SPECIAL APPLICATIONS

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1. Evaluate:

$$\boldsymbol{I = \int_0^\infty \frac{\ln(1+x)Li_2(-x)}{x^{\frac{3}{2}}}dx}$$

Proof.

$$\textit{Making change of variable,}$$
$$x = y^2$$
$$\Rightarrow 2\int_0^\infty \frac{\ln(1+y^2)Li_2(-y^2)}{y^2}dy$$
$$\Rightarrow 2\Big[Li_2(-y^2)\Big]_0^\infty \int_0^\infty \frac{\ln(1+y^2)}{y^2}dy - \int_0^\infty \left(\frac{d}{dx}\Big(Li_2(-y^2)\Big)\int_0^\infty \frac{\ln(1+y^2)}{y^2}dy\right)$$
$$\textit{As we know,}$$
$$\int \frac{\ln(1+y^2)}{y^2}dy = 2J\Big(\ln(1+iy)\Big) - \frac{\ln(1+y^2)}{y} - \pi$$
$$\textit{And,}$$
$$Li_2(-y^2) = \frac{-2\ln(1+y^2)}{y}$$
$$I = 4\int_0^\infty \left(2J\Big(\ln(1+iy)\Big) - \frac{\ln(1+y^2)}{y} - \pi\right)\frac{\ln(1+y^2)}{y}dy$$
$$I = \frac{-3\pi}{3}(\pi^2 + 24\ln 2)$$

□

2. Evaluate:

$$I = \int_{\frac{1}{2}}^1 \frac{\ln^3 x \ln(1-x)}{x}dx$$

Proof.

$$\Rightarrow -\int_{\frac{1}{2}}^1 \sum_{n=1}^\infty \frac{x^{n-1}}{n}\ln^3 x dx$$
$$\Rightarrow -\sum_{n=1}^\infty \frac{1}{n}\int_{\frac{1}{2}}^1 x^{n-1}\ln^3 x dx$$
$$\Rightarrow -\sum_{n=1}^\infty \frac{1}{n}\frac{\partial^3}{\partial n^3}\left[\int_{\frac{1}{2}}^1 x^{n-1}dx\right]$$
$$\Rightarrow -\sum_{n=1}^\infty \frac{1}{n}\left[\frac{x^n\ln^3 x}{n} - 3\frac{x^n\ln^2 x}{n^2} + 6\frac{x^n\ln x}{n^3} - 6\frac{x^n}{n^4}\right]_{\frac{1}{2}}^1$$

$$\Rightarrow \Big[6Li_5(x) - 6Li_4(x)\ln x + 3Li_3(x)\ln^2 x - Li_2(x)\ln^3 x\Big]_{\frac{1}{2}}^{1}$$

$$I \Rightarrow \frac{\pi^2}{6}\ln^3 2 - \frac{21}{8}\zeta(3)\ln^2 2 - 6Li_4\Big(\frac{1}{2}\Big)\ln 2 - 6Li_5\Big(\frac{1}{2}\Big) + 6\zeta(5)$$

□

3. Evaluate:

$$\boldsymbol{S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+z)^2}}$$

Proof.

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{(6n+z)^2} - \sum_{n=0}^{\infty} \frac{1}{(6n+z+z+2)^2}$$

$$\Rightarrow \frac{1}{36}\sum_{n=0}^{\infty} \frac{1}{(n+\frac{z}{6})^2} - \frac{1}{36}\sum_{n=0}^{\infty} \frac{1}{(n+\frac{z}{6}+\frac{1}{3})^2}$$

$$\Rightarrow \frac{1}{36}\psi'\Big(\frac{z}{6}\Big) - \frac{1}{36}\psi'\Big(\frac{z}{6}+\frac{1}{3}\Big)$$

$$S \Rightarrow \frac{1}{36}\Bigg[\psi'\Big(\frac{z}{6}\Big) - \psi'\Big(\frac{z}{6}+\frac{1}{3}\Big)\Bigg]$$

□

4. Evaluate:

$$\boldsymbol{I = \int_0^1 \frac{\ln^3 x}{2-x}dx}$$

Proof.

$$Let\ x = e^{-t}$$

$$dx = -e^{-t}dt$$

$$I = \frac{1}{2}\int_{\infty}^{0} \frac{-t^3(-e^{-t})}{1-\frac{e^{-t}}{2}}dt$$

$$\Rightarrow -\frac{1}{2}\int_0^{\infty} t^3 e^{-t}\sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^n e^{-nt}dt$$

$$\Rightarrow -\frac{1}{2}\sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^n \int_0^{\infty} t^3 e^{-(n+1)t}dt$$

$$\Rightarrow -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})^n}{(n+1)^4}\int_0^{\infty} t^3 e^{-t}dt$$

$$\Rightarrow -6\sum_{n=0}^{\infty}\frac{(\frac{1}{2})^n}{n^4}$$

$$I = -6Li_4\Big(\frac{1}{2}\Big)$$

□

5. Evaluate:

$$\boldsymbol{I = \int_0^1 \frac{\zeta(3) - Li_3(x)}{1-x} dx}$$

Proof.

$$\Rightarrow -\Big[-\zeta(3) - Li_3(x)\ln(1-x)\Big]_0^1 - \int_0^1 \frac{Li_2(x)\ln(1-x)}{x} dx$$

$$\Rightarrow -\int_0^1 \frac{Li_2(x)\ln(1-x)}{x} dx$$

$$\Rightarrow \int_0^1 Li_2(x) d\Big(Li_2(x)\Big) dx$$

$$\Rightarrow \left[\frac{(Li_2(x))^2}{2}\right]_0^1$$

$$\Rightarrow \frac{\zeta^2(2)}{2}$$

$$I = \frac{\pi^4}{72}$$

□

6. Prove that:

$$\boldsymbol{\frac{\zeta'(x)}{x\zeta(\ln x)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\ln x}}}$$

where $\boldsymbol{\Lambda(x)}$ denotes von mangoldt function.

Proof.

As we know,

$$\zeta(s) = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1}$$

$$\ln\Big(\zeta(s)\Big) = -\sum_p \ln\Big(1 - \frac{1}{p^s}\Big)$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\ln(p)}{p^s}\Big(1 - \frac{1}{p^s}\Big)^{-1}$$

$$\Rightarrow -\sum_p \frac{\ln(p)}{p^s} \sum_{k\ge 1} \frac{1}{p^{sk}}$$

$$\Rightarrow -\sum_{n\ge 1} \frac{\Lambda(n)}{n^s}$$

Put $s = \ln x$, we get,

$$\frac{\zeta'(\ln x)}{x\zeta(\ln x)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\ln x}}$$

□

7. Evaluate:

$$\boldsymbol{I=\int_0^1\left(Li_2\Big(-\frac{1}{x^2}\Big)\right)^2dx}$$

Proof.

As we know,

(1) $$Li_2(u)=\frac{u}{\Gamma(2)}\int_0^\infty\frac{t}{e^t-u}dt$$

Then the integral can be written as,

$$I=\int_0^\infty\int_0^\infty\int_0^\infty\frac{y_1y_2}{(x^2e^{y_1}+1)(x^2e^{y_2}+1)}dy_1dy_2dx$$

$$\Rightarrow 8\pi\int_0^\infty\frac{y_1y_2}{(e^{y_1}+e^{y_2})}dy_1dy_2$$

Using equation 1, we get,

$$\Rightarrow 8\pi\int_0^\infty y_2e^{-y_2}Li_2\Big(-e^{-y_2}\Big)dy_2$$

Let $y_2\to y$

Then apply I.B.P, we finally get,

$$I=\frac{4\pi^3}{3}+32\pi\ln 2$$

□

8. Evaluate:

$$\boldsymbol{I=-2\int_0^1\frac{\ln^3 x}{1-x^2}\ln\Big(\frac{1-x}{1+x}\Big)dx}$$

Proof.

$$\Rightarrow -2\sum_{n=1}^\infty\Big(2H_{2n}-H_n\Big)\int_0^1x^{2n-1}(-\ln x)^3dx$$

$$\Rightarrow -12\sum_{n=1}^\infty\frac{2H_{2n}-H_n}{(2n)^4}$$

$$\Rightarrow -12\sum_{n=1}^\infty\frac{((-1)^{2n}+1)H_{2n}}{(2n)^4}-\frac{3}{8}\sum_{n=1}^\infty\frac{H_n}{n^4}$$

$$\Rightarrow (6\times -2)\sum_{n=1}^\infty\frac{((-1)^n+1)H_n}{n^4}-\frac{3}{8}\sum_{n=1}^\infty\frac{H_n}{n^4}$$

$$\Rightarrow \frac{45}{8}\sum_{n=1}^\infty\frac{H_n}{n^4}+6\sum_{n=1}^\infty\frac{(-1)^nH_n}{n^4}$$

$$\Rightarrow \frac{45}{8}[3\zeta(5)-\zeta(2)\zeta(3)\Big]+6\Big[-\frac{59}{32}\zeta(5)+\frac{1}{2}\zeta(2)\zeta(3)\Big]$$

$$I=\frac{21}{4}\zeta(2)\zeta(3)-\frac{93}{8}\zeta(5)$$

□

9. Evaluate:

$$\boldsymbol{S = \sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n}{n^2}}$$

Proof.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)} H_{n-1}}{n^2} + \sum_{n=1}^{\infty} \frac{\sum_{n-1}^{(2)}}{n^3} + \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^4} + \sum_{n=1}^{\infty} \frac{1}{n^5}$$

(1)
$$\Rightarrow \sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)} H_{n-1}}{n^2} + \zeta(3,2) + \zeta(4,1) + \zeta(5)$$

$$Let\ A = \sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)} H_{n-1}}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \Big(\zeta_{n-1}(2)\Big)\Big(\zeta_{n-1}(1)\Big)$$

$$\Rightarrow \sum_{n=1}^{\infty} \Big[\Big(\zeta_{n-1}(2,1)\Big) + \Big(\zeta_{n-1}(1,2)\Big) + \Big(\zeta_{n-1}(3)\Big)\Big]$$

$$\Rightarrow \zeta(2,2,1) + \zeta(2,1,2) + \zeta(2,3)$$

$$A = 2\zeta(2,3) + \zeta(3,2)$$

Now put the value in equation 1, we get,

$$S = 2\zeta(2,3) + \zeta(3,2) + \zeta(3,2) + \zeta(5) - \zeta(3,2) - \zeta(2,3) + \zeta(5)$$

$$\Rightarrow \zeta(2,3) + \zeta(3,2) + 2\zeta(5)$$

$$S = \zeta(2)\zeta(3) + \zeta(5)$$

□

10. Evaluate:

$$\boldsymbol{S = \sum_{n=2}^{\infty} \frac{1}{n(n+1)^2(n+2)}}$$

Proof.

$$\Rightarrow \frac{1}{2} \int_0^1 \left(\sum_{n=2}^{\infty} \frac{x^{n-1}}{n+1} \right) (1-x)^2 dx$$

$$\Rightarrow -\frac{1}{2} \int_0^1 \left(x + \frac{x^2}{2} + \ln(1-x) \right) \frac{(1-x)^2}{x^2} dx$$

$$\Rightarrow -\frac{1}{4} \int_0^1 \frac{(2+x)(1-x)^2}{x} dx - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{x}\right)^2 \ln(1-x) dx$$

$$\Rightarrow -\frac{1}{4} \left[\frac{x^3}{3} - 3x + 2\ln x \right]_0^1 - \frac{1}{2} \Big[(x-1)\ln(1-x) - x \Big]_0^1 + \Big[-Li_2(x) \Big]_0^1 - \frac{1}{2} \left[\frac{(x-1)\ln(1-x) - x\ln x}{x} \right]_0^1$$

$$S = \frac{5}{3} - \zeta(2)$$

□

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