COMMENTED PROBLEM - 3

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In Mathematical Gazette nr. 11/2016, problem 27298 has the following content: Prove that in any triangle ΔABC we have

$$\sum \frac{a}{b+c} + \frac{r}{R} \leq 2$$

Florin Stănescu, Găești, Dâmbovița

a)$\sum \frac{a}{b+c} + \frac{r}{R} \leq 2$.

Mathematical Reflections 4/2016, Florin Stănescu, Găești, Romania

Solution:

Using the known identity in triangle $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}$, we write the inequality

$$\frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} + \frac{r}{R} \leq 2 \Leftrightarrow 2R(p^2-r^2-Rr) \leq (2R-r)(p^2+r^2+2Rr)$$

$$\Leftrightarrow p^2 \leq 6R^2+2Rr-r^2, \text{ which follows from Gerresten's inequality}$$

$\Leftrightarrow p^2 \leq 4R^2+4Rr+3r^2$. It remains to prove that $\Leftrightarrow 4R^2+4Rr+3r^2 \leq 6R^2+2Rr-r^2$ $\Leftrightarrow R^2-Rr-2r^2 \geq 0 \Leftrightarrow (R-2r)(R+r) \geq 0$, obviously from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

The article proposes to strengthen this inequality, and developments of some inequalities with sums having the form $\sum \frac{a}{b+c}$. □

b) $\sum \frac{a}{b+c} + \frac{3r}{2R+2r} \leq 2$.

Solution:

Using the known identity in triangle $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2r}$, we write the inequality

$$\frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} + \frac{3r}{2R+2r} \leq 2 \Leftrightarrow 3p^2 \leq 12R^2+14Rr+5r^2,$$

which follows from Gerretsen's inequality $\Leftrightarrow p^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that $\Leftrightarrow 3(4R^2+4Rr+3r^2) \leq 12R^2+14Rr+5r^2 \Leftrightarrow R \geq 2r$ obviously from Euler's inequality.

The equality holds if and only if the triangle is equilateral. □

c) $\sum \frac{a}{b+c} + \frac{r}{R} \leq \sum \frac{a}{b+c} + \frac{3r}{2R+2r} \leq 2$.

Solution:

The first inequality is equivalent with Euler's inequality $R \geq 2r$, the second is b). Obviously b) is stronger than a).

The equality holds if and only if the triangle is equilateral. □

d) $\sum \frac{a}{b+c} + n \cdot \frac{r}{R} \leq \frac{n+3}{2}$, where $n \geq 1$.

Solution:

We use $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}$, Gerretsen's inequality.

$$16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2 \text{and Euler's inequality} R \geq 2r.$$

The equality holds if and only if the triangle is equilateral. □

e) $\sum \frac{a}{b+c} + \frac{3n}{2} \cdot \frac{r}{R+r} \leq \frac{n+3}{2}$, unde $n \geq 1$.

Solution:

Analogous d).
The equality holds if and only if the triangle is equilateral. □

f) $\sum \frac{a}{b+c} + n \cdot \frac{r}{R} \leq \sum \frac{a}{b+c} + \frac{3n}{2} \cdot \frac{r}{R+r} \leq \frac{n+3}{2}$, where $n \geq 1$.

Developments, M. Chirciu

Solution:

Analogous c).
The equality holds if and only if the triangle is equilateral. □

g) $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \geq \frac{3}{2}$.

Solution:

$$\sum \frac{a}{b+c} = \frac{\sum a(a+b)(a+c)}{\prod(b+c)} = \frac{\sum a^3 + \sum a \sum bc}{\prod(b+c)} = \frac{2p(p^2-3r^2-6Rr)+2p(p^2+r^2+4Rr)}{2p(p^2+r^2+2Rr)} =$$

$$= \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}.$$

The inequality $\frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \geq \frac{3}{2}$ is equivalent with $p^2 \geq 10Rr + 7r^2$, which follows

from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.
Its Nesbitt's inequality in triangle. □

h) $\sum \frac{a}{b+c} = \frac{11p^2-15r^2-60Rr}{6p^2-6r^2-24Rr} \geq \frac{3}{2}$.

Mathematical Recreations 2/2009, Marius Olteanu, Rm. Vâlcea

Solutions:

See g). □

i) $\sum \frac{a}{b+c} \geq \frac{4p^2-6Rr}{2p^2+5Rr} \geq \frac{3}{2}$.

Solution:

We use $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}$, Gerretsen's inequality
$p^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$.
It is a strengthening of Nesbitt's inequality in triangle.
The equality holds if and only if the triangle is equilateral. □

j)$\sum \frac{a^2}{b^2+c^2} \geq 2 - \frac{r}{R} \geq \sum \frac{a}{b+c} \geq \frac{3}{2}$.

Solution:

For the first inequality we use Bergstrom, Gerretsen and Euler.
We obtain

$$\sum \frac{a^2}{b^2+c^2} = \sum \frac{a^4}{a^2b^2+a^2c^2} \geq \frac{(\sum a^2)^2}{2\sum b^2c^2} = \frac{[2(p^2-r^2-4Rr)]^2}{2[p^4-2p^2(4Rr-r^2)+r^2(4R+r)^2]} \geq$$

$\geq 2 - \dfrac{r}{R}$, the last inequality is equivalent to $p^4 + p^2(2r^2-16Rr) + r^2(4R+r)^2 \geq 0$, which follows from Gerretsen's inequality.

For the second inequality we use $\displaystyle\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}$ and Gerretsen.

The equality holds if and only if the triangle is equilateral. □

k) $\sum \frac{a}{b+c} + \frac{9r}{4R+r} \leq \frac{5}{2}$.

Solution:

$$\text{We use } \sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \text{ and Gerretsen.}$$

The equality holds if and only if the triangle is equilateral. □

l) $\sum \frac{a}{b+c} + \frac{nr}{4R+r} \leq \frac{3}{2} + \frac{n}{9}$, unde $n \geq \frac{9}{2}$.

Solution:

$$\text{We use } \sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \text{ and Gerretsen.}$$

The equality holds if and only if the triangle is equilateral. □

m) $\sum \frac{a}{b+c} + \frac{3abc}{\sum bc(b+c)} \geq 2$.

Solution:

We use $\displaystyle\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}, \sum bc(b+c) = 2p(p^2+r^2-2Rr)$ and Gerretsen.

The equality holds if and only if the triangle is equilateral. □

n) $\sum \frac{a}{b+c} + n \cdot \frac{abc}{\sum bc(b+c)} \geq \frac{n+9}{6}$, where $n \leq 3$.

Solution:

Analogous m). □

o) $\sum \frac{a}{b+c} + 4 \prod \frac{a}{b+c} \geq 2$.

Solution:

We use $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}, \prod \frac{a}{b+c} = \frac{2Rr}{p^2+r^2+2Rr}$ and Gerretsen. The equality holds if and only if the triangle is equilateral. □

p) $\sum \frac{a}{b+c} + n \cdot \prod \frac{a}{b+c} \geq \frac{n+12}{8}$, where $n \leq 4$.

Solution:

Analogous o).

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□

Other inequalities with sums having the form $\sum \frac{a}{b+c}$.

1) $\sum \frac{b+c}{a} \geq 4 \sum \frac{a}{b+c}$.
2) $\sum \frac{b+c}{a} \geq 6 - \frac{3n}{2} + n \sum \frac{a}{b+c}$, where $n \leq 4$.
3) $3 \sum \frac{a}{b+c} \geq \sum a \cdot \sum \frac{1}{b+c} \geq 9$.
4) $\sum a^2 \cdot \sum \frac{1}{a^2} \geq 6 \sum \frac{a}{b+c}$.
5) $\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{2}{3} \sum \frac{a}{b+c} \geq 1$.
6) $\frac{a^3+b^3+c^3}{abc} \geq 2 \sum \frac{a}{b+c} \geq 3$.
7) $\frac{a^3+b^3+c^3}{abc} + n \geq \frac{2}{3}(n+2) \sum \frac{a}{b+c}$, where $n \leq \frac{3}{4}$.
8) $\sum a \cdot \sum \frac{a}{bc} \geq 6 \sum \frac{a}{b+c} \geq 9$.
9) $2 \sum \frac{a}{b+c} \geq \frac{(a+b+c)^2}{ab+bc+ca} \geq 3$.
10) $\sum a \cdot \sum \frac{1}{a} \geq \frac{12r}{R} \sum \frac{a}{b+c}$.
11) $\sum \frac{b+c}{a} - 2 \sum \frac{a}{b+c} \geq 3$; b) $\sum \frac{b+c}{a} - \sum \frac{a}{b+c} \geq 6n - \frac{3}{2}$, where $n \geq \frac{1}{4}$.
12) $\sum \frac{a}{b+c} + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 2$;
13) $\sum \frac{a}{b+c} + n \cdot \frac{abc}{(a+b)(b+c)(c+a)} \geq \frac{n+12}{8}$, where $n \leq 4$.
14) $\sum \frac{a}{b+c} + 3 \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \leq \frac{9}{2}$.
15) $\sum \frac{a}{b+c} + n \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \leq n + \frac{3}{2}$, where $n \geq 1$.
16) $\sum \frac{a}{b+c} + \frac{1}{3} \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \geq n + \frac{3}{2}$.
17) $\sum \frac{a}{b+c} + n \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \geq n + \frac{3}{2}$, where $n \leq \frac{1}{3}$.
18) $\sum \frac{a}{b+c} + 3 \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{11}{2}$.
19) $\sum \frac{a}{b+c} + n \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \geq n + \frac{3}{2}$, where $n \geq 3$.

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Solution:

We use the known identities in triangle:

$$\sum \frac{b+c}{a} = \frac{p^2+r^2-2Rr}{2Rr}; \sum \frac{1}{b+c} = \frac{5p^2+r^2+4Rr}{2p(p^2+r^2+2Rr)}; \sum a^2 = 2(p^2-r^2-4Rr)$$

$$\sum \frac{1}{a^2} = \frac{p^2-2p^2(4Rr-r^2)+r^2(4R+r)^2}{16R^2r^2p^2}; \sum bc = p^2+r^2+4Rr; abc = 4Rrp;$$

$$\sum a^3 = 2p(p^2 - 3r^2 - 6Rr); \sum \frac{a}{bc} = \frac{p^2 - r^2 - 4Rr}{2Rrp}; \sum \frac{1}{a} = \frac{p^2 + r^2 + 4Rr}{4Rrp}$$

$$\prod (b + c) = 2p(p^2 + r^2 + 2R).$$

Then we use like the proves before:
Gerresten's inequality: $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$.
Euler's inequality: $R \geq 2r$.
To each proposed inequalities, the equality is realised if and only if the triangle is equilateral. □

References

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