)}{f(x)} \leq 0$ give us $g'(x) \leq 0$.
Hence, we have:

(3) $$g(x_1) + g(x_2) + \dots + g(x_n) \geq ng(m_2); g(x_1x_2\dots + x_n) \leq g(m_1^n)$$

Rewrite 2 as:

$$\ln k + g(x_1) + g(x_2) + \ldots + g(n) \geq g(x_1 x_2 \ldots x_n).$$

Since 2 and 3 we have to prove: $\ln k + ng(m_2) \geq g(m_1^n)$

$$\leftrightarrow e^{\ln k + ng(m_2)} \geq e^{g(m_1^n)}$$

$$\leftrightarrow kf(m_2)^n \geq f(m_1^n) \text{ which is the condition.}$$

□

Theorem 3: *Let $f(x)$ be a function defined on $\mathbb{I}$ such that:* $f(x) \geq 0; f'(x) \geq 0; f''(x) \geq 0$. *Given $x_1, x_2, \ldots, x_n \in \mathbb{I}$. Then we will have the following true inequality:*
$f(x_1)f(x_2)\ldots f(x_n) \geq f(\sqrt[n]{x_1 x_2 \ldots x_n})^n$

Proof. Rewrite the inequality as:

$$g(x_1) + g(x_2) + \ldots + g(x_n) \geq g(\sqrt[n]{x_1 x_2 \ldots x_n})^n \text{ with } g(x) = \ln f(x)$$

Applying Jensen's Inequality and AM-GM, we obtain:

$$g(x_1) + g(x_2) + \ldots + g(x_n) \geq ng\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \geq ng(\sqrt[n]{x_1 x_2 \ldots x_n})$$

□

Theorem 4: *Let $f(x)$ be defined on $\mathbb{I}$ such that: $f(x) \geq 0; f'(x) \leq 0; f''(x) \leq 0$. Given $x_1, x_2, \ldots, x_n \in \mathbb{I}$.*
Then we will have the following true inequality: $f(x_1)f(x_2)\ldots f(x_n) \leq f(\sqrt[n]{x_1 x_2 \ldots x_n})^n$

Proof. Rewrite the inequality as:

$$g(x_1) + g(x_2) + g(x_2) + \ldots + g(x_n) \leq g(\sqrt[n]{x_1 x_2 \ldots x_n)^n} \text{ with } g(x) = \ln f(x).$$

Applying Jensen's Inequality and AM-GM, we obtain:

$$g(x_1) + g(x_2) + \ldots + g(x_n) \leq ng\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \leq ng(\sqrt[n]{x_1 x_2 \ldots x_n})$$

You can see that Theorem 4, Theorem 5 are the similar with Theorem 1. □

2. Inequalities between $f(x_1)f(x_2)\ldots f(x_n)$ and $f\left((\frac{x_1+x_2+\ldots+x_n}{n})^n\right), f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)$

Theorem 5: *Let $f(x)$ be a function defined on $\mathbb{I}$ such that: $f''(x) \cdot f(x) \geq f'(x)^2$. Given $x_1, x_2, \ldots x_n \in \mathbb{I}$. Then we will have the following true inequality:*
$f(x_1)f(x_2)\ldots f(x_n) \geq f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^n$

Proof. Define $g(x) = \ln f(x)$. Rewrite the inequality as:
$g(x_1) + g(x_2) + \ldots + g(x_n) \geq ng\Big(\frac{x_1+x_2+\ldots+x_n}{n})$ which is true since $g''(x) \geq 0$ and Jensen's Inequality.
The inequality we need to prove is rewritten as:

$$\leftrightarrow \ln f(x_1) + \ln f(x_2) + \ldots + \ln f(x_n) \geq n \ln f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right).$$

$$\leftrightarrow e^{\ln f(x_1) + \ln f(x_2) + \ldots + \ln f(x_n)} \geq e^{n \ln f(\frac{x_1+x_2+\ldots+x_n}{n})}.$$

$$\leftrightarrow f(x_1)f(x_2)\ldots f(x_n) \geq f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n$$

□

Theorem 6: *Let $f(x)$ be a function defined on $\mathbb{I}$ such that:* $f(x) \geq 0; f'(x) \geq 0; f''(x) \cdot f(x) \geq f'(x)^2 .k$ *is a positive real numbers satisfied* $kf(x_n)^n \geq f(x^n)$. *Given* $x_1, x_2, \ldots, x_n \in \mathbb{I}$. *Then we will have the following true inequality:*
$kf(x_1)f(x_2)\ldots f(x_n) \geq f\Big((\frac{x_1+x_2+\ldots+x_n}{n})^n\Big)$

Proof. Define $g(x) = \ln f(x)$. These conditions: $\frac{f'(x)}{f(x)} \geq 0; f''(x)f(x) \geq f'(x)^2$ give us: $g'(x) \geq 0$ and $g''(x) \geq 0$. Applying Jensen's Inequality and AM-GM, we have:

$$\ln k+g(x_1^n)+g(x_2^n)+\ldots+g(x_n^n) \geq ng\Big(\frac{x_1^n+x_2^n+\ldots+x_n^n}{n}\Big) \geq ng\Big((\frac{x_1+x_2+\ldots+x_n}{n})^n\Big)$$

We need to prove this:

$$n\ln k + ng(x_1) + ng(x_1) + ng(x_2) + \ldots + ng(x_n) \geq g(x_1^n) + g(x_2^n) + \ldots + g(x_n^n)$$

We will prove this one: $\ln k + ng(x_n) \geq g(x_n^n)$ with all n.

$$(6) \qquad \leftrightarrow e^{\ln k + ng(x_n)} \geq e^{g(x_n^n)} \leftrightarrow e^{n\ln f(x_1)\cdot e^{\ln k}} \geq e^{\ln f(x_n^n)} \text{ or } kf(x_n)^n \geq f(x_n^n)$$

But it is true because we have 1 form the condition.
This proof is similar with the proof of Theorem 1. □

3. Inequalities between $f(x_1)f(x_2)\ldots f(x_n)$ and $f\Big(\frac{x_1^n+x_2^n+\ldots+x_n^n}{n}\Big), f\Big(\sqrt[n]{\frac{x_1^n+x_2^n+\ldots+x_n^n}{n}}\Big)^n$

Theorem 7: *Let $f(x)$ be a function defined on $\mathbb{I}$ such that:* $f(x) \geq 0$; $f'(x) \leq 0; f''(x) \cdot f(x) \geq f'(x)^2$. *Given* $x_1, x_2, \ldots, x_n \in \mathbb{I}$. *Then we will have the following true inequality:*

Proof. The inequality is equivalent to:

$$\ln f(x_1) + \ln f(x_2) + \ldots + \ln f(x_1) \geq n\ln f\Big(\sqrt[n]{\frac{x_1^n+x_2^n+\ldots+x_n^n}{n}}\Big)$$

Define $g(x) = \ln f(x)$. Those conditions give us: $g'(x) \leq 0$ and $g''(x) \geq 0$.
Rewrite the inequality as:

$$g(x_1) + g(x_2) + \ldots + g(x_n) \geq ng\Big(\sqrt[n]{\frac{x_1^n+x_2^n+\ldots+x_n^n}{n}}\Big)$$

Applying Jensen's Inequality and AM-GM, we have:

$$g(x_1)+g(x_2)+\ldots+g(x_n) \geq ng\Big(\frac{x_1+x_2+\ldots+x_n}{n}\Big) \geq ng\Big(\sqrt[n]{\frac{x_1^n+x_2^n+\ldots+x_n^n}{n}}\Big).$$

□

Theorem 8: *Let $f(x)$ be a function defined on $\mathbb{I}$ such that: $f(x) \geq 0$; $f''(x) \cdot f(x) \geq f'(x)^2$.$k$ is a positie real numbers satisfied $kf(x)^n \geq f(x^n)$. Given $x_1, x_2, \ldots, x_n \in \mathbb{I}$. Then we will have the following true inequality:* $kf(x_1)f(x_2)\ldots f(x_n) \geq f\Big(\frac{x_1^n+x_2^n+\ldots+x_n^n}{n}\Big)$

Proof. The inequality is equivalent to:

$$\ln k + \ln f(x_1) + \ln f(x_2) + \ldots + \ln f(x_n) \geq \ln f\Big(\frac{x_1^n + x_2^n + \ldots + x_n^n}{n}\Big)$$

Define $g(x) = \ln f(x)$. The conditions give us: $g''(x) \geq 0$.
Rewrite the inequality as:

$$\ln k + g(x_1) + g(x_2) + \ldots + g(x_n) \geq g\Big(\frac{x_1^n + x_2^n + \ldots + x_n^n}{n}\Big)$$

Applying Jensen's Inequality and AM-GM, we have:

$$g(x_1^n) + g(x_2^n) + \ldots + g(x_n^n) \geq ng\Big(\frac{x_1^n + x_2^n + \ldots + x_n^n}{n}\Big).$$

We need to prove this:

$$n\ln k + ng(x_1) + ng(x_2) + \ldots + ng(x_n) \geq g(x_1^n) + g(x_2^n) + \ldots + g(x_n^n)$$

We will prove this one: $\ln k + ng(x_n) \geq g(x_n^n)$
Or $e^{\ln k + ng(x_n)} \geq e^{g(x_n^n)}$ or $e^{n \ln f(x_1)} \cdot e^{\ln k} \geq e^{\ln f(x_n^n)}$ or $kf(x_n)^n \geq f(x_n^n)$. ☐

4. Corollaries

Corollary 1: *Let $x_1, x_2, \ldots, x_n$ be a positive real numbers such that: $x_1x_2\ldots x_n \leq 1$ and a positive constant k. A function $f(x)$ satisfied: $\frac{f'(x)}{f(x)} \geq 0; f''(x)f(x) \geq f'(x)^2; f(x) \geq 0$ and $kf(x)^{n-1} \geq 1$. Then we will have the following true inequality: $kf(x_1)f(x_2)\ldots f(x_n) \geq f(x_1x_2\ldots x_n)$*

Proof. The inequality is equivalent to:

$$\ln k + \ln f(x_1) + \ln f(x_2) + \ldots + \ln f(x_n) \geq \ln f(x_1x_2\ldots x_n).$$

Define $g(x) = \ln f(x)$. These conditions: $\frac{f'(x)}{f(x)} \geq 0; f''(x)f(x) \geq f'(x)^2$ gives us: $g'(x) \geq 0$ and $g''(x) \geq 0$.
Rewrite the inequality as:

$$\ln k + g(x_1) + g(x_2) + \ldots + g(x_n) \geq g(x_1x_2\ldots x_n).$$

Applying Jensen's Inequality and AM-GM, since $g'(x) \geq 0$ and $g''(x) \geq 0$, we have:

$$g(x_1) + g(x_2) + \ldots + g(x_n) \geq ng\Big(\frac{x_1 + x_2 + \ldots + x_n}{n}\Big) \geq ng(\sqrt[n]{x_1x_2\ldots x_n})$$

Since $x_1x_2\ldots x_n \leq 1$, we have $\sqrt[n]{x_1x_2\ldots x_n} \geq x_1x_2\ldots x_n$. Hence, we have:

$$g(\sqrt[n]{x_1x_2\ldots x_n}) \geq (x_1x_2\ldots x_n).$$

We have to prove this:

$$n\ln k + ng(x_2) + ng(x_2) + \ldots + ng(x_n) \geq g(x_1) + g(x_2) + \ldots + g(x_n)$$

Or:

$$n\ln k + (n-1)g(x_1) + (n-1)ng(x_2) + \ldots + (n-1)g(x_n) \geq 0.$$

We will prove this:

$$\ln k + (n-1)g(x_n) \geq 0 \text{ with all } n.$$

$\leftrightarrow e^{\ln k+(n-1)g(x_n)} \geq 1 \leftrightarrow k \cdot f(x_n)^{n-1} \geq 1$ which is true since we considered the condition.

GPF Inequalities isn't the best way to prove a product of functions Inequality. It must be used with another methods or another inequalities. Because it's only true in some range of variables. I will show you! We will start from this condition: $f''(x)f(x) \geq f'(x)^2$ □

Corollary 2: *Let $f(x)$ be a polynomial: $f(x) = a_1x^n + a_2x^{n-1} + \ldots + a_nx + a_{n+1}$ such that $a_1 > 0, x \geq 0$ and $n \geq 0$.*
We will have $f''(x) \cdot f(x)$ isn't always bigger than $f'(x)^2$ with all $x \geq 0$

Proof. We have $f'(x)^2 = (na_1x^{n-1} + (n-1)a_2x^{n-2} + \ldots + a_n)^2 = n^2a_1^2x^{2n-2} + G(x)$ with $G(x)$ is a polynomial after squaring $f'(x)$.

$$f''(x)\cdot f(x) = \Big(n(n-1)a_1x^{n-2} + (n-1)(n-2)a_2x^{n-3} + \ldots + 2a_{n-1}\Big)(a_1x^n + a_2x^{n-1} + \ldots + a_nx)$$

with $H(x)$ is a polynomial after expanding $f''(x) \cdot f(x)$.
We always have $n^2 \geq n(n-1)$. So we obtain:

$$S = f''(x) \cdot f(x) - f'(x)^2 = -a_1^2x^{2n-2} + H(x) - G(x)$$

Then we can conclude that S isn't always bigger than zero. □

That is the biggest problem of GPF Inequality. GPF can only help us on some interval $\mathbb{I}$ that satisfied the condition. But GPF is the good way to find the best estimation for inequality. Next part of this article is the applications of GPF. (Note: There are also solutions of following example using uvw method or Cauchy - Schwarz, etc).

5. Applications

As I said, GPF Inequality isn't always the best way to prove a Inequality with product of functions. When using this theorem, you have to consider some cases. In my opinion, GPF Inequality can be only a lemma. It isn't strong enough to be a theorem. But I will show you some applications of this inequality. I hope you enjoy these examples!

1/(Michael Rozenberg)
Let a, b and c be non-negative numbers. Prove that:

$$\Big(1 + \frac{2}{\sqrt{3}}\Big)(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq a^2b^2c^2 - abc + 1$$

Solution: This inequality can be found in [1]. This is a very hard inequality. The original one was a problem in USA TST 2006. I will show you the solution for this case: $a,b,c \in \left[\frac{1}{2}; \frac{1+\sqrt{3}}{2}\right]$

Define $f(x) = x^2 - x + 1$. Since $a,b,c \in \left[\frac{1}{2}; \frac{1+\sqrt{3}}{2}\right]$, we have $x \geq \frac{1}{2}$ and $f'(x) \geq 0$.
The inequality is equivalent to:
$kf(a)f(b)f(c) \geq f(abc)$ with $k = 1 + \frac{2}{\sqrt{3}}$.
We have: $\frac{f'(x)}{f(x)} = \frac{2x-1}{x^2-x+1}; f''(x)f(x) \geq f'(x)^2$.

$$\left(1 + \frac{2}{\sqrt{3}}\right)(x^2 - x + 1)^3 - (x^6 - x^3 + 1) \geq 0 \tag{1}$$

1 can be checked by computer.
So we have: $kf(x)^3g \geq f(x^3)$.
Applying GPF1 Inequality, we obtain: $kf(a)f(b)f(c) \geq f(abc)$. □

2/ (Unknown origin)
Let a,b,c be a real numbers. Determine the positive constant k such that the following inequality is true:

$$k(a^2+1)(b^2+1)(c^2+1) \geq a^2b^2c^2 + 1$$

Solution. For $a,b,c \in [0,1]$. We have:
$f'(x) = 2x; f''(x) = 2$. Hence, $f''(x) \cdot f(x) \geq f'(x)^2$.
$k \cdot f(x)^3 \geq f(x^3)$ if and only if $(k-1)x^6 + 3kx^4 + 3kx^2 + (k-1) \geq 0$.
Applying the first GPF Inequality and we will have $k = 1$ is the best constant. □

3/ (own)
Let a,b,c be positive real numbers such that: $a,b,c \in \left[\frac{1}{3}; \frac{2}{3}\right]$. Prove that:

$$(a^3 - 2a^2 + a + 1)(b^3 - 2b^2 + b + 1)(c^3 - 2c^2 + c + 1) \leq (abc - 2\sqrt[3]{a^2b^2c^2} + \sqrt[3]{abc} + 1)^3$$

Solution: We have: $f(x) = x^3 - 2x^2 + x + 1 \geq 0$.
$f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1) \leq 0$
$f''(x) = 6x - 4 \leq 0$.
Applying the fourth GPF Inequality, we have: $f(a)f(b)f(x) \leq f(\sqrt[3]{abc})^3$ □

4/ (Holder's Inequality)
Let $x_1; x_2; \ldots; x_n$ be nonnegative real numbers. Prove that:

$$(1+x_1^3)(1+x_2^3)\ldots(1+x_n^3) \geq \left(1 + \sqrt[n]{(x_1x_2\ldots x_n)^3}\right)^n$$

Solution: This isn't a good solution. GPF can only prove Holder's inequality for only one cases!!! For Holder's inequality, GPF is the weakest. This example will show you the weakness of GPF.
We have the function $f(x) = 1 + x^3$.

$$f'(x) = 3x^2 \geq 0; f''(x) = 6x \geq 0; f''(x) \cdot f(x) - f'(x)^2 = 6x - 3x^4 = 3x(2 - x^3)$$

Hence, we have 2 cases:
If $x^3 \leq 2$.

We will have $f''(x) \cdot f(x) \geq f'(x)^2$.
Applying GPF3 Inequality, we will have:

$$f(x_1)f(x_2)\dots f(x_n) \geq f(\sqrt[n]{x_1x_2\dots x_n}).$$

Actually, the inequality is true for all positive real numbers! As I said, GPF inequality isn't worked in some cases! Maybe you want to use another method! In addition, GPF inequality is a good choice for a product of functions inequality with ranges of variables. When the inequality doesn't have any condition, it's hard to find the range of variables that GPF inequality is worked. □

6. Problems for practicing

1. Let a, b, c be positive real numbers such that: $a, b, c \geq \sqrt[3]{3}$. Prove that:

$$(a^3 - 3)(b^3 - 3)(c^3 - 3) \leq (abc - 3)^3$$

2. Let $a, b, c \in \left[\frac{1}{e}; 1\right]$ be positive real numbers, $k = e^{\frac{3}{3}(1-\frac{1}{e^2})}$. Prove that:

$$ka^2b^bc^c \geq (abc)^{abc}$$

3. Let $a, b, c \in \left[m; \frac{1}{e}\right]$ be positive real numbers with $m \neq 1$ be the root of the equation: $x = e^{\frac{1-x^2}{3x^2-1}}$. Let $k = e^{3m(m^2-1)\ln m}$. Prove that:

$$ka^ab^bc^c \geq (abc)^{abc}$$

4. Let $a, b, c \geq 1$. Prove that:

$$\frac{3}{4}\left(1 + \frac{1}{a}\right)^a \left(1 + \frac{1}{b}\right)^b \left(1 + \frac{1}{c}\right)^c \geq \left(1 + \frac{1}{abc}\right)^{abc}$$

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