RMM - Cyclic Inequalities Marathon 1 - 100

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

RMM

CYCLIC INEQUALITIES

MARATHON

1 – 100

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Proposed by

Daniel Sitaru – Romania, George Apostolopoulos - Messolonghi - Greece

Nguyen Viet Hung – Hanoi – Vietnam , Babis Stergioiu – Greece

Erdene Natsagdorj – Ulanbaatar-Mongolia

Dorin Marghidanu – Romania, Vaggelis Stamatiadis-Greece

Le Viet Hung –Hai Lang-Vietnam, Hoang Le Nhat Tung – Hanoi – Vietnam

Anish Ray-Santaragachi-India, Marin Chirciu – Romania

Nguyen Phuc Tang - Dong Thap – Vietnam , Mihalcea Andrei Ștefan – Romania, Adil Abdullayev – Baku – Azerbaidian

V.Lokot – Russia, S. Phenicheva – Russia, Pham Quoc Sang-Ho Chi Minh – Vietnam , Do Quoc Chinh-Ho Chi Minh – Vietnam

Solutions by

Daniel Sitaru *– Romania,* ***Henry Ricardo*** *- New York – USA*

Kunihiko Chikaya *– Tokyo – Japan,* ***Imad Zak*** *– Saida – Lebanon*

Fotini Kaldi *– Greece,* ***Do Huu Duc Thinh-Ho Chi Minh*** *– Vietnam*

Redwane El Mellass*-Casablanca-Morroco,* ***Sk Rejuan****-West Bengal-India*

Myagmarsuren Yadamsuren *– Mongolia ,* ***Safal Das Biswas*** *– Chinsurah – India*

Kevin Soto Palacios *– Huarmey-Peru ,* ***Nguyen Viet Hung*** *– Hanoi – Vietnam*

Soumitra Mandal*-Chandar Nagore-India,* ***Abhay Chandra*** *– India*

Marian Dincă *– Romania ,* ***Ravi Prakash*** *- New Delhi – India*

Seyran Ibrahimov *– Maasilli – Azerbaidian*

Abdallah El Farisi*-Bechar-Algerie,* ***Anh Tai Tran*** *– Hanoi – Vietnam*

Le Viet Hung*- Quang Tri – Vietnam ,* ***Nirapada Pal****-India,*

Soumava Chakraborty *– Kolkata – India,* ***Anas Adlany****-El Jadida-Morroco,*

Phan Loc So'n*-Quy - Nhon City – Vietnam,* ***Saptak Bhattacharya****-Kolkata-India,*

Nguyen Minh Triet*-Quang Ngai-Vietnam,*

Mihalcea Andrei Stefan*-Romania,* ***Soumitra Mandal****-Chandar Nagore-India*

Hoang Le Nhat Tung *– Hanoi – Vietnam,*

Khung Long Xanh *- Da Nang – Vietnam,* ***Do Quoc Chinh-Ho Chi Minh*** *– Vietnam*

Nguyen Phuc Tang *- Dong Thap – Vietnam*

Aditya Narayan Sharma *– Kanchrapara – India,*

Richdad Phuc *– Hanoi – Vietnam,* ***Christos Eythimiou*** *– Greece*

Pham Quy –Quang Ngai*- Vietnam,* ***Hamza Mahmood****-Lahore-Pakistan*

Sanong Huayrerai-Nakon Pathom *– Thailand*

1.

$$n\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 \geq (n-1)\left(\sqrt[n]{\prod_{k=1}^{n} a_k}\right)^2 + \left(\sqrt{\frac{1}{n}\sum_{k=1}^{n} a_k^2}\right)^2, a_k > 0, n \geq 3, n \in \mathbb{N}$$

Proposed by V.Lokot, S. Phenicheva-Russia

Solution by Ravi Prakash-New Delhi-India

$$\left(\sum_{k=1}^{n} a_k\right)^2 = \sum_{k=1}^{n} a_k^2 + \underbrace{\sum_{j\neq k} a_j a_k}_{n(n-1)\ terms}$$

$$\left[Each\ a_i\ occurs\ exactly\ 2(n-1) times\right]$$

$$\geq \sum_{k=1}^{n} a_k^2 + n(n-1)\left[\prod_{k=1}^{n} a_k^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$[AM \geq GM]$$

$$\Rightarrow \frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2 \geq \frac{1}{n}\sum_{k=1}^{n} a_k^2 + (n-1)\left(\prod_{k=1}^{n} a_k^2\right)^{\frac{1}{n}}$$

$$\Rightarrow n\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 \geq (n-1)\left(\sqrt[n]{\prod_{k=1}^{n} a_k}\right)^2 + \left(\sqrt{\frac{1}{n}\sum_{k=1}^{n} a_k^2}\right)^2$$

2. If $x, y, z, t, a, b, c \in (0, \infty), xyzt = a^4$ then:

$$3a\sum \frac{x^b + y^c}{y + z + t} \geq 4(a^b + a^c)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Daniel Sitaru-Romania

$$x+y+z+t\ge 4\sqrt[4]{xyzt}=4\sqrt[4]{a^4}=4a \textit{ (AM-GM)}$$

$$\sum\frac{x^b}{y+z+t}\ge\frac{(\sum x)^b}{4^{b-2}\cdot 3(\sum x)}=\frac{(\sum x)^{b-1}}{3\cdot 4^{b-2}}\ge$$

$$\ge\frac{(4a)^{b-1}}{3\cdot 4^{b-2}}=\frac{4^{b-1}\cdot a^{b-1}}{3\cdot 4^{b-2}}=\frac{4\cdot a^{b-1}}{3}\quad \textit{(1)}$$

$$\sum\frac{y^c}{y+z+t}\ge\frac{(\sum x)^c}{4^{c-2}\cdot 3(\sum x)}=\frac{(\sum x)^{4-1}}{4^{c-2}\cdot 3}\ge$$

$$\ge\frac{(4a)^{c-1}}{3\cdot 4^{c-2}}=\frac{4^{c-1}\cdot a^{c-1}}{3\cdot 4^{c-2}}=\frac{4\cdot a^{c-1}}{3}\quad \textit{(2)}$$

By adding (1); (2):

$$\sum\frac{x^a+y^c}{y+z+t}\ge\frac{4}{3}\left(a^{b-1}+a^{c-1}\right)$$

Solution 2 by Myagmarsuren Yadamsuren – Mongolia

$$x,y,z,t,a,b\in(0,\infty)\quad xyzt=a^4$$

Prove that: $LHS=3a\cdot\sum\frac{x^b+y^c}{y+z+t}\ge 4\cdot\left(a^b+a^c\right)$

$$\sum\frac{x^b+y^c}{y+z+t}\overset{Chebyshev}{\geq}\frac{1}{4}\cdot\left(\sum x^b+\sum x^c\right)\cdot\left(\sum\frac{1}{x+y+z}\right)$$

$$\overset{Cauchy-Schwarz}{\geq}\frac{1}{4}\cdot\left(\sum x^b+\sum x^c\right)\cdot\frac{4^2}{3\cdot(x+y+z+t)}=$$

$$=\frac{4}{3}\cdot\left(\frac{\sum x^b+\sum x^c}{x+y+z+t}\right)\overset{Chebyshev}{\geq}$$

$$\ge\frac{4}{3}\cdot\frac{1}{4}\cdot\frac{(x+y+z+t)\cdot\left(\sum x^{b-1}+\sum x^{c-1}\right)}{x+y+z+t}=$$

$$=\frac{1}{3}\cdot\left(\sum x^{b-1}+\sum x^{c-1}\right)\overset{Cauchy}{\geq}\frac{1}{3}\cdot 4\left[\left(\sqrt[4]{(xyzt)}\right)^{b-1}+\left(\sqrt[4]{xyzt}\right)^{c-1}\right]=$$

$$=\frac{4}{3}\cdot\left(a^{b-1}+a^{c-1}\right)$$

$$LHS=3\cdot a\cdot\sum\frac{x^b+y^c}{y+z+t}\geq 3a\cdot\frac{4}{3}\cdot\left(a^{b-1}+a^{c-1}\right)=4\cdot\left(a^b+a^c\right)$$

3. If a,b,c be positive real number and $k\geq 2$ then

$$\frac{a+kb}{ka+b}+\frac{b+kc}{kb+c}+\frac{c+ka}{kc+a}\geq 3$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

Solution by Christos Eythimiou-Greece

$$a,b,c>0\wedge k\geq 2$$

$$\frac{a+kb}{ka+b}+\frac{b+kc}{kb+c}+\frac{c+ka}{kc+a}\geq 3\Leftarrow$$

$$\frac{(k+1)(a+b)-(ka+b)}{ka+b}+\frac{(k+1)(b+c)-(kb+c)}{kb+c}+\frac{(k+1)(c+a)-(kc+a)}{kc+a}\geq 3\Leftarrow$$

$$(k+1)\left(\frac{a+b}{ka+b}+\frac{b+c}{kb+c}+\frac{c+a}{kc+a}\right)-3\geq 3\Leftarrow$$

$$\frac{a+b}{ka+b}+\frac{b+c}{kb+c}+\frac{c+a}{kc+a}\geq\frac{6}{k+1}\Leftarrow$$

$$\frac{ka+b-(k-1)a}{ka+b}+\frac{kb+c-(k-1)b}{kb+c}+\frac{kc+a-(k-1)c}{kc+a}\geq\frac{6}{k+1}\Leftarrow$$

$$3-(k-1)\left(\frac{a}{ka+b}+\frac{b}{kb+c}+\frac{c}{kc+a}\right)\geq\frac{6}{k+1}\Leftarrow$$

$$\frac{3}{k+1}\geq\frac{a}{ka+b}+\frac{b}{kb+c}+\frac{c}{kc+a}\Leftarrow$$

$$\frac{3}{k+1}\geq\frac{1}{k+\frac{b}{a}}+\frac{1}{k+\frac{c}{b}}+\frac{1}{k+\frac{a}{c}}\Leftarrow$$

$$3\left(k+\frac{b}{a}\right)\left(k+\frac{c}{b}\right)\left(k+\frac{a}{c}\right)\geq(k+1)\left(\left(k+\frac{c}{b}\right)\left(k+\frac{a}{c}\right)+\left(k+\frac{b}{a}\right)\left(k+\frac{a}{c}\right)+\left(k+\frac{b}{a}\right)\left(k+\frac{c}{b}\right)\right)\Leftarrow$$

$$3k^3+3k^2\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)+3k\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\geq 3k^2(k+1)+2k(k+1)\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)+(k+1)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{b}\right)\Leftarrow$$

$$k(k-2)\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)+(2k-1)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-3k^2+3\geq 0\Leftarrow$$

$$3k(k-2)\sqrt[3]{\frac{a}{c}\cdot\frac{b}{a}\cdot\frac{c}{b}}+3(2k-1)\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}-3k^2+3\geq 0\Leftarrow$$

$$0\geq 0$$

4. If $a,b,c\in(0,\infty)$ then:

$$\frac{a^2b^2}{a^5+b^5}+\frac{b^2c^2}{b^5+c^5}+\frac{c^2a^2}{c^5+a^5}\leq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

Si: $a,b,c\in<0,\infty>$**. Probar que:**

$$\frac{a^2b^2}{a^5+b^5}+\frac{b^2c^2}{b^5+c^5}+\frac{c^2a^2}{c^5+a^5}\leq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

La desigualdad es equivalente:

$$\frac{1}{a+b}-\frac{a^2b^2}{a^5+b^5}+\frac{1}{b+c}-\frac{b^2c^2}{b^5+c^5}+\frac{1}{a+c}-\frac{a^2c^2}{a^5+c^5}\geq 0$$

Tener presente lo siguiente:

$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)\Leftrightarrow\text{(Cocientes Notables)}$$

Sea: $A=\frac{1}{a+b}-\frac{a^2b^2}{a^5+b^5}=\frac{a^5+b^5-(a+b)(a^2b^2)}{(a^5+b^5)(a+b)}$

$$A=\frac{(a+b)\left(a^4+b^4-ab(a^2+b^2)\right)}{(a^5+b^5)(a+b)}=\frac{a^4+b^4-ab(a^2+b^2)}{a^5+b^5}$$

Desde que: $a,b,c \in <0,\infty>$. **Por:** $MA \geq MG$

$$a^4+a^4+a^4+b^4 \geq 4a^3b \wedge b^4+b^4+b^4+b^4 \geq 4b^3a$$

Sumando: $a^4+b^4 \geq ab(a^2+b^2) \rightarrow a^4+b^4-ab(a^2+b^2) \geq 0$

Por la tanto: $A = \frac{a^4+b^4-ab(a^2+b^2)}{a^5+b^5} \geq 0$

$$\rightarrow \sum \frac{1}{a+b} - \sum \frac{a^2b^2}{a^5+b^5} \geq 0$$

5. If $a,b,c \in (0,\infty)$ then:

$$a^8b^8+b^8c^8+c^8a^8 \geq a^5b^5c^5\sqrt[4]{27(a^4+b^4+c^4)}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

Si: $a,b,c \in <0,\infty>$. **Probar que:**

$$a^8b^8+b^8c^8+c^8a^8 \geq a^5b^5c^5\sqrt[4]{27(a^4+b^4+c^4)}$$

Desde que: $a,b,c \in <0,\infty>$

$$\Rightarrow a^8b^8+b^8c^8+c^8a^8 \geq a^4b^4c^4(a^4+b^4+c^4)$$

Por lao cual nos falta probar que:

$$a^4b^4c^4(a^4+b^4+c^4) \geq a^5b^5c^5\sqrt[4]{27(a^4+b^4+c^4)}$$

$$\Rightarrow a^4+b^4+c^4 \geq abc\sqrt[4]{27(a^4+b^4+c^4)}$$

$$\Rightarrow (a^4+b^4+c^4)^4 \geq (abc)^4 \cdot 27(a^4+b^4+c^4)$$

$\Rightarrow (a^4+b^4+c^4)^3 \geq 27(abc)^4 \rightarrow$ **Válido por: (MA≥MG).**

Por la tanto: $a^8b^8+b^8c^8+c^8a^8 \geq a^5b^5c^5\sqrt[4]{27(a^4+b^4+c^4)}$

La igualdad se alcanza cuando: $a=b=c$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

6. Let a, b, c be positive real numbers such that:

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1.$$

Prove that

$$(a-1)(b-1)(c-1)\leq 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sean: a, b, c **números reales positivos tal que :** $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1$ **(A)**

Probar que: $(a-1)(b-1)(c-1)\leq abc$ **(B). Sea:**

$a=\frac{x+y}{z}>0, b=\frac{z+x}{y}>0, c=\frac{y+z}{x}>0$. **Reemplazando en (A)**

$$\frac{1}{1+\frac{x+y}{z}}+\frac{1}{1+\frac{z+x}{y}}+\frac{1}{1+\frac{y+z}{x}}=\frac{z}{x+y+z}+\frac{y}{x+y+z}+\frac{x}{x+y+z}=1$$

Por la tanto (B) es equivalente:

$$(a-1)(b-1)(c-1)=\left(\frac{x+y}{z}-1\right)\left(\frac{z+x}{y}-1\right)\left(\frac{y+z}{x}-1\right)\leq 1$$

$$\Rightarrow (x+y-z)(z+x-y)(y+z-x)\leq xyz$$

$$\rightarrow (x+y-z)(z+x-y)(y+z-x)=$$

$$=(x^2-(y-z)^2)(y+z-x)=(x^2-y^2-z^2+2yz)(y+z-x)$$

$$\rightarrow (x^2-y^2-z^2+2yz)(y+z-x)=x^2y+x^2z-x^3-y^3-y^2z+y^2x-$$

$$-z^2y-z^3+z^2x+2y^2z+2yz^2-2xyz$$

$$\Rightarrow (x+y-z)(z+x-y)(y+z-x)=-x^3-y^3-z^3+$$

$$+xy(x+y)+yz(y+z)+zx(z+x)-2xyz\leq xyz$$

$$\Rightarrow x^3+y^3+z^3+3xyz\geq xy(x+y)+yz(y+z)+zx(z+x)\rightarrow$$

→(Válido por desigualdad de Schur)

7. If $a, b, c \geq 1$, then prove that:

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\geq\frac{1}{1+\sqrt[4]{ab^3}}+\frac{1}{1+\sqrt[4]{bc^3}}+\frac{1}{1+\sqrt[4]{ca^3}}$$

Proposed by Erdene Natsagdorj - Ulanbaatar Mongolia

Solution by Hung Nguyen Viet – Hanoi – Vietnam

We have known that if $x\geq 0, y\geq 0$ and $xy\geq 1$ then $\frac{1}{1+x}+\frac{1}{1+y}\geq\frac{2}{1+\sqrt{xy}}$

Applying this result step by step we obtain

$$\frac{1}{1+a}+\frac{3}{1+b}=\frac{1}{1+a}+\frac{1}{1+b}+\frac{2}{1+b}\geq\frac{2}{1+\sqrt{ab}}+\frac{2}{1+b}$$

$$\geq\frac{4}{1+\sqrt{\sqrt{ab}\cdot b}}=\frac{4}{1+\sqrt[4]{ab^3}}$$

Similarly, $\frac{1}{1+b}+\frac{3}{1+c}\geq\frac{4}{1+\sqrt[4]{bc^3}}, \frac{1}{1+c}+\frac{3}{1+a}\geq\frac{4}{1+\sqrt[4]{ca^3}}$

Adding up these three relations we get the desired inequality.

8. Prove the inequality holds for all positive real numbers a, b, c

$$(b+c)a^2+(c+a)b^2+(a+b)c^2\geq 3abc+a\sqrt{\frac{b^4+c^4}{2}}+b\sqrt{\frac{c^4+a^4}{2}}+c\sqrt{\frac{a^4+b^4}{2}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c números R^+. Probar la siguiente desigualdad:

$$a^2(b+c)+b^2(c+a)+c^2(a+b)\geq 3abc+a\sqrt{\frac{b^4+c^4}{2}}+b\sqrt{\frac{c^4+a^4}{2}}+c\sqrt{\frac{a^4+b^4}{2}}$$

Partimos de la siguiente desigualdad:

$$(a-b)^4\geq 0\rightarrow a^4+b^4+6a^2b^2-4b^3a-4a^3b\geq 0$$

$$\Rightarrow 2(a^4+b^4+a^2b^2-2a^3b-2b^3a+2a^2b^2)\ge a^4+b^4$$

$$\Rightarrow 2(a^2-ab+b^2)^2\ge a^4+b^4 \to a^2-ab+b^2\ge\sqrt{\frac{a^4+b^4}{2}} \text{ ... (A)}$$

Análogamente para los siguientes términos:

$$b^2-bc+c^2\ge\sqrt{\frac{b^4+c^4}{2}} \text{ ... (B)} \wedge c^2-ca+a^2\ge\sqrt{\frac{c^4+a^4}{2}} \text{ ... (C)}$$

En consecuencia, la desigualdad inicial es equivalente:

$$3abc+a\sqrt{\frac{b^4+c^4}{2}}+b\sqrt{\frac{c^4+a^4}{2}}+c\sqrt{\frac{a^4+b^4}{2}}\le$$

$$\le 3abc+a(b^2-bc+c^2)+b(c^2-ca+a^2)+c(a^2-ab+b^2)$$

$$\Rightarrow 3abc+a\sqrt{\frac{b^4+c^4}{2}}+b\sqrt{\frac{c^4+a^4}{2}}+c\sqrt{\frac{a^4+b^4}{2}}\le$$

$$\le a^2(b+c)+b^2(c+a)+c^2(a+b) \text{ ... (LQQD)}$$

9. If $a,b,c,x,y,z\ge 0, a+b+c=1$ then:

$$\frac{1}{1+x^ay^bz^c}+\frac{1}{1+x^by^cz^a}+\frac{1}{1+x^cy^az^b}\ge\frac{9}{3+x+y+z}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{1}{1+x^ay^bz^c}\ge\frac{(1+1+1)^2}{3+\sum_{cyc}x^ay^bz^c}\ge$$

(Bergstrom's inequality)

$$\ge\frac{(1+1+1)^2}{3+\sum_{cyc}\left(\frac{ax+by+cz}{a+b+c}\right)^{a+b+c}}=$$

$$= \frac{9}{3 + \sum_{cyc}(ax + by + cz)} = \frac{9}{3 + (x + y + z)(a + b + c)} =$$

$$= \frac{9}{3+x+y+z} \text{ (Proved)}$$

10. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sean a, b, c números $\mathbb{R}^+$ tal que: $ab + bc + ac = 3$. Probar que:

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ac + a^2} + \frac{c^3}{a^2 - ba + b^2} \geq 3$$

La desigualdad es equivalente:

$$\frac{a^3(b + c)}{b^3 + c^3} + \frac{b^3(a + c)}{a^3 + c^3} + \frac{c^3(a + b)}{a^3 + b^3} \geq 3$$

Para todos los $\mathbb{R}^+$: "a, b, c, x, y, z", se cumple la siguiente desigualdad:

$$(b + c)x + (c + a)y + (a + b)z \geq 2\sqrt{(ab + bc + ac)(xy + yz + zx)}$$

(Demostrado anteriormente)

Sean: $x = \frac{a^3}{b^3+c^3}, y = \frac{b^3}{a^3+c^3}, z = \frac{c^3}{a^3+b^3}$

Asimismo:

$$xy + yz + zx = \frac{a^3}{b^3 + c^3} \cdot \frac{b^3}{a^3 + c^3} + \frac{b^3}{a^3 + c^3} \cdot \frac{c^3}{a^3 + b^3} + \frac{c^3}{a^3 + b^3} \cdot \frac{a^3}{b^3 + c^3}$$

Sean: $m = a^3, n = b^3, p = c^3$

$$\to xy + yz + zx = \frac{mn}{(p + n)(p + m)} + \frac{np}{(m + p)(m + n)} + \frac{pm}{(n + m)(n + p)} \geq \frac{3}{4}$$

(Demostrado anteriormente)

Por la tanto:

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ac+a^2}+\frac{c^3}{a^2-ba+b^2}\geq 2\sqrt{(3)(xy+yz+zx)}\geq$$

$$\geq 2\sqrt{\frac{9}{4}}=3$$

11. Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that:

$$(1+ab+bc+ca)\left(\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}\right)\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Proposed by Nguyen Viet Hung – Hanoi - Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c números $\mathbb{R}^+$, de tal manera que: $a+b+c=1$. Probar que:

$$(1+ab+bc+ac)\left(\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}\right)\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

$$a+bc=(1-b-c)+bc=(1-b)(1-c)=(a+c)(a+b)$$

$$b+ac=(1-a-c)+ac=(1-c)(1-a)=(a+b)(c+b)$$

$$c+ab=(1-a-b)+ab=(1-a)(1-b)=(b+c)(a+c)$$

La desigualdad se puede expresar de la siguiente manera:

$$\big((a+bc)+(b+ca)+(c+ab)\big)\left(\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}\right)\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

$$\Rightarrow\left(1+\frac{a+bc}{b+ca}+\frac{a+bc}{c+ab}\right)+\left(\frac{b+ca}{a+bc}+1+\frac{b+ca}{c+ab}\right)+\left(\frac{c+ab}{a+bc}+\frac{c+ab}{b+ca}+1\right)\geq$$

$$\geq\left(1+\frac{b+c}{a}\right)+\left(1+\frac{a+c}{b}\right)+\left(1+\frac{a+b}{c}\right)$$

$$\Rightarrow\left(\frac{a+c}{c+b}+\frac{a+b}{c+b}\right)+\left(\frac{b+c}{a+c}+\frac{a+b}{a+c}\right)+\left(\frac{b+c}{a+b}+\frac{a+c}{a+b}\right)\leq$$

$$\leq \frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}$$

$$\Rightarrow (b+c)\left(\frac{1}{a}-\frac{1}{a+c}\right)-(a+b)\frac{1}{a+c}+(a+c)\left(\frac{1}{b}-\frac{1}{a+b}\right)-(b+c)\frac{1}{a+b}+$$

$$+(a+b)\left(\frac{1}{c}-\frac{1}{c+b}\right)-(a+c)\frac{1}{c+b}\geq 0$$

$$\Rightarrow \left(\frac{b+c}{a}\right)\frac{c}{a+c}-(a+b)\frac{1}{a+c}+\left(\frac{a+c}{b}\right)\frac{a}{a+b}-(b+c)\frac{1}{a+b}+$$

$$+\left(\frac{a+b}{c}\right)\frac{b}{c+b}-(a+c)\frac{1}{c+b}\geq 0$$

$$\Rightarrow \left(\frac{1}{a+c}\right)\left(\frac{(b+c)c}{a}-(a+b)\right)+\left(\frac{1}{a+b}\right)\left(\frac{(a+c)a}{b}-(b+c)\right)+$$

$$+\left(\frac{1}{b+c}\right)\left(\frac{(a+b)b}{c}-(a+c)\right)\geq 0$$

$$\Rightarrow \frac{1}{(a+c)}\left(\frac{bc+c^2-a^2-ab}{a}\right)+\frac{1}{a+b}\left(\frac{ac+a^2-b^2-bc}{b}\right)+$$

$$+\frac{1}{b+c}\left(\frac{ab+b^2-c^2-ac}{c}\right)\geq 0$$

$$\Rightarrow \frac{(c+a)(c-a)+b(c-a)}{a(a+c)}+\frac{(a+b)(a-b)+c(a-b)}{b(a+b)}+$$

$$+\frac{(b+c)(b-c)+a(b-c)}{c(b+c)}\geq 0$$

$$\Rightarrow \frac{(c-a)(a+b+c)}{a(a+c)}+\frac{(a-b)(a+b+c)}{b(a+b)}+\frac{(b-c)(a+b+c)}{c(b+c)}\geq 0$$

$$\Rightarrow \left(\frac{c-a}{a+c}\right)\frac{1}{a}+\left(\frac{a-b}{a+b}\right)\frac{1}{b}+\left(\frac{b-c}{b+c}\right)\frac{1}{c}\geq 0$$

$$\Rightarrow \left(1-\frac{2a}{a+c}\right)\frac{1}{a}+\left(1-\frac{2b}{a+b}\right)\frac{1}{b}+\left(1-\frac{2c}{b+c}\right)\frac{1}{c}\geq 0$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge \frac{2}{a+c}+\frac{2}{a+b}+\frac{2}{b+c}\quad \text{(Lo cual probaremos)}$$

Desde que $a, b, c > 0$: Por: $MA \ge MG$

$$\frac{1}{a}+\frac{1}{b}\ge\frac{4}{a+b}\ \text{(A)};\ \frac{1}{b}+\frac{1}{c}\ge\frac{4}{b+c}\ \text{(B)};\ \frac{1}{a}+\frac{1}{c}\ge\frac{4}{a+c}\ \text{(C)}$$

Sumando: (A) + (B) + (C) $\rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge \frac{2}{a+c}+\frac{2}{a+b}+\frac{2}{b+c}$ **(LQQD)**

12. Let a, b, c be non-negative real numbers such that $a+b+c=3$.

Prove that:

$$\sqrt{a^3+1}+\sqrt{b^3+1}+\sqrt{c^3+1}\le\sqrt{6(a^2+b^2+c^2)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c números reales no negativos, de tal manera que: $a+b+c=3$.

Probar que: $\sqrt{a^3+1}+\sqrt{b^3+1}+\sqrt{c^3+1}\le\sqrt{6(a^2+b^2+c^2)}$

La desigualdad se puede expresar como:

$$\sqrt{(a+1)}\sqrt{a^2-a+1}+\sqrt{b+1}\sqrt{b^2-b+1}+\sqrt{c+1}\sqrt{c^2-c+1}\le\sqrt{6(a^2+b^2+c^2)}$$

Por la desigualdad de Cauchy Schwarz:

$$\left(\sqrt{(a+1)}\sqrt{a^2-a+1}+\sqrt{b+1}\sqrt{b^2-b+1}+\sqrt{c+1}\sqrt{c^2-c+1}\right)^2\le$$

$$\le(a+b+c+3)(a^2+b^2+c^2-(a+b+c)+3)$$

$$\left(\sqrt{(a+1)}\sqrt{a^2-a+1}+\sqrt{b+1}\sqrt{b^2-b+1}+\sqrt{c+1}\sqrt{c^2-c+1}\right)^2\le$$

$$\le 6(a^2+b^2+c^2)$$

$$\Rightarrow \sqrt{a^3+1}+\sqrt{b^3+1}+\sqrt{c^3+1}\le\sqrt{6(a^2+b^2+c^2)}\ \ldots \text{(LQQD)}$$

13. Prove that for all positive real numbers a, b, c :

$$\frac{(a+1)^2(b+1)^2}{ab+1}+\frac{(b+1)^2(c+1)^2}{bc+1}+\frac{(c+1)^2(a+1)^2}{ca+1}\ge 8(a+b+c)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los números $\mathbb{R}^+$ a, b, c la siguiente desigualdad:

$$\frac{(1+a)^2(1+b)^2}{ab+1}+\frac{(1+b)^2(1+c)^2}{bc+1}+\frac{(1+c^2)(1+a^2)}{ca+1}\ge 8(a+b+c)$$

Realizamos los siguientes cambios de variables:

$$p=a+b+c \wedge q=ab+bc+ac$$

Desde que: $a, b, c > 0$. Por la desigualdad de Bergstrom's:

$$\frac{(1+a)^2(1+b)^2}{ab+1}+\frac{(1+b)^2(1+c)^2}{bc+1}+\frac{(1+c)^2(1+a)^2}{ca+1}\ge$$

$\ge \frac{(\sum(1+a)(1+b))^2}{3+q}\ge 8p$. **Por lo cual nos queda demostrar que:**

$$\Rightarrow \left(\sum(1+a)(1+b)\right)^2\ge 8p(3+q)\Leftrightarrow(3+2p+q)^2\ge 24p+8pq$$

$$\Rightarrow 9+4p^2+q^2+12p+4pq+6q\ge 24p+8pq\Leftrightarrow$$

$$\Leftrightarrow 4p^2+q^2+9-4pq-12p+6q\ge 0$$

Pero ... $4p^2+q^2+9-4pq-12p+6q=(2p-q-3)^2\ge 0$ **... (LQQD)**

14. Prove that for all real numbers a, b, c:

$$\frac{|a+b+c|}{1+|a+b+c|}\le\frac{|a|}{1+|b|+|c|}+\frac{|b|}{1+|c|+|a|}+\frac{|c|}{1+|a|+|b|}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Abhay Chandra – India

For real numbers we have $|a| + |b| + |c| \geq |a + b + c|$ **which implies**

$$\frac{|a| + |b| + |c|}{|a| + |b| + |c| + 1} \geq \frac{|a + b + c|}{|a + b + c| + 1}$$

Hence we are required to prove the following for all non-negative real numbers

$$x = |a|, y = |b|, z = |c|$$

$$\sum \frac{x}{1 + y + z} \geq \frac{x + y + z}{x + y + z + 1}$$

But from Cauchy – Schwarz on LHS we have

$$LHS \geq \frac{(x + y + z)^2}{x + y + z + 2(xy + yz + zx)}$$

we are left to prove

$$(x + y + z)^2 \geq 2(xy + yz + zx) \Rightarrow x^2 + y^2 + z^2 \geq 0$$

which is obviously true. Equality at $x = y = z = 0$ **or** $a = b = c = 0$.

Solution 2 by Marian Dincă – Romania

$$\frac{|a|}{1 + |b| + |c|} + \frac{|b|}{1 + |c| + |a|} + \frac{|c|}{1 + |a| + |b|} =$$

$$= \left(\frac{|a|}{1 + |b| + |c|} + 1\right) + \left(\frac{|b|}{1 + |c| + |a|}\right) + \left(\frac{|c|}{1 + |a| + |b|} + 1\right) - 3 =$$

$$= (1 + |a| + |b| + |c|)\left(\frac{1}{1 + |b| + |c|} + \frac{1}{1 + |c| + |a|} + \frac{1}{1 + |a| + |b|}\right) - 3 \geq$$

$$\geq (1 + |a| + |b| + |c|)\left(\frac{9}{\sum_{ciclic} 1 + |a| + |b|}\right) - 3 \text{ use harmonic inequality}$$

$$= \frac{9 + 9(|a| + |b| + |c|)}{3 + 2(|a| + |b| + |c|)} - 3 = \frac{3(|a| + |b| + |c|)}{3 + 2(|a| + |b| + |c|)} = \frac{|a| + |b| + |c|}{1 + \frac{2}{3}(|a| + |b| + |c|)} \geq$$

$$\geq \frac{|a| + |b| + |c|}{1 + (|a| + |b| + |c|)} \geq \frac{|a + b + c|}{1 + |a + b + c|}$$

because the function $f(x) = \frac{x}{1+x}$, is increasing for $x \geq 0$ and

$|a| + |b| + |c| \geq |a + b + c|$. done!

15. If $a, b, c, d \in (0, 1)$, prove that:

$$\left(\frac{a+b}{2}\right)^{\frac{c+d}{2}} + \left(\frac{b+c}{2}\right)^{\frac{d+a}{2}} + \left(\frac{c+d}{2}\right)^{\frac{a+b}{2}} + \left(\frac{d+a}{2}\right)^{\frac{b+c}{2}} > 2$$

Proposed by Dorin Marghidanu – Romania

Solution by Marian Dincă – Romania

$$a, b, c, d \in (0, 1); \sum_{cyc} \left(\frac{a+b}{2}\right)^{\frac{c+d}{2}} > 2$$

use Lema by Dorin Marghidanu $x^y \geq \frac{x}{x+y}$, for $x, y \in (0, 1)$

$$\frac{a+b}{2} \in (0, 1), \frac{c+d}{2} \in (0, 1) \Rightarrow \left(\frac{a+b}{2}\right)^{\frac{c+d}{2}} \geq \frac{\frac{a+b}{2}}{\frac{a+b}{2} + \frac{c+d}{2}} = \frac{a+b}{a+b+c+d}$$

and similarly, we shall obtain: $\sum_{cyc} \left(\frac{a+b}{2}\right)^{\frac{c+d}{2}} \geq \sum_{cyc} \frac{a+b}{a+b+c+d} = 2$

16. If $a, b, c \in (0, \infty)$ then:

$$\sqrt[3]{\prod \left(1 + \frac{2}{a} + \frac{3}{a^2}\right)} \geq 1 + \frac{2}{\sqrt[6]{abc}} + \frac{3}{\sqrt[3]{(abc)^2}}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal - Chandar Nagore – India

Applying Holder's Inequality

$$\left(1+\frac{2}{a}+\frac{3}{a^2}\right)\left(1+\frac{2}{b}+\frac{3}{b^2}\right)\left(1+\frac{2}{c}+\frac{3}{c^2}\right)\geq\left(1+\sqrt[3]{\frac{2}{a}\cdot\frac{2}{b}\cdot\frac{2}{c}}+\sqrt[3]{\frac{3}{a^2}\cdot\frac{3}{b^2}\cdot\frac{3}{c^2}}\right)^3$$

$$\Leftrightarrow\sqrt[3]{\prod_{cyc}\left(1+\frac{2}{a}+\frac{3}{a^2}\right)}\geq 1+\frac{2}{\sqrt[3]{abc}}+\frac{3}{\sqrt[3]{(abc)^2}}$$

17. $a, b, c > 0 \wedge ab+bc+ca = a+b+c \Rightarrow \frac{1}{a^2+b+1}+\frac{1}{b^2+c+1}+\frac{1}{c^2+a+1}\leq 1$

Proposed by Vaggelis Stamatiadis-Greece

Solution by Nguyen Viet Hung – Hanoi – Vietnam

By Cauchy – Schwarz inequality we have

$$\sum_{cyc}\frac{1}{a^2+b+1}=\sum_{cyc}\frac{1+b+c^2}{(a^2+b+1)(1+b+c^2)}=\sum_{cyc}\frac{1+b+c^2}{(a+b+c)^2}=$$

$$=\frac{3+a+b+c+a^2+b^2+c^2}{(a+b+c)^2}$$

It suffices to show that $3+a+b+c+a^2+b^2+c^2\leq(a+b+c)^2$

or $3\leq a+b+c$ (since $a+b+c=ab+bc+ca$)

But this is true by $a+b+c=ab+bc+ca\leq\frac{1}{3}(a+b+c)^2$

The proof is completed.

18. If $a, b, c\in(0,\infty), a+b+c=1$ then:

$$\frac{a^3}{5b+7c}+\frac{b^3}{5c+7a}+\frac{c^3}{5a+7b}\geq\frac{1}{36}$$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren – Mongolia

$$I=\frac{a^3}{5b+7c}+\frac{b^3}{5c+7a}+\frac{c^3}{5a+7b}\geq\frac{1}{36}$$

$$I\overset{Chebyshev\ (3)}{\geq}\frac{1}{27}\cdot(a+b+c)^3\cdot\left(\frac{1}{5b+7c}+\frac{1}{5c+7a}+\frac{1}{5a+7b}\right)$$

$$=\frac{1}{27}\cdot\left(\frac{1}{5a+7b}+\frac{1}{5b+7c}+\frac{1}{5c+7a}\right)\overset{Cauchy-Schwarz}{\geq}$$

$$\geq\frac{1}{27}\cdot\frac{(1+1+1)^2}{12\cdot(a+b+c)}=\frac{1}{36}$$

19. If $a,b,c\in(0,\infty)$ then:

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}+\sqrt{\frac{a^2}{c}}+\sqrt{\frac{c^2}{a}}+\sqrt{\frac{b^2}{c}}+\sqrt{\frac{c^2}{b}}\geq 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Ravi Prakash - New Delhi – India

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}+\sqrt{\frac{b^2}{c}}+\sqrt{\frac{c^2}{b}}+\sqrt{\frac{c^2}{a}}+\sqrt{\frac{a^2}{c}}\geq 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}-\left(\sqrt{a}+\sqrt{b}\right)=\frac{1}{\sqrt{ab}}\{a\sqrt{a}+b\sqrt{b}-b\sqrt{a}-a\sqrt{b}\}=$$

$$=\frac{1}{\sqrt{ab}}\left((a-b)\left(\sqrt{a}-\sqrt{b}\right)\right)\geq 0$$

Thus, $\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}\geq\sqrt{a}+\sqrt{b}$ **(1). Similarly** $\sqrt{\frac{b^2}{c}}+\sqrt{\frac{c^2}{b}}\geq\sqrt{b}+\sqrt{c}$ **(2)**

and $\sqrt{\frac{c^2}{a}}+\sqrt{\frac{a^2}{c}}\geq\sqrt{a}+\sqrt{c}$ **(3)**

Adding (1), (2) and (3) we get the desired inequality.

Solution 2 by Seyran Ibrahimov – Maasilli – Azerbaidian

$$\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}}+\frac{a}{\sqrt{c}}+\frac{c}{\sqrt{a}}+\frac{b}{\sqrt{c}}+\frac{c}{\sqrt{b}}=\frac{b+c}{\sqrt{a}}+\frac{a+c}{\sqrt{b}}+\frac{a+b}{\sqrt{c}}$$

$$\left.\begin{array}{l} b+c\ge 2\sqrt{bc} \\ a+c\ge 2\sqrt{ac} \\ a+b\ge 2\sqrt{ab} \end{array}\right\} AM-GM$$

$$2\sqrt{\frac{bc}{a}}+2\sqrt{\frac{ac}{b}}\ge 4\sqrt{c};\ 2\sqrt{\frac{ac}{b}}+2\sqrt{\frac{ab}{c}}\ge 4\sqrt{a};\ 2\sqrt{\frac{bc}{a}}+2\sqrt{\frac{ab}{c}}\ge 4\sqrt{b}$$

$$LHS\ge 2\sqrt{\frac{bc}{a}}+2\sqrt{\frac{ac}{b}}+2\sqrt{\frac{ab}{c}}\ge 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

20. If $a,b,c\in(0,\infty)$ and $m,n\in\mathbb{N}^*$, prove that:

$$\frac{a^{m+n}}{b^m c^{n-1}}+\frac{b^{m+n}}{c^m a^{n-1}}+\frac{c^{m+n}}{a^m b^{n-1}}\ge a+b+c$$

Proposed by Dorin Mărghidanu – Romania

Solution by Soumitra Mandal - Chandar Nagore – India

Applying A.M. $\ge$ G.M. : $\frac{\frac{a^{m+n}}{b^m c^{n-1}}+mb+(n-1)c}{m+n}\ge a$, similarly

$\frac{\frac{b^{m+n}}{c^m a^{n-1}}+mc+(n-1)a}{m+n}\ge b$ and $\frac{\frac{c^{m+n}}{a^m b^{n-1}}+ma+(n-1)b}{m+n}\ge c$. Now,

$$\sum_{cycl}\frac{a^{m+n}}{b^m c^{n-1}}+m\left(\sum_{cycl}a\right)+(n-1)\left(\sum_{cycl}a\right)\ge(m+n)\sum_{cycl}a$$

so,

$$\sum_{cycl}\frac{a^{m+n}}{b^m c^{n-1}}\ge a+b+c$$

21. **If $a,b,c\in(-1,1)$ then:**

$$\frac{|a|+|b|}{1-c^2}+\frac{|b|+|c|}{1-a^2}+\frac{|c|+|a|}{1-b^2}\ge\frac{2|a|}{1-bc}+\frac{2|b|}{1-ca}+\frac{2|c|}{1-ab}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si: $a,b,c\in<-1,1>$

$$\frac{|a|+|b|}{1-c^2}+\frac{|b|+|c|}{1-a^2}+\frac{|c|+|a|}{1-b^2}\ge\frac{2|a|}{1-bc}+\frac{2|b|}{1-ac}+\frac{2|c|}{1-ab}$$

Por desigualdad de Cauchy:

$$\Rightarrow\frac{|a|}{1-c^2}+\frac{|a|}{1-b^2}=|a|\left(\frac{1}{1-c^2}+\frac{1}{1-b^2}\right)\ge|a|\left(\frac{4}{(2-b^2-c^2)}\right)\ge$$

$$\ge|a|\left(\frac{4}{2-2bc}\right)=\frac{2|a|}{1-bc}\ \dots\text{(I)}$$

$$\Rightarrow\frac{|b|}{1-c^2}+\frac{|b|}{1-a^2}\ge\frac{2|b|}{1-ac}\ \dots\text{(II)}\ \wedge\ \frac{|c|}{1-a^2}+\frac{|c|}{1-b^2}\ge\frac{2|c|}{1-ab}\ \dots\text{(III)}$$

Sumando ... (I)+(II)+(III):

$$\frac{|a|+|b|}{1-c^2}+\frac{|b|+|c|}{1-a^2}+\frac{|c|+|a|}{1-b^2}\ge\frac{2|a|}{1-bc}+\frac{2|b|}{1-ac}+\frac{2|c|}{1-ab}\ \dots\text{(LQQD)}$$

Solution 2 by Ravi Prakash - New Delhi – India

Rewrite the inequality as:

$$|a|\left(\frac{1}{1-b^2}+\frac{1}{1-c^2}\right)+|b|\left(\frac{1}{1-c^2}+\frac{1}{1-a^2}\right)+|c|\left(\frac{1}{1-a^2}+\frac{1}{1-b^2}\right)\ge$$

$$\ge\frac{2|a|}{1-bc}+\frac{2|b|}{1-ca}+\frac{2|c|}{1-ab}$$

For $-1<b,c<1$

$$\frac{1}{1-b^2}+\frac{1}{1-c^2}-\frac{2}{1-bc}=\left(\frac{1}{1-b^2}-\frac{1}{1-bc}\right)+\left(\frac{1}{1-c^2}-\frac{1}{1-bc}\right)=$$

$$=\frac{b(b-c)}{(1-b)^2(1-bc)}+\frac{c^2-bc}{(1-c^2)(1-bc)}=\frac{b-c}{1-bc}\left[\frac{b(1-c^2)-c(1-b^2)}{(1-b^2)(1-c^2)}\right]=$$

$=\frac{(b-c)^2}{(1-b^2)(1-c^2)}\ge 0$. Thus, $|a|\left(\frac{1}{1-b^2}+\frac{1}{1-c^2}\right)\ge\frac{2|a|}{1-bc}$

Similarly for the second and third expressions on both sides.

22. If $a,b,c>0$ then:

$$a^4c^2e^{\frac{a}{b}}+b^4a^2e^{\frac{b}{c}}+c^4b^2e^{\frac{c}{a}}\ge 3ea^2b^2c^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 Abdallah El Farisi-Bechar-Algerie

The function x^2e^x is convex for $x\ge 0$ then we have for all $a,b,c>0$.

$$\frac{1}{3}\left(\frac{a}{b}\right)^2e^{\frac{a}{b}}+\frac{1}{3}\left(\frac{b}{c}\right)^2e^{\frac{b}{c}}+\frac{1}{3}\left(\frac{c}{a}\right)^2e^{\frac{c}{b}}$$

$$\ge\left(\frac{1}{3}\frac{a}{b}+\frac{1}{3}\frac{b}{c}+\frac{1}{3}\frac{c}{a}\right)^2e^{\frac{1a}{3b}+\frac{1b}{3c}+\frac{1c}{3a}}\ge e.\ \text{ by MA-MG}$$

Solution 2 by Myagmarsuren Yadamsuren – Mongolia

$$a^4c^2e^{\frac{a}{b}}+b^4c^2e^{\frac{b}{c}}+c^4b^2c^{\frac{c}{a}}\overset{AM-GM}{\ge}$$

$$\ge 3\cdot\sqrt[3]{a^6\cdot b^6\cdot c^6\cdot e^{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}}=3a^2b^2c^2\cdot\sqrt[3]{e^{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}}\overset{AM-GM}{\ge}$$

$$\ge 3a^2b^2c^2\cdot\sqrt[3]{e^{3\cdot\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}}}=3e\cdot a^2b^2c^2$$

Solution 3 by Safal Das Biswas – Chindahar – India

$a^4c^2e^{\frac{a}{b}}+b^4a^2e^{\frac{b}{c}}+c^4b^2e^{\frac{c}{a}}=k$. **Then by A.M ≥ G.M we have**

$$k\ge 3\sqrt[3]{a^6b^6c^6e^{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}}$$

$$= 3a^2b^2c^2e^{\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}} \text{ again A.M} \geq \text{G.M } \frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \geq 1 \text{ then } e^{\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}} \geq e, \text{ thus}$$

$$k \geq 3ea^2b^2c^2$$

Solution 4 by Soumitra Moukherjee - Chandar Nagore – India

Applying A.M $\geq$ G.M,

$$\sum_{cyc} a^4c^2e^{\frac{a}{b}} \geq 3\sqrt[3]{\left(\prod_{cyc} a^4c^2\right)e^{\sum_{cyc}\frac{a}{b}}} = 3a^2b^2c^2e^{\frac{\sum_{cyc}\frac{a}{b}}{3}} \geq 3ea^2b^2c^2$$

23. In $\Delta\ ABC$ the following relationship holds:

$$\sum \frac{5a^2 + 8ab + 5b^2}{2c^2 + ab} \geq 4\left(3 + \sum \frac{a}{b+c}\right)$$

Proposed by Do Quoc Chinh-Ho Chi Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$3 + \sum \frac{a}{b+c} = \sum\left(1 + \frac{a}{b+c}\right) = 2s\sum \frac{1}{b+c} = 2s\frac{\sum a^2 + 3\sum ab}{(a+b)(b+c)(c+a)} =$$

$$= 2s\frac{4s^2 + \sum ab}{(a+b)(b+c)(c+a)} \rightarrow \text{(1). Let } x = s-a, y = s-b, z = s-c \therefore s = x+y+z$$

So, $a = y+z, b = z+x, c = x+y \quad (x,y,z > 0)$. ***Using this substitution and (1), given inequality transforms into:*** $\sum \frac{5(y+z)^2 + 5(z+x)^2 + 8(y+z)(z+x)}{2(x+y)^2 + (y+z)(z+x)} \geq$

$$\geq \frac{8\sum x}{\prod(2x+y+z)}\left(4\left(\sum x\right)^2 + \sum(y+z)(z+x)\right) = \frac{8\sum x}{\prod(2x+y+z)}\left(5\left(\sum x\right)^2 + \sum xy\right)$$

$$\Leftrightarrow \left[\sum_{cyc}\left[\begin{matrix}\{5(y+z)^2 + 5(z+x)^2 + 8(y+z)(z+x)\}\{2(y+z)^2 + (z+x)(x+y)\} \\ \{2(z+x)^2 + (x+y)(y+z)\}\end{matrix}\right]\right] \cdot$$

$$\cdot (2x+y+z)(2y+z+x)(2z+x+y) \geq$$

$$\geq 8\left(\sum x\right)\left\{5\left(\sum x\right)^2 + \sum xy\right\}\left[\prod_{cyc}\{2(x+y)^2 + (y+z)(z+x)\}\right] \Leftrightarrow$$

$$\Leftrightarrow 24\sum x^9+272\left(\sum x^8y+\sum xy^8\right)+1004\left(\sum x^7y^2+\sum x^2y^7\right)+$$
$$+175xyz\left(\sum x^6\right)+1755\left(\sum x^6y^3+\sum x^3y^6\right)+$$
$$+3611xyz\left(\sum x^5y+\sum xy^5\right)+1913\left(\sum x^5y^4+\sum x^4y^5\right)+$$
$$+2636xyz\left(\sum x^4y^2+\sum x^2y^4\right)+854x^2y^2z^2\left(\sum x^3\right)+902xyz\left(\sum x^3y^3\right)\geq$$

$\geq 9465x^2y^2z^2(\sum x^2y+\sum xy^2)+21072x^3y^3z^3\to$ ***(2). Now,*** $\forall u,v,w>0$,

$2\sum u^3\geq\sum u^2v+\sum uv^2\to$ ***(a). Let,*** $\sum x^2y+\sum xy^2=P$

$$854x^2y^2z^2(\sum x^3)\overset{by\,(a)}{\geq}427x^2y^2z^2(P)\to \textit{(i)}$$
$$2636xyz(\sum x^4y^2+\sum x^2y^4)\overset{A-G}{\geq}2636xyz(2\sum x^3y^3)\overset{by\,(a)}{\geq}2636x^2y^2z^2(P)\to \textit{(ii)}$$
$$3611xyz(\sum x^5y+\sum xy^5)\overset{A-G}{\geq}3611xyz(2\sum x^3y^3)\overset{by\,(a)}{\geq}3611x^2y^2z^2(P)\to \textit{(iii)}$$
$$902xyz(\sum x^3y^3)\overset{by(a)}{\geq}451x^2y^2z^2(P)\to \textit{(iv)}$$
$$1752xyz\left(\sum x^6\right)\overset{by(a)}{\geq}876xyz\left(\sum x^4y^2+\sum x^2y^4\right)\overset{A-G}{\geq}1752xyz\sum x^3y^3$$
$$\overset{by(a)}{\geq}876x^2y^2z^2(P)\to \textit{(v)}$$
$$1444\left(\sum x^6y^3+\sum x^3y^6\right)=1444\sum x^6(y^3+z^3)\geq 1444\sum x^6yz(y+z)$$
$$=1444xyz(\sum x^5y+\sum xy^5)\overset{A-G}{\geq}1444xyz(2\sum x^3y^3)\overset{by\,(a)}{\geq}1444x^2y^2z^2(P)\to \textit{(vi)}$$

Again, $24\sum x^9+272(\sum x^8y+\sum xy^8)+1004(\sum x^7y^2+\sum x^2y^7)+$

$$+311(\sum x^6y^3+\sum x^3y^6)+1913(\sum x^5y^4+\sum x^4y^5)\overset{A-G}{\geq}21072x^3y^3z^3\to \textit{(vii)}$$

(i)+(ii)+(iii)+(iv)+(v)+(vi)+(vii) $\Rightarrow$ ***(2) is true (proved)***

24. If $a,b,c\geq 0$ then:

$$a\left(2^b+2^{-c}\right)+b(2^c+2^{-a})+c\left(2^a+2^{-b}\right)\geq 2(a+b+c)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdallah El Farisi-Bechar – Algerie

The functions 2^x and 2^{-x} are convex for $x \geq 0$ then for all $a, b, c > 0$ we have:

$$\left(\frac{a}{a+b+c}2^b + \frac{b}{a+b+c}2^c + \frac{c}{a+b+c}2^a\right) +$$

$$+\left(\frac{a}{a+b+c}2^{-c} + \frac{b}{a+b+c}2^{-a} + \frac{c}{a+b+c}2^{-b}\right) \geq$$

$$\geq 2^{\left(\frac{ab}{a+b+c}+\frac{bc}{a+b+c}+\frac{ca}{a+b+c}\right)} + 2^{-\left(\frac{ab}{a+b+c}+\frac{bc}{a+b+c}+\frac{ca}{a+b+c}\right)} \geq 2 \text{ by MA-MG}$$

Solution 2 by Ravi Prakash - New Dehi – India

$$a(2^b) + b(2^c) + c(2^a) \geq (a+b+c)2^{\frac{ab+bc+ca}{a+b+c}} \quad (1)$$

$$a(2^{-c}) + b(2^{-a}) + c(2^{-b}) \geq (a+b+c)2^{-\frac{ac+ab+bc}{(a+b+c)}} \quad (2)$$

Adding (1), (2) we get

$$a(2^b + 2^{-c}) + b(2^c + 2^{-a}) + c(2^a + 2^{-b}) \geq (a+b+c)(2^\alpha + 2^{-\alpha}) \geq$$

$$\geq 2(a+b+c)$$

where $\alpha = \frac{ab+bc+ca}{a+b+c}$

Solution 3 by Soumitra Mandal - Chandar Nagore – India

We know, $e^x \geq x+1$ and $e^{-x} \geq 1-x$ for all $x \geq 0$

$$\sum_{cyc} a(2^b + 2^{-c}) = \sum_{cyc} a(e^{b\log 2} + e^{-c\log 2}) \geq \sum_{cyc} a(b\log 2 + 1 + 1 - c\log 2) =$$

$$= \sum_{cyc}\{2a + \log 2\,(ab - ac)\} = 2(a+b+c) + \log 2\{a(b-c) + b(c-a) + c(a-b)\} =$$

$$= 2(a+b+c)$$

(proved)

25. Prove that for all positive real numbers a, b, c

$$\sqrt{\frac{a^2+2}{b+c+1}}+\sqrt{\frac{b^2+2}{c+a+1}}+\sqrt{\frac{c^2+2}{a+b+1}}\ge 3.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los numerous R^+ a, b, c:

$$\sqrt{\frac{a^2+2}{b+c+1}}+\sqrt{\frac{b^2+2}{c+a+1}}+\sqrt{\frac{c^2+2}{a+b+1}}\ge 3$$

Por la desigualdad de Cauchy:

$$(a^2+1+1)(1+b^2+1)\ge(a+b+1)^2 \text{ ... (A)}$$

De forma análoga:

$$(b^2+1+1)(1+c^2+1)\ge(b+c+1)^2 \text{ ... (B)}$$

$$(c^2+1+1)(1+a^2+1)\ge(c+a+1)^2 \text{ ... (C)}$$

Multiplicando (A) (B) (C):

$$(a^2+2)^2(b^2+2)^2(c^2+2)^2\ge(b+c+1)^2(c+a+1)^2(a+b+1)^2$$

$$\Rightarrow (a^2+2)(b^2+2)(c^2+2)\ge(b+c+1)(c+a+1)(a+b+1)$$

De la desigualdad propuesta ... Por: MA $\ge$ MG

$$\sqrt{\frac{a^2+2}{b+c+1}}+\sqrt{\frac{b^2+2}{c+a+1}}+\sqrt{\frac{c^2+2}{a+b+1}}\ge 3\sqrt[3]{\sqrt{\frac{(a^2+2)(b^2+2)(c^2+2)}{(b+c+1)(c+a+1)(a+b+1)}}}\ge 3 \text{ ... (LQQD)}$$

26. If $a, b\in(0,\infty); m, n\in\mathbb{N}^*$ then:

$$\sum\frac{1}{\sqrt{(ma+nb)(na+mb)}}\ge\frac{9}{(m+n)(a+b+c)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash - New Delhi – India

$$\frac{(ma+nb)+(na+mb)}{2}\ge\sqrt{(ma+nb)(na+mb)}$$

$$\Rightarrow\frac{1}{2}(m+n)(a+b)\ge\sqrt{(ma+nb)(na+mb)}$$

$$\Rightarrow\frac{1}{\sqrt{(ma+nb)(na+mb)}}\ge\frac{2}{(m+n)(a+b)}$$

$$\sum\frac{1}{\sqrt{(ma+nb)(na+mb)}}\ge\frac{2}{m+n}\sum\frac{1}{a+b}\ge$$

$$\ge\frac{2}{m+n}\cdot\frac{a}{\sum(a+b)}=\frac{a}{(m+n)(a+b+c)}$$

[AM ≥ HM]

Solution 2 by Myagmarsuren Yadamsuren – Mongolia

$$\sum\frac{1}{\sqrt{(ma+nb)\cdot(na+mb)}}\overset{Cauchy}{\ge}$$

$$\ge\sum\frac{2}{(m+n)\cdot a+(m+n)\cdot b}=\frac{2}{m+n}\cdot\left(\sum\frac{1}{a+b}\right)\ge$$

$$\overset{Cauchy-Schwarz}{\ge}\frac{2}{m+n}\cdot\frac{9}{2\cdot(a+b+c)}=\frac{9}{(m+n)\cdot(a+b+c)}$$

27. If $a,b,c\in(0,\infty)$ then:

$$\frac{a^3}{b^2(5a+2b)}+\frac{b^3}{c^2(5b+2c)}+\frac{c^3}{a^2(5c+2a)}\ge\frac{3}{7}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Imad Zak – Saida – Lebanon

$$f(x)=\frac{1}{x(5+2x)}+\frac{9}{49}\ln x-\frac{1}{7}$$

$$f'(x) = \frac{(x-1)(36x^2+256x+245)}{49x^2(2x+5)^2}$$

a	0		1		$+\infty$
$f'(x)$	/	$- - - - - - - -$	0	$+ + + + + + + + + + + +$	
$f(x)$	/	↘	0	↗	

$$\Rightarrow f(x) \geq 0 \quad \forall\, x > 0$$

Now consider the inequality

$$\sum \frac{a^3}{b^2(5a+2b)} \overset{??}{\geq} \frac{3}{7}$$

Let $x = \frac{b}{a} \quad y = \frac{c}{b} \; z = \frac{a}{c} \Rightarrow xyz = 1$

We get $\sum g(x) \overset{??}{\geq} \frac{3}{7}$

$$\sum g(x) = \sum\left(f(x) - \frac{9}{49}\ln x + \frac{1}{7}\right) = \left(\sum f(x)\right) - \frac{9}{49}(\ln(xyz)) + \frac{3}{7}$$

$$= \left(\sum f(x)\right) - 0 + \frac{3}{7} \geq 0 - 0 + \frac{3}{7}$$

$\ll = \gg$ **at** $x = y = z = 1$ **or** $a = b = c$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Siendo: $a, b, c \in <0, \infty>$**. Probar que:**

$$\frac{a^3}{b^2(5a+2b)} + \frac{b^3}{c^2(5b+2c)} + \frac{c^3}{a^2(5c+2a)} \geq \frac{3}{7}$$

Por la desigualdad de Cauchy:

$$\left(\frac{a^3}{b^2(5a+2b)} + \frac{b^3}{c^2(5b+2c)} + \frac{c^3}{a^2(5c+2a)}\right)\left(ab^2(5a+2b) + bc^2(5b+2c) + ca^2(5c+2a)\right) \geq$$

$$\geq (a^2+b^2+c^2)^2$$

$$\Rightarrow \frac{a^3}{b^2(5a+2b)} + \frac{b^3}{c^2(5b+2c)} + \frac{c^3}{a^2(5c+2a)} \geq \frac{\left(a^2+b^2+c^2\right)^2}{5\sum a^2b^2+2b^3a+2c^3b+2a^2c} \geq \frac{3}{7}$$

$$\Rightarrow 7(a^2+b^2+c^2)^2 \geq +3\left(5\sum a^2b^2 + 2b^3a + 2c^3b + 2a^2c\right)$$

$$\Rightarrow 6(a^4+b^4+c^4) + (a^4+b^4+c^4) + 14\sum a^2b^2 \geq 15\sum a^2b^2 + 6b^3a + 6c^3b + 6a^3c$$

$$\Rightarrow a^4+b^4+c^4 \geq b^3a + c^3b + a^3c$$

Por: $MA \geq MG: \rightarrow a^4+a^4+a^4+c^4 \geq 4a^3c$ **... (A)**

$b^4+b^4+b^4+a^4 \geq 4b^3c$ **... (B)**

$c^4+c^4+c^4+b^4 \geq 4c^3b$ **... (C)**

Sumando ... (A) + (B) + (C):

$\Rightarrow a^4+b^4+c^4 \geq b^3a + c^3b + a^3c$ **... (LQQD)**

Solution 3 by Soumitra Mandal – Chandar Nagore – India

$$\sum_{cyc} \frac{a^3}{b^2(5a+2b)} \overset{\text{Radon's Inequality}}{\geq} \frac{(a+b+c)^3}{\left(\sum_{cyc}\sqrt{b}\cdot\sqrt{5ab+2b^2}\right)^2}$$

$$\geq \frac{(a+b+c)^3}{\left(\sqrt{(a+b+c)\{\sum_{cyc}(2b^2+5ab)\}}\right)^2}$$

[Applying Cauchy – Schwarz]

$$\geq \frac{(a+b+c)^3}{\left(\sqrt{(a+b+c)\{2(a+b+c)^2+ab+bc+ca\}}\right)^2}$$

$$\geq \frac{(a+b+c)^3}{\left(\sqrt{(a+b+c)\left\{2(a+b+c)^2+\frac{1}{3}(a+b+c)^2\right\}}\right)^2} = \frac{3}{7} \text{ (proved)}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

28. Prove that for any positive real numbers a, b, c:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los numerous R^+:

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}$$

Por la desigualdad de Nesbitt: (R^+):

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{1}{2}\left(\frac{a}{b+c}\right)+\frac{1}{2}\left(\frac{b}{a+c}\right)+\frac{1}{2}\left(\frac{c}{a+b}\right)+\frac{3}{4}$$

$$\Rightarrow\frac{1}{2}\left(\frac{a}{b+c}\right)+\frac{1}{2}\left(\frac{b}{a+c}\right)+\frac{1}{2}\left(\frac{c}{a+b}\right)\geq\frac{3}{4}\rightarrow\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}$$

Por transitividad:

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{1}{2}\left(\frac{a}{b+c}\right)+\frac{1}{2}\left(\frac{b}{a+c}\right)+\frac{1}{2}\left(\frac{c}{a+b}\right)+\frac{3}{4}\geq$$

$$\geq\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}$$

Es suficiente probar:

$$\frac{1}{2}\left(\frac{a}{b+c}\right)+\frac{1}{2}\left(\frac{b}{a+c}\right)+\frac{1}{2}\left(\frac{c}{a+b}\right)+\frac{3}{4}\geq\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}$$

$$\Rightarrow\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}+\frac{3}{2}\geq\frac{2bc}{a^2+bc}+\frac{2ca}{b^2+ca}+\frac{2ab}{c^2+ab}$$

$$\Rightarrow\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}+\frac{3}{2}\geq\left(2-\frac{2a^2}{a^2+bc}\right)+\left(2-\frac{2b^2}{b^2+ca}\right)+\left(2-\frac{2c^2}{c^2+ab}\right)$$

$$\Rightarrow\left(\frac{a^2}{ab+ac}+\frac{b^2}{ba+bc}+\frac{c^2}{ca+cb}\right)+\left(\frac{2a^2}{a^2+bc}+\frac{2b^2}{b^2+ca}+\frac{2c^2}{c^2+ab}\right)\geq\frac{9}{2}$$

Por desigualdad de Cauchy:

$$\Rightarrow \frac{a^2}{ab+ac}+\frac{a^2}{a^2+bc}+\frac{a^2}{a^2+bc} \geq \frac{9a^2}{(2bc+2a^2)+(ac+ab)} \text{ ... (A)}$$

$$\Rightarrow \frac{b^2}{ba+bc}+\frac{b^2}{b^2+ca}+\frac{b^2}{b^2+ca} \geq \frac{9b^2}{(2ac+2b^2)+(ab+bc)} \text{ ... (B)}$$

$$\Rightarrow \frac{c^2}{ca+cb}+\frac{c^2}{c^2+ab}+\frac{c^2}{c^2+ab} \geq \frac{9c^2}{(2a+2c^2)+(bc+ac)} \text{ ... (C)}$$

Sumando ... (A) + (B) + (C):

$$\Rightarrow \left(\frac{a^2}{ab+ac}+\frac{b^2}{ba+bc}+\frac{c^2}{ca+cb}\right)+\left(\frac{2a^2}{a^2+bc}+\frac{2b^2}{b^2+ca}+\frac{2c^2}{c^2+ab}\right) \geq$$

$$\geq \sum \frac{9a^2}{(2bc+2a^2)+(ac+ab)}$$

$$\Rightarrow \sum \frac{9a^2}{(2bc+a^2)+(ac+bc+ab)} \geq \frac{9(\sum a)^2}{2\sum(bc+a^2)+\sum(ab+ac)} = \frac{9}{2}$$

(LQQD)

29. Let a,b,x,y,z,u,v,w be positive real numbers such that $x+y+z=3$.

Prove that:

$$\frac{u^2}{uw}\cdot\frac{1}{(ay+bz)^x}+\frac{v^2}{wu}\cdot\frac{1}{(az+bx)^y}+\frac{w^2}{uv}\cdot\frac{1}{(ax+by)^z} \geq \frac{3}{a+b}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Soumitra Mandal – Chandar Nagore – India

$$x+y+z=3,\ 3 \geq xy+yz+zx$$

$$\sum_{cyc}\frac{u^2}{uw}\cdot\frac{1}{(ay+bz)^x} \geq \frac{3}{\sqrt[3]{\prod_{cyc}(ay+bz)^x}} \geq \frac{9}{\sum_{cyc} x(ay+bz)}$$

Applying Weighted A.M ≥ G.M

$$\frac{\sum_{cyc} x(ay+bz)}{x+y+z} \geq \sqrt[x+y+z]{\prod_{cyc}(ay+bz)^x} = \sqrt[3]{\prod_{cyc}(ay+bz)^x}$$

$$\Rightarrow \frac{1}{\sqrt[3]{\prod_{cyc}(ay+bz)^x}} \geq \frac{3}{\sum_{cyc} x(ay+bz)}$$

$$\sum_{cyc} \frac{u^2}{uw} \cdot \frac{1}{(ay+bz)^x} \geq \frac{9}{\sum_{cyc} x(ay+bz)} = \frac{9}{(a+b)(xy+yz+zx)} \geq \frac{3}{a+b}$$

(proved)

30. If $a, b, c > 0, abc = 1$ then:

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \geq \frac{3}{(ab)^2+(bc)^2+(ca)^2}$$

Proposed by Babis Stergioiu – Greece

Solution 1 by Imad Zak – Saida – Lebanon

$$a, b, c > 0, \forall\, abc = 1$$

prove that

$$\sum \frac{1}{a^2+ab+b^2} \overset{??}{\geq} \frac{3}{\sum a^2b^2}$$

when $abc = 1$ it is known that $p = \sum a \geq 3, q^2 \geq 3pr = 3pc$

$$LHS = \frac{c^2}{a^2c^2+(abc)c+b^2c^2} + \frac{a^2}{a^2b^2+(abc)a+a^2c^2} + \frac{b^2}{b^2c^2+(abc)b+a^2b^2}$$

$$= \frac{c^2}{a^2c^2+c+b^2c^2} + \frac{a^2}{a^2b^2+a+a^2c^2} + \frac{b^2}{b^2c^2+b+a^2b^2} \overset{C-B-S}{\geq} \frac{(a+b+c)^2}{2\sum a^2b^2 + \sum a}$$

$$= \frac{p^2}{2\sum a^2b^2 + p} \geq \frac{9}{2\sum a^2b^2 + p}$$

We need to prove that $\frac{9}{2\sum a^2b^2+p} \overset{??}{\geq} \frac{3}{\sum a^2b^2}$ *or* $3\sum a^2b^2 \overset{??}{\geq} 2\sum a^2b^2 + p$

or $\sum a^2b^2 \overset{??}{\geq} p$ or $(\sum ab)^2 - 2pr \overset{??}{\geq} p \Leftrightarrow$

$q^2 \geq 3\text{p}$ true Q.E.D

$\ll=\gg$ ***at*** $a = b = c = 1$

Solution 2 by Soumitra Mandal - Chandar Nagore – India

Let $x = ab, y = bc$ and $= ca\,; xyz = 1$

$$\sum_{cyc}\frac{1}{a^2 + ab + b^2} \geq \frac{3}{\sum_{cyc}(ab)^2}$$

$$\Leftrightarrow \sum_{cyc}\frac{1}{\frac{zx}{y} + \frac{xy}{z} + x} \geq \frac{3}{x^2 + y^2 + z^2} \ldots (1)$$

Now,

$$\sum_{cyc}\frac{1}{\frac{zx}{y} + \frac{xy}{z} + x} = \sum_{cyc}\frac{yz}{x(y^2 + z^2 + yz)} \geq 3\sqrt[3]{\frac{xyz}{\prod_{cyc}(x^2 + y^2 + xy)}}$$

$$= \frac{3}{\sqrt[3]{\prod_{cyc}(x^2 + y^2 + xy)}} \geq \frac{9}{2(x^2 + y^2 + z^2) + xy + yz + zx} \geq$$

$$\geq \frac{9}{2(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2)} \geq \frac{3}{x^2 + y^2 + z^2}$$

Hence statement (1) is established

$$\sum_{cyc}\frac{1}{a^2 + ab + b^2} \geq \frac{3}{\sum_{cyc}(ab)^2}$$

(proved)

Solution 3 by Le Viet Hung- Quang Tri – Vietnam

First, we have:

$$3(a^2-ab+b^2)\ge a^2+ab+b^2 \Leftrightarrow 2(a-b)^2\ge 0$$

$$\Rightarrow \frac{1}{a^2+ab+b^2}\ge \frac{1}{3}\cdot\frac{1}{a^2-ab+b^2}$$

$$3(b^2-bc+c^2)\ge b^2+bc+c^2 \Leftrightarrow 2(b-c)^2\ge 0$$

$$\Rightarrow \frac{1}{b^2+bc+c^2}\ge \frac{1}{3}\cdot\frac{1}{b^2-bc+c^2}$$

$$3(c^2-ca+a^2)\ge c^2+ca+a^2 \Leftrightarrow 2(c-a)^2\ge 0$$

$$\Rightarrow \frac{1}{c^2+ca+a^2}\ge \frac{1}{3}\cdot\frac{1}{c^2-ca+a^2}$$

$$\sum\frac{1}{a^2+ab+b^2}\ge\frac{1}{3}\cdot\sum\frac{1}{a^2-ab+b^2}\ge\frac{1}{3}\cdot\frac{9}{\sum(a^2-ab+b^2)}$$

$$=\frac{3}{2(a^2+b^2+c^2)-(ab+bc+ca)}\ \textbf{Cauchy – Schwarz}$$

$$\ge\frac{3(ab+bc+ca)}{[2(a^2+b^2+c^2)-(ab+bc+ca)](ab+bc+ca)}\ge\frac{3\cdot 3\sqrt[3]{a^2b^2c^2}}{\left[\frac{2(a^2+b^2+c^2)}{2}\right]^2}=\frac{9}{(a^2+b^2+c^2)}\ \textbf{AM – GM}$$

$$\ge\frac{9}{3(a^2b^2+b^2c^2+c^2a^2)}=\frac{3}{a^2b^2+b^2c^2+c^2a^2}$$

Solution 4 by Fotini Kaldi – Greece

$$B=(ab)+(bc)+(ca)\ge 3\sqrt[3]{(ab)(bc)(ca)}\Rightarrow B\ge 3$$

$$A=(ab)^2+(bc)^2+(ca)^2\ge 3\sqrt[3]{(ab)^2(bc)^2(ca)^2}\Rightarrow A\ge 3$$

$$3(x^2+y^2+z^2)\ge(x+y+z)^2\ge 3(xy+yz+zx),$$

$$A\ge(ab)(bc)+(ab)(ac)+(ac)(bc)\Rightarrow A\ge a+b+c\Rightarrow$$

$$\Rightarrow 3A\ge 2A+(a+b+c)\Rightarrow\frac{1}{2A+(a+b+c)}\ge\frac{1}{3A}$$

$$LHS=\frac{c^2}{(ac)^2+c+(bc)^2}+\frac{a^2}{(ab)^2+a+(ac)^2}+\frac{b^2}{(bc)^2+b+(ab)^2}\ge$$

$$\frac{(a+b+c)^2}{2((ab)^2+(bc)^2+(ca)^2)+(a+b+c)}\ge\frac{3(ab+bc+ca)}{2A+(a+b+c)}\ge$$

$$\ge\frac{9}{2A+(a+b+c)}\ge\frac{9}{3A}=\frac{3}{A}\Rightarrow$$

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\ge\frac{3}{(ab)^2+(bc)^2+(ca)^2}$$

31. Let a,b,c be positive real numbers such that $a\le 2$, $a+b\le 5$ and $a+b+c\le 11$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a,b,c números R^+ de tal manera que

$$a\le 2,\ a+b\le 5,\ a+b+c\le 11.$$

Probar que:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 1$$

Por la desigualdad de Cauchy:

$$\frac{1}{3a}+\frac{1}{3a}+\frac{1}{3a}+\frac{1}{2b}+\frac{1}{2b}+\frac{1}{c}\ge\frac{36}{9a+4b+c}=\frac{36}{(a+b+c)+3(a+b)+5a}\ge\frac{36}{11+15+10}=1\ \ldots$$

(LQQD)

La igualdad se alcanza cuando $a=2$, $b=3$, $c=6$.

32. **Let $x; y; z > 0$. Prove that:**

$$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right) \geq \sum \frac{x^2+xy+y^2}{z}$$

Proposed by Le Viet Hung –Hai Lang-Vietnam

Solution 1 by Ravi Prakash - New Delhi – India

$$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right) \geq \sum \frac{x^2+xy+y^2}{z}$$

$$LHS = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)$$

$$= \sum \frac{x^2+xy}{z} + \sum \frac{x^3}{y^2}$$

$$= \sum \frac{x^2+xy+y^2}{z} + \sum \left(\frac{x^3}{y^2}-\frac{x^2}{y}\right) \quad \textbf{(1)}$$

Let $E = \sum \frac{x^2}{y^2}(x-y)$

$$= \sum \left(\frac{x^2}{y^2}-1\right)(x-y) \quad \left[\because \sum (x-y) = 0\right]$$

$$= \sum \frac{(x^2-y^2)(x-y)}{y^2}$$

$$= \sum \frac{(x+y)(x-y)}{y^2} \geq 0$$

Thus, form (1)

$$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right) \geq \sum \frac{x^2+xy+y^2}{z}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$LHS = \frac{x^3}{y^2}+\frac{xy}{z}+\frac{z^2}{y}+\frac{x^2}{z}+\frac{y^3}{z^2}+\frac{yz}{x}+\frac{zx}{y}+\frac{y^2}{x}+\frac{z^3}{x^2}$$

$$RHS = \frac{x^2}{z} + \frac{xy}{z} + \frac{y^2}{z} + \frac{y^2}{x} + \frac{yz}{x} + \frac{z^2}{x} + \frac{z^2}{y} + \frac{zx}{y} + \frac{x^2}{y}$$

$$LHS \geq RHS \Leftrightarrow \frac{x^3}{y^2} + \frac{y^3}{z^2} + \frac{z^3}{x^2} \geq \frac{y^2}{z} + \frac{z^2}{x} + \frac{x^2}{y} \quad \textbf{(a)}$$

Now, $\frac{x^3}{y^2} + x \overset{AM-GM}{\geq} 2\frac{x^2}{y}, \frac{y^3}{z^2} + y \overset{AM-GM}{\geq} \frac{2y^2}{z}, \frac{z^3}{x^2} + z \overset{AM-GM}{\geq} \frac{2z^2}{x}$

Adding the 3:

$$\frac{x^3}{y^2} + \frac{y^3}{z^2} + \frac{z^3}{x^2} + x + y + z \geq 2\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}\right) \quad \textbf{(1)}$$

Again $\frac{x^3}{y^2} + \frac{y^3}{z^2} + \frac{z^3}{x^2} \underset{(2)}{\overset{Radon}{\geq}} \frac{(x+y+z)^3}{(x+y+z)^2} = x + y + z$

$$\textbf{(1)+(2)} \Rightarrow 2\left(\frac{x^3}{y^2} + \frac{y^3}{z^2} + \frac{z^3}{x^2}\right) \geq 2\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}\right)$$

$$\Rightarrow \frac{x^3}{y^2} + \frac{y^3}{z^2} + \frac{z^3}{x^2} \geq \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$$

$\Rightarrow$ **(a) is true (Proved)**

Solution 3 by Imad Zak – Saida – Lebanon

By Cauchy – Schwarz $B(x + y + z) \geq (x + y + z)^2 \Rightarrow B \geq (x + y + z)$ **(1)**

$$A(x + y + z) \geq B^2 \Rightarrow A \geq \frac{B^2}{\sum x}$$

So it is sufficient to prove $\frac{B^2}{\sum x} \geq B \Leftrightarrow B \geq \sum x$ **which is true by (1)**

$\ll = \gg$ **holds for** $x = y = z$

Solution 4 by Daniel Sitaru – Romania

$$LHS = \frac{\sum x^2 z}{xyz} \cdot \frac{\sum x^3 z}{xyz} \geq RHS$$

$$\frac{\sum x^2 z \cdot \sum x^3 z}{(xyz)^2} \geq \frac{\sum xy(x^2 + xy + y^2)}{xyz}$$

$$\sum x^2z\cdot\sum x^3z\ge xyz\cdot\sum xy(x^2+xy+y^2)$$

$$\sum x^5z^2\ge\sum xy^2z^4$$

$$(5,0,2)\succcurlyeq(1,2,4)\ (Muirhead)$$

33. If $a,b,c\in(0,\infty)$ then:

$$\frac{10a^3}{3a^2+7bc}+\frac{10b^3}{3b^2+7ca}+\frac{10c^3}{3c^2+7ab}\ge a+b+c$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Anas Adlany-Morroco

WLOG, let $\sum a^2=1$.

$$\sum\frac{10a^3}{3a^2+7bc}\ge\sum\left(\frac{10a^3}{3a^2+7\left(\frac{b^2+c^2}{2}\right)}\right)=$$

$$=20\sum\frac{a^3}{6a^2+7b^2+7c^2}=20\sum\left(\frac{a^3}{7-a^2}\right)\ge$$

[Using Chebyshev's inequality two times]

$$\ge\frac{20}{9}\left(\sum a\right)\left(\sum a^2\right)\left(\sum\frac{1}{7-a^2}\right)\ge\frac{20}{9}\left(\sum a\right)\left(\frac{9}{\sum(7-a^2)}\right)=\sum a$$

as desired.

Solution 2 by Imad Zak – Saida – Lebanon

Homogeneous $\Rightarrow$ let $abc=1$ the inequality is re-written as

$$\sum f(a)\ge 0$$

where $f(x)=\frac{10x^4}{3x^3+7}-x=\frac{7x(x^3-1)}{3x^3+7}$ which is convex

$\Rightarrow f(x) \ge g(x) = \frac{21}{10}(x-1)$ **the tangent at** $x = 1$**, we get**

$$\sum f(a) \ge \sum g(a) = \frac{63}{10} - \frac{63}{10} = 0$$

q.e.d

equality at $a = b = c$

Solution 3 by Kevin Soto Palacios – Huarmey – Peru

By: Inequality Holder's:

$$\left(\frac{10a^3}{3a^2+7bc} + \frac{10b^3}{3b^2+7ac} + \frac{10c^3}{3c^2+7ab}\right)\left((3a^2+7bc)+(3b^2+7ac)+(3c^2+8ab)\right)(1+1+1) \ge$$

$$\ge 10(a+b+c)^3$$

It is enough prove that:

$$(3a^2+7bc)+(3b^2+7ac)+(3c^2+7ab) \le \frac{10(a+b+c)^2}{3} \Rightarrow$$

$$\Rightarrow a^2+b^2+c^2 \ge ab+bc+ac$$

$$\Rightarrow \left(\frac{10a^3}{3a^2+7bc} + \frac{10b^3}{3b^2+7ac} + \frac{10c^3}{3c^2+7ab}\right) \ge \frac{10(a+b+c)^3}{3\left((3a^2+7bc)+(3b^2+7ac)+(3c^2+7ab)\right)} \ge$$

$$\ge \frac{10(a+b+c)^3}{10(a+b+c)^2} = a+b+c$$

Solution 4 by Soumitra Mandal -Kolkata – India

$$\Rightarrow \sum_{cyc} \frac{10a^3}{3a^2+7bc} \ge a+b+c \Leftrightarrow \sum_{cyc}\left(\frac{10a^3}{3a^2+7bc} - a\right) \ge 0$$

$$\Leftrightarrow \frac{7}{2}\sum_{cyc} \frac{a(a+b)(a-c)+a(a-b)(a+c)}{3a^2+7bc} \ge 0$$

$$\Leftrightarrow \frac{7}{2}\sum_{cyc}\left\{\frac{a(a-b)(a+c)}{3a^2+7bc} + \frac{a(a+b)(a-c)}{3a^2+7bc}\right\} \ge 0$$

$$\Leftrightarrow \frac{7}{2}\sum_{cyc}\left\{\frac{a(a-b)(a+c)}{3a^2+7bc}+\frac{b(b+c)(b-a)}{3b^2+7ac}\right\}\geq 0$$

$$\Leftrightarrow \frac{7}{2}\sum_{cyc}(a-b)\left\{\frac{a(a+c)}{3a^2+7bc}-\frac{b(b+c)}{3b^2+7ac}\right\}\geq 0$$

$$\Leftrightarrow \frac{7}{2}\sum_{cyc}(a-b)\left\{\frac{7c(a^3-b^3)+3abc(b-a)+7c^2(a^2-b^2)}{(3a^2+7bc)(3b^2+7ac)}\right\}\geq 0$$

$$\Leftrightarrow \frac{7}{2}\sum\nolimits_{cyc}(a-b)^2\left\{\frac{7c(a^2+b^2)+7c^2(a+b)+4abc}{(3a^2+7bc)(3b^2+7ac)}\right\}\geq 0, \text{ which is true}$$

Hence,

$$\sum_{cyc}\frac{10a^3}{3a^2+7bc}\geq a+b+c$$

(proved)

34. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{(b+c)^n}+\frac{b}{(c+a)^n}+\frac{c}{(a+b)^n}\geq\left(\frac{3}{2}\right)^n$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c números R^+ de tal manera que: $a+b+c=1$. Probar que:

$$\frac{a}{(b+c)^n}+\frac{b}{(c+a)^n}+\frac{c}{(a+b)^n}\geq\left(\frac{3}{2}\right)^n$$

La desigualdad puede ser equivalente:

$$\Rightarrow \frac{a^{n+1}}{[a(b+c)]^n}+\frac{b^{n+1}}{[b(c+a)]^n}+\frac{c^{n+1}}{[c(a+b)]^n} \ldots \text{(A)}$$

De la desigualdad de Radon:

Si: $x_k, a_k > 0, k \in \{1,2,3,\dots,n\}, n \in \mathbb{N} \wedge m > 0$**, se cumple lo siguiente:**

$$\Rightarrow \frac{x_1^{m+1}}{a_1^m} + \frac{x_2^{m+1}}{a_2^m} + \cdots + \frac{x_n^{m+1}}{a_n^m} \ge \frac{(x_1 + x_2 + \cdots + x_n)^{m+1}}{(a_1 + a_2 + \cdots + a_n)^m}$$

La igualdad se alcanza cuando: $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \frac{x_3}{a_3} = \cdots = \frac{x_n}{a_n}$

Por la desigualdad de Radon en (A) ... :

$$\frac{a^{n+1}}{[a(b+c)]^n} + \frac{b^{n+1}}{[b(c+a)]^n} + \frac{c^{n+1}}{[c(a+b)]^n} \ge \frac{(a+b+c)^{n+1}}{[a(b+c)+b(c+a)+c(a+b)]^n} \ge$$

$$\ge \frac{1}{\left[\frac{2}{3}(a+b+c)^2\right]^n} = \left(\frac{3}{2}\right)^n$$

Solution 2 by Phan Loc So'n-Quy Nhon City – Vietnam

Asume: $a \ge b \ge c$.

$$\frac{a}{b+c} \ge \frac{b}{c+a} \ge \frac{c}{a+b} \ \& \ \frac{1}{b+c} \ge \frac{1}{c+a} \ge \frac{1}{a+b}.$$

By Cebyshev's inequality:

$$\frac{a}{b+c} \cdot \frac{1}{(b+c)^n} + \frac{b}{c+a} \cdot \frac{1}{(c+a)^n} + \frac{c}{a+b} \cdot \frac{1}{(c+a)^n}$$

$$\ge \frac{1}{3}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)\left(\frac{1}{(a+b)^n} + \frac{1}{(b+c)^n} + \frac{1}{(c+a)^n}\right) \ge \left(\frac{3}{2}\right)^{n+1}$$

35. If $a, b, c > 0$ then:

$$2\sum(a+b)^3 + 5\sum a^3 \ge 21(a^2b + b^2c + c^2a)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si: $a, b, c > 0$**. Probar que:**

$$2\sum(a+b)^3+5\sum a^3 \geq 21(a^2b+b^2c+c^2a)$$

$$\Rightarrow 2(a+b)^3+2(b+c)^3+2(c+a)^3+5(a^3+b^3+c^3) \geq 21(a^2b+b^2c+c^2a)$$

$$\Rightarrow 9(a^3+b^3+c^3)+6ab(a+b)+6bc(b+c)+6ca(c+a) \geq$$

$$\geq 21(a^2b+b^2c+c^2a)$$

$$\Rightarrow 9(a^3+b^3+c^3)+6b^2a+6c^2b+6a^2c \geq 15(a^2b+b^2c+c^2a)$$

Desde que: $a, b, c > 0$.

Por: $\mathrm{MA} \geq \mathrm{MG}$

$$\Rightarrow 6a^3+6b^2a \geq 12a^2b \text{ ... (A)}$$

$$6b^3+6c^2b \geq 12b^2c \text{ ... (B),}$$

$$6c^3+6a^2c \geq 12c^2a \text{ ... (C)}$$

Sumando: (A) + (B) + (C):

$$\Rightarrow 6(a^3+b^3+c^3)+6b^2a+6c^2b+6a^2c \geq 12\,(a^2b+b^2c+c^2a) \text{ ... (D)}$$

Por otro lado. Por: $\mathrm{MA} \geq \mathrm{MG}$

$$\Rightarrow (a^3+a^3+b^3)+(b^3+b^3+c^3)+(c^3+c^3+a^3)=$$

$$=3(a^3+b^3+c^3) \geq 3a^2b+3b^2c+3c^2a \text{ ... (E)}$$

Finalmente sumando: (D) + (E) ...

$$\Rightarrow 9(a^3+b^3+c^3)+6b^2a+6c^2b+6a^2c \geq 15(a^2b+b^2c+c^2a) \text{ ... (LQQD)}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

Given inequality $\Leftrightarrow 2(2\sum a^3+3\sum a^2b+3\sum ab^2)+5\sum a^3 \geq 21\sum a^2b$

$$\Leftrightarrow 3\sum a^3+2\sum a^2b+2\sum ab^2 \geq 7\sum a^2b \quad \text{(A)}$$

$$\left.\begin{array}{l} a^3+a^2b+ab^2 \overset{A-G}{\geq} 3a^2b \\ b^3+b^2c+bc^2 \overset{A-G}{\geq} 3b^2c \\ c^3+c^2a+ca^2 \overset{A-G}{\geq} 3c^2a \end{array}\right\} \Rightarrow \begin{array}{c} \sum a^3+\sum a^2b+\sum ab^2 \geq 3\sum a^2b \\ 2\sum a^3+2\sum a^2b+2\sum ab^2 \geq 6\sum a^2b \quad (1) \end{array}$$

Again, (A-G) $\Rightarrow$

$$\left.\begin{array}{r}a^3+a^3+b^3\geq 3a^2b\\ b^3+b^3+c^3\geq 3b^2c\\ c^3+c^3+a^3\geq 3c^2a\end{array}\right\}\Rightarrow \begin{array}{c}3\sum a^3\geq 3\sum a^2b\\ \sum a^3\geq \sum a^2b\quad (2)\end{array}$$

$$\textbf{(1) + (2)}\Rightarrow 3\sum a^3+2\sum a^2b+2\sum ab^2\geq 7\sum a^2b$$

$\Rightarrow$ **(A) is true (Proved)**

Solution 3 by Seyran Ibrahimov – Maasilli – Azerbaidjian

$$\left.\begin{array}{r}a^3+a^3+ab^2+ab^2\geq 4a^2b\\ b^3+b^3+bc^2+bc^2\geq 4b^2c\\ c^3+c^3+ca^2+ca^2\geq 4c^2a\end{array}\right\rangle +$$

$$-----------------$$

$$x=a^3+b^3+c^3+ab^2+bc^2+ca^2\geq 2cc^2+2b^2c+2c^2a$$

$$2\sum_{cycl}(a+b)^3+5\sum_{cycl}a^3\geq 21(a^2b+b^2c+c^2a)$$

$$9a^3+9b^3+9c^3+6ab^2+6bc^2+6ca^2\geq 15a^2b+15b^2c+15c^2a$$

$$3a^3+3b^3+3c^3+2ab^2+2bc^2+2ca^2\geq 5a^2b+5b^2c+5c^2a$$

$$a^3+b^3+c^3+2x\geq 5a^2b+5b^2c+5c^2a$$

$$a^3+b^3+c^3\geq a^2b+b^2c+c^2a$$

(proved)

36. If $a,b,c>0$ then:

$$\sum\left(ac+bc-c\sqrt{ab}\right)^2\geq a^2b^2+b^2c^2+c^2a^2$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

$$\sum\left(ac+bc-c\sqrt{ab}\right)^2\geq a^2b^2+b^2c^2+c^2a^2$$

$$\Leftrightarrow \sum\left[2c^2\left(a+b-\sqrt{ab}\right)^2-c^2a^2-c^2b^2\right]\geq 0 \quad \textbf{(1)}$$

Consider

$$2c^2\left(a+b-\sqrt{ab}\right)^2-c^2a^2-c^2b^2$$

$$=c^2\left[2(a+b)^2+2ab-4\sqrt{ab}(a+b)-a^2-b^2\right]$$

$$=c^2\left[a^2+b^2+6ab-4\sqrt{ab}(a+b)\right]$$

$$=c^2\left[(a+b)^2-4\sqrt{ab}(a+b)+4ab\right]$$

$$=c^2\left(a+b-2\sqrt{ab}\right)^2$$

$$=c^2\left(\sqrt{a}-\sqrt{b}\right)^4\geq 0$$

Similarly for other two expressions.

Thus (1) is true.

37. If $x,y,z\in(0,\infty), xyz=1$ then:

$$\sum\left(x^4+y^3+z\right)\geq\sum\left(\frac{x^2+y^2}{z}\right)+3$$

Proposed by Daniel Sitaru – Romania

Solution by Imad Zak – Saida – Lebanon

Schur $\Rightarrow \sum x^4+xyz(x+y+z)\geq\sum xy\left(x^2+y^2\right)$ **but** $xyz=1\Rightarrow xy=\frac{1}{z}$

$$\sum x^4+\sum x\geq\sum\left(\frac{x^2+y^2}{z}\right) \quad \textbf{... (1)}$$

Our inequality: $\sum x^4+\sum x^3+\sum x\geq\sum\left(\frac{x^2+y^2}{z}\right)+3$ **!!**

but (1) $\Rightarrow \sum\frac{x^2+y^2}{z}\leq\sum x^4+\sum x$

so we need to prove that

$$\sum x^4 + \sum x^3 + \sum x \overset{??}{\geq} \sum x^4 + \sum x + 3 \Leftrightarrow \sum x^3 \overset{??}{\geq} 3 \text{ True}$$

by *AM-GM* $\ll = \gg$ $x = y = z = 1$

38. If $a, b, c > 0$ then:

$$\frac{2(a+b+c)}{\sqrt{abc}} \geq \sum\left(\sqrt{\frac{a+b}{2ac}} + \sqrt{\frac{2a}{c(a+b)}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si: $a, b, c > 0$. Probar que:

$$\frac{2(a+b+c)}{\sqrt{abc}} \geq \sum\sqrt{\frac{a+b}{2ac}} + \sum\sqrt{\frac{2a}{c(a+b)}}$$

$$\sum\sqrt{\frac{a+b}{2ac}} + \sum\sqrt{\frac{2a}{c(a+b)}} \leq \sum\sqrt{\frac{(a+b)b}{2abc}} + \sum\sqrt{\frac{(a+b)a}{2abc}} \leq$$

$$\leq \frac{\sum\sqrt{(a+b)}(\sqrt{a}+\sqrt{b})}{\sqrt{2abc}} \leq \frac{\sum\sqrt{2}\left(\sqrt{a+b}\right)^2}{\sqrt{2abc}}$$

$$\Rightarrow \frac{2(a+b+c)}{\sqrt{abc}} \leq \frac{\sum\sqrt{2}\left(\sqrt{a+b}\right)^2}{\sqrt{2abc}} = \frac{2\sqrt{2}(a+b+c)}{\sqrt{2abc}} = \frac{2(a+b+c)}{\sqrt{abc}}$$

LQQD

Solution 2 by Soumava Chakraborty – Kolkata-India

$$\sum\sqrt{\frac{a+b}{2ac}} = \sqrt{\frac{a+b}{2ac}} + \sqrt{\frac{b+c}{2ab}} + \sqrt{\frac{c+a}{2bc}} =$$

$$= \frac{\sqrt{b}\sqrt{a+b} + \sqrt{c}\sqrt{b+c} + \sqrt{a}\sqrt{c+a}}{\sqrt{2abc}} \underset{(1)}{\overset{CBS}{\leq}} \frac{\sqrt{\sum a}\sqrt{2\sum a}}{\sqrt{2abc}} = \frac{a+b+c}{\sqrt{a+b+c}}$$

$$\sum\sqrt{\frac{2a}{c(a+b)}}=\sqrt{\frac{2a}{c(a+b)}}+\sqrt{\frac{2b}{a(b+c)}}+\sqrt{\frac{2c}{b(c+a)}}\le$$

$$\overset{CBS}{\le}\sqrt{2\sum a}\sqrt{\frac{1}{c(a+b)}+\frac{1}{a(b+c)}+\frac{1}{b(c+a)}}\le$$

$$\le\sqrt{2\sum a}\sqrt{\frac{1}{c(2\sqrt{ab})}+\frac{1}{a(2\sqrt{bc})}+\frac{1}{b(2\sqrt{ca})}}=$$

$$\left(\because a+b\overset{AM-GM}{\ge}2\sqrt{ab}\right)$$

$$\left(\therefore\frac{1}{a+b}\le\frac{1}{2\sqrt{ab}}\right)$$

etc

$$=\sqrt{\sum a}\sqrt{\frac{1}{\sqrt{abc}}\left(\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)}=$$

$$=\sqrt{\sum a}\sqrt{\frac{1}{\sqrt{abc}}\left(\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{\sqrt{abc}}\right)}=\sqrt{\frac{\sum a}{abc}}\sqrt{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}\le$$

$$\underset{(2)}{\overset{C-B-S}{\le}}\sqrt{\frac{\sum a}{abc}}\sqrt{\sqrt{\sum a}\sqrt{\sum a}}=\frac{\sum a}{\sqrt{abc}}=\frac{a+b+c}{\sqrt{abc}}$$

$$(1)+(2)\Rightarrow\sum\left(\sqrt{\frac{a+b}{2ac}}+\sqrt{\frac{2a}{c(a+b)}}\right)\le\frac{2(a+b+c)}{\sqrt{abc}}$$

(Proved)

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

39. Let $x, y, z \in (0, 1)$ such that $xy + yz + zx = 1$. Prove that:

$$\frac{x}{(1-y^2)(1-z^2)} + \frac{y}{(1-z^2)(1-x^2)} + \frac{z}{(1-x^2)(1-y^2)} \geq \frac{9\sqrt{3}}{4}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: $x, y, z \in < 0{,}1 >$ **de tal manera que:**

$xy + yz + zx = 1$**. Probar que:**

$$\frac{x}{(1-y^2)(1-z^2)} + \frac{y}{(1-z^2)(1-x^2)} + \frac{z}{(1-x^2)(1-y^2)} \geq \frac{9\sqrt{3}}{4}$$

Por la desigualdad de Cauchy:

$$\frac{x^2}{x(1-y^2)(1-z^2)} + \frac{y^2}{y(1-z^2)(1-x^2)} + \frac{z^2}{z(1-x^2)(1-y^2)} \geq \frac{9\sqrt{3}}{4}$$

Por la desigualdad de Cauchy:

$$\sum \frac{x^2}{x(1-y^2)(1-z^2)} = \frac{(x+y+z)^2}{(x+y+z)+xyz(xy+yz+zx)-(x+y+z)(xy+yz+zx)+3xyz}$$

$$\Rightarrow \frac{x^2}{x(1-y^2)(1-z^2)} + \frac{y^2}{y(1-z^2)(1-x^2)} + \frac{z^2}{z(1-x^2)(1-y^2)} \geq$$

$$\geq \frac{(x+y+z)^2}{4xyz} \geq \frac{3}{3 \times \frac{1}{3\sqrt{3}}} = \frac{9\sqrt{3}}{4}$$

(LQQD)

Lo cual es cierto ya que, por MA ≥ MG:

$$xy + yz + zx \geq 3\sqrt[3]{(xyz)^2} \rightarrow \frac{1}{3\sqrt{3}} \geq xyz \wedge x + y + z \geq \sqrt{3(xy + yz + zx)} \rightarrow$$

$$\rightarrow x + y + z \geq \sqrt{3}$$

La igualdad se alcanza cuando, $x = y = z = \frac{1}{\sqrt{3}}$

Solution 2 by Abhay Chandra – India

From $AM \geq GM$, we have

$$\frac{x}{(1-y^2)(1-z^2)} + \frac{27x}{8}(1-y^2) + \frac{27x}{8}(1-z^2) \geq \frac{27x}{4}$$

which implies

$$\frac{x}{(1-y^2)(1-z^2)} \geq \frac{27x}{8}(y^2+z^2)$$

Summing it up, we get

$$\sum \frac{x}{(1-y^2)(1-z^2)} \geq \sum \frac{27x}{8}(y^2+z^2) = \frac{27}{8}\left((x+y+z)(xy+yz+zx) - 3xyz\right)$$

We are left to prove that

$$\frac{27}{8}(x+y+z-3xyz) \geq \frac{9\sqrt{3}}{4}$$

but $x+y+z \geq \sqrt{3}$ and $xyz \leq \frac{1}{3\sqrt{3}}$. Hence we are done. Equality at

$$x = y = z = \frac{1}{\sqrt{3}}.$$

40. Let a, b, c be positive real numbers such that:

$$(a^2+b^2+c^2)(ab+bc+ca) = abc(a+b+c+2).$$

Prove that:

$$\sqrt[9]{\frac{a^9+b^9}{2}} + \sqrt[9]{\frac{b^9+c^9}{2}} + \sqrt[9]{\frac{c^9+a^9}{2}} \leq 1.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c números R^+ de tal manera que:

$$(a^2+b^2+c^2)(ab+bc+ca) = abc(a+b+c+2)$$

Probar que:

$$\sqrt[9]{\frac{a^9+b^9}{2}}+\sqrt[9]{\frac{b^9+c^9}{2}}+\sqrt[9]{\frac{c^9+a^9}{2}}\leq 1$$

De la condición, se tiene lo siguiente:

$$\Rightarrow a^2bc+b^2ac+c^2ab+ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)=$$

$$=a^2bc+b^2ac+c^2ab+2abc$$

$$\Rightarrow \frac{a^2+b^2}{c}+\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}=2\rightarrow \frac{a^3+b^3}{ab}+\frac{b^3+c^3}{bc}+\frac{c^3+a^3}{ca}=2$$

Demostraremos lo siguiente:

$$\sqrt[9]{\frac{a^9+b^9}{2}}\leq\frac{a^3+b^3}{2ab}\rightarrow (a^3+b^3)^8\geq 2^8(a^6+b^6-a^3b^3)a^9b^9$$

$$\Rightarrow a^{24}+8a^{21}b^3+28a^{18}b^6+56a^{15}b^9+70a^{12}b^{12}+56a^9b^{15}+$$

$$+28a^6b^{18}+8a^3b^{21}+b^{24}\geq 256a^{15}b^9+256b^{15}a^9-256(ab)^{12}$$

$$\Rightarrow a^{24}+8a^{21}b^3+28a^{18}b^6-200a^{15}b^9+326a^{12}b^{12}-200a^9b^{15}+$$

$$+28a^6b^{18}+8a^3b^{21}+b^{24}\geq 0$$

Dividendo ($\div a^{12}b^{12}$), de tal manera a que el sentido no se altere, ya que:

$a, b > 0$:

$$\Rightarrow \left(\frac{a^{12}}{b^{12}}+\frac{a^{12}}{b^{12}}\right)+8\left(\frac{a^9}{b^9}+\frac{b^9}{a^9}\right)+28\left(\frac{a^6}{b^6}+\frac{b^6}{a^6}\right)-200\left(\frac{a^3}{b^3}+\frac{b^3}{a^3}\right)+326\geq 0$$

Sea: $\frac{a^3}{b^3}+\frac{b^3}{a^3}=x\geq 2$ **(Válido por: MA $\geq$ MG). Por lo tanto:**

$$\frac{a^6}{b^6}+\frac{b^6}{a^6}=x^2-2,$$

$$\frac{a^9}{b^9}+\frac{b^9}{a^9}=x^3-3x,$$

$$\frac{a^{12}}{b^{12}}+\frac{a^{12}}{b^{12}}=(x^2-2)^2-2$$

$$\Rightarrow (x^2-2)^2-2+8(x^3-3x)+28(x^2-2)-200x+326\geq 0$$

$$\Rightarrow x^4+8x^3+24x^2-224x+272\geq 0$$

Lo cual se puede expresar de la siguiente forma:

$$(x^4-4x^3+4x^2)+(12x^3-48x^2+48x)+(68x^2-272x+272)\geq 0$$

$$\Rightarrow x^2(x-2)^2+12x(x-2)^2+68(x-2)^2=(x-2)^2(x^2+12x+68)\geq 0$$

Luego:

$$\sqrt[9]{\frac{a^9+b^9}{2}}+\sqrt[9]{\frac{b^9+c^9}{2}}+\sqrt[9]{\frac{c^9+a^9}{2}}\leq \frac{a^3+b^3}{2ab}+\frac{b^3+c^3}{2bc}+\frac{c^3+a^3}{2ca}=1$$

(LQQD)

41. Let a,b,c be the side – lengths of a triangle. Prove that

$$abc(b+c-a)(c+a-b)(a+b-c)\geq$$

$$\geq (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2).$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c *los* lados de un triángulo. Probar que:

$$abc(b+c-a)(c+a-b)(a+b-c)\geq$$

$$\geq (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2).$$

Recordar lo siguiente:

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{(b+c-a)(c+a-b)(a+b-c)}{8abc}$$

Dividiendo $\div 8(abc)^2$ *a* la desigualdad inicial se tiene:

$$\frac{(b+c-a)(c+a-b)(a+b-c)}{8abc} \geq$$

$$\geq \left(\frac{b^2+c^2-a^2}{2bc}\right)\left(\frac{c^2+a^2-b^2}{2ca}\right)\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$\Rightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \geq \cos A\cos B\cos C$$

1) Supongamos que sea un triángulo acutángulo:

$$\cos A\cos B = \frac{\cos(A+B)+\cos(A-B)}{2} \leq \frac{1-\cos C}{2} = \sin^2\frac{C}{2},$$

$$\cos B\cos C \leq \sin^2\frac{A}{2} \wedge \cos C\cos A \leq \sin^2\frac{B}{2}$$

Luego, multiplicando: $\Rightarrow (\cos A\cos B\cos C)^2 \leq \left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2 \Leftrightarrow$

$\Leftrightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \geq \cos A\cos B\cos C$ **(LQQD)**

2) Supongamos que sea un triángulo obtusángulo:

$$C \geq B \geq A, C \geq 90, A, B \leq 90$$

$$\cos A\cos B\cos C \leq 0 \wedge \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} > 0 \rightarrow$$

$\rightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - \cos A\cos B\cos C \geq 0$ **(LQQD)**

42. If $a, b, c > 0$, $a^2+b^2+c^2 = 3$ then:

$$\sum a\sqrt{b^2+c^2} \leq \sqrt{6(ab+bc+ca)}$$

Proposed by Le Viet Hung - Quang Tri – Vietnam

Solution 1 by Hung Nguyen Viet – Hanoi – Vietnam

Squaring both sides, the desired inequality becomes

$$\sum_{cyc} a^2(b^2+c^2)+\sum_{cyc} 2ab\sqrt{(b^2+c^2)(c^2+a^2)}\le 6(ab+bc+ca)$$

By AM-GM inequality we obtain

$$LHS\le 2(a^2b^2+b^2c^2+c^2a^2)+\sum_{cyc} ab(a^2+b^2+2c^2)$$

$$=2(a^2b^2+b^2c^2+c^2a^2)+\sum_{cyc} ab(3+c^2)$$

$$=2(a^2b^2+b^2c^2+c^2a^2)+3(ab+bc+ca)+abc(a+b+c)$$

It's enough to show that

$$2(a^2b^2+b^2c^2+c^2a^2)+abc(a+b+c)\le 3(ab+bc+ca).$$

Indeed, we have

$$3(ab+bc+ca)=(a^2+b^2+c^2)(ab+bc+ca)$$

$$=ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)+abc(a+b+c)$$

$$\ge 2(a^2b^2+b^2c^2+c^2a^2)+abc(a+b+c)$$

and we are done.

Solution 2 by Imad Zak – Saida – Lebanon

$a,b,c>0,\sum a^2=3$ **Prove that**

$$\sum a\sqrt{a^2+b^2}\le\sqrt{6\sum ab}\quad \text{... (E)}$$

First let's prove

$$2\left(\sum a^2\right)\left(\sum ab\right)\ge\left(\sum a\right)\left(\sum ab\,(a+b)\right)=\sum a\left[\left(\sum a\right)\left(\sum ab\right)-3abc\right]\text{... (F)}$$

$$LHS-RHS=(a^3b+ab^3-2a^2b^2)+(a^3c+ac^3-2a^2c^2)+$$

$$+(b^3c+bc^3-2b^2c^2)\overset{AM-GM}{\ge}\sum(2a^2b^2-2a^2b^2)=0$$

(F) is proved.

Now by weighted Jensen's of weights a, b, c on the concave function

$f(x) = \sqrt{x}$**, we have**

$$\sum a\sqrt{a^2+b^2} \le (a+b+c)f\left(\frac{a(c^2+b^2)+b(a^2+c^2)+c(a^2+b^2)}{a+b+c}\right) =$$

$$= \sqrt{\left(\sum a\right)\left(\sum ab(a+b)\right)}$$

according to (F).

$$\sum a\sqrt{a^2+b^2} \le \sqrt{2\left(\sum a^2\right)\left(\sum ab\right)} = \sqrt{6\cdot\sum ab}$$

Q.E.D.

$\ll=\gg$ ***at*** $(a,b,c) = (1,1,1)$

43. If $a,b,c \ge 1, ab+bc+ca = 9$ then:

$$a^{a^2}\cdot b^{b^2}\cdot c^{c^2} \ge \left(\frac{abc}{3}\right)^{\frac{a^2b^2c^2}{3}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

Weighted GM-HM inequality ⇒

$$\left(a^{a^2}\cdot b^{b^2}\cdot c^{c^2}\right)^{\frac{1}{\sum a^2}} \ge \frac{\sum a^2}{\frac{a^2}{a}+\frac{b^2}{b}+\frac{c^2}{c}} = \frac{a^2}{\sum a}+\frac{b^2}{\sum a}+\frac{c^2}{\sum a}$$

$$\overset{Bergstrom}{\ge} \frac{(\sum a)^2}{3\sum a} = \frac{\sum a}{3} \overset{A-G}{\ge} \sqrt[3]{abc}$$

$$\therefore a^{a^2}\cdot b^{b^2}\cdot c^{c^2} \ge \left(\sqrt[3]{abc}\right)^{\sum a^2} \overset{A-G}{\ge} \left(\sqrt[3]{abc}\right)^{3\sqrt[3]{a^2b^2c^2}}$$

$$\left(\because \sqrt[3]{abc} \ge 1, \sum a^2 > 1, 3\sqrt[3]{a^2b^2c^2} > 1, \text{ as } a,b,c \ge 1\right)$$

$\therefore$ **it suffices to prove:**

$$\left(\sqrt[3]{abc}\right)^{3\sqrt[3]{a^2b^2c^2}} \geq \left(\frac{abc}{3}\right)^{\frac{a^2b^2c^2}{3}}$$

$$\Leftrightarrow x^{3x^2} \geq \left(\frac{x^3}{3}\right)^{\frac{x^6}{3}} \Leftrightarrow \left(x^{x^2}\right)^3 \geq \left\{\left(\frac{x^3}{3}\right)^{\left(\frac{x^3}{3}\right)^2}\right\}^3 \quad \left(x = \sqrt[3]{abc}\right)$$

$$\Leftrightarrow x^{\{x^2\}} \geq \left(\frac{x^3}{3}\right)^{\left\{\left(\frac{x^3}{3}\right)^2\right\}} \Leftrightarrow x^2 \ln x \geq \left(\frac{x^3}{3}\right)^2 \ln\left(\frac{x^3}{3}\right) \quad \textbf{(1)}$$

Case (1)

$$\frac{1}{3} \leq \frac{x^3}{3} < 1 \quad \left(x = \sqrt[3]{abc} \geq 1 \Rightarrow \frac{x^3}{3} \geq \frac{1}{3}\right)$$

$$\left.\begin{matrix} \because x \geq 1, \therefore \ln x \geq 0 \Rightarrow LHS\ of\ (1) \geq 0 \\ \because \frac{x^3}{3} < 1, \therefore \ln\left(\frac{x^3}{3}\right) < 0 \Rightarrow RHS\ of\ (1) < 0 \end{matrix}\right\} \Rightarrow LHS > RHS \Rightarrow \textbf{(1) is true}$$

Case 2

$$\frac{x^3}{3} \geq 1$$

Now, $ab + bc + ca \overset{A-G}{\geq} 3\sqrt[3]{a^2b^2c^2}$

$$\Rightarrow 3 \geq x^2 \Rightarrow 1 \geq \frac{x^2}{3} \Rightarrow x \geq \frac{x^3}{3}$$

$\therefore x^2 \geq \left(\frac{x^3}{3}\right)^2$ **and also,** $\ln x \geq \ln\left(\frac{x^3}{3}\right) \Rightarrow x^2 \ln x \geq \left(\frac{x^3}{3}\right)^2 \ln\left(\frac{x^3}{3}\right)$

($\because x^2, \ln x, \left(\frac{x^3}{3}\right)^2 > 0$ **and** $\ln\left(\frac{x^3}{3}\right) \geq 0$**)**$\Rightarrow$ ***(1)*** **is true (Proved)**

(**Equality when** $a = b = c = \sqrt{3}$***, i.e.,*** **at** $x = \sqrt{3}$***)***

Solution 2 by Soumitra Mandal – Kolkata – India

$$\text{Let } f(x) = x^2 \log x \text{ for all } x \in [1, \infty)$$

$$f'(x) = 2x\log x + x > 0 \text{ for all } x \in [1, \infty)$$

$$f''(x) = 2\log x + 3 > 0$$

Applying Jensen's Inequality

$$\frac{\sum_{cyc} a^2 \log a}{3} \ge \left(\frac{a+b+c}{3}\right)^2 \log\left(\frac{a+b+c}{3}\right)$$

$$\text{now } (a+b+c)(ab+bc+ca) \ge 9abc \Rightarrow a+b+c \ge abc$$

$$\therefore \log\left\{a^{a^2} b^{b^2} c^{c^2}\right\} \ge \log\left(\frac{abc}{3}\right)^{\frac{(a+b+c)^2}{3}} \Rightarrow a^{a^2} \cdot b^{b^2} \cdot c^{c^2} \ge \left(\frac{abc}{3}\right)^{\frac{(abc)^2}{3}}$$

(proved)

Solution 3 by Myagmarsuren Yadamsuren – Mongolia

$$\textit{1)}\ a^2\ln a + b^2 \ln b + c^2 \ln c \ge \frac{a^2 \cdot b^2 \cdot c^2}{3} \ln\left(\frac{abc}{3}\right)$$

$$LHS \Rightarrow f(x) = x^2 \cdot \ln x \Rightarrow f''(x) \ge 0$$

$$\text{JENSEN: } a^2\ln a + b^2 \ln b + c^2 \ln c \ge 3\left(\frac{a^2+b^2+c^2}{3}\right) \cdot \ln\frac{a+b+c}{3}$$

$$\textit{2)}\ RHS: \quad \underbrace{\underbrace{(a+b+c)}_{Cauchy} \cdot \underbrace{(ab+bc+ca)}_{Cauchy}} \ge 9abc \text{ (ASSURE)}$$

$$\left.\begin{matrix} a+b+c \ge 3\sqrt[3]{abc} \\ ab+bc+ca \ge 3\sqrt[3]{(abc)^2} \end{matrix}\right\} \Rightarrow (a+b+c)\cdot(ab+bc+ca) \ge 9abc$$

$$(a+b+c)\cdot(ab+bc+ca) \ge 9abc$$

$$9(a+b+c) \ge 9abc \Rightarrow (a+b+c) \ge abc\ (*)$$

$$(*) \Rightarrow (a+b+c)^2 \ge a^2b^2c^2$$

$$\underbrace{a^2+b^2+c^2+2(ab+bc+ca)}_{\Downarrow \atop Cauchy} \ge a^2b^2c^2$$

$$a^2+b^2+c^2+2(ab+bc+ca)\le 3\cdot(a^2+b^2+c^2)$$

$$3(a^2+b^2+c^2)\ge a^2b^2c^2 \text{ (**)}$$

$$\text{(*), (**)}\Rightarrow RHS:\frac{a^2b^2c^2}{3}\cdot\left(\ln\frac{abc}{3}\right)\overset{(*),(**)}{\le}\frac{3\cdot(a^2+b^2+c^2)}{3}\cdot\ln\left(\frac{a+b+c}{3}\right)=LHS$$

44. Prove the inequality holds for all positive real numbers a,b,c

$$\frac{1}{(a^2-ab+b^2)(b^2-bc+c^2)}+\frac{1}{(b^2-bc+c^2)(c^2-ca+a^2)}+$$
$$+\frac{1}{(c^2-ca+a^2)(a^2-ab+b^2)}\le\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los números R^+: a,b,c

$$\frac{1}{(a^2-ab+b^2)(b^2-bc+c^2)}+\frac{1}{(b^2-bc+c^2)(c^2-ca+a^2)}+$$
$$+\frac{1}{(c^2-ca+a^2)(a^2-ab+b^2)}\le\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}$$

Ahora bien $\forall: x,y,z\in\mathbb{R}$, se cumple la siguiente desigualdad:

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\le\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\rightarrow \text{Siendo:}$$

$$x=a^2-ab+b^2,$$

$$y=b^2-bc+c^2,$$

$$z=c^2-ca+c^2$$

Por lo cual solo hace falta demostrar lo siguiente:

$$\frac{1}{(a^2-ab+b^2)^2}+\frac{1}{(b^2-bc+c^2)^2}+\frac{1}{(c^2-ca+a^2)^2}\le$$

$$\leq \frac{2}{a^4+b^4}+\frac{2}{b^4+c^4}+\frac{2}{c^4+a^4} \leq \frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}$$

Lo cual es cierto ya que:

$$(a-b)^4 \geq 0 \to a^4+b^4+6a^2b^2-4a^3b-4b^3a \geq 0$$

$$\Rightarrow 2(a^4+b^4+a^2b^2-2a^3b-2b^3a+2a^2b^2) \geq a^4+b^4 \to$$

$$\to 2(a^2-ab+b^2)^2 \geq a^4+b^4$$

Además por desigualdad de Cauchy: $x, y, z > 0$**, se cumple lo siguiente:**

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x}$$

45. Prove the inequality holds for all positive real numbers a, b, c

$$\frac{a^2b^2}{(b^2-bc+c^2)(c^2-ca+a^2)}+\frac{b^2c^2}{(c^2-ca+a^2)(a^2-ab+b^2)}+$$

$$+\frac{c^2a^2}{(a^2-ab+b^2)(b^2-bc+c^2)} \leq 3$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Le Viet Hung-Vietnam

Using AM-GM inequality:

$$a^2-ab+b^2=\frac{1}{2\sqrt{2}}\left(\frac{a^2-ab\sqrt{2}+b^2}{\sqrt{2}-1}+\frac{a^2+ab\sqrt{2}+b^2}{\sqrt{2}+1}\right) \geq$$

$$\geq \frac{1}{2\sqrt{2}}2\cdot\sqrt{(a^2+b^2)-2a^2b^2}=\sqrt{\frac{a^4+b^4}{2}}$$

Similarly, we have:

$$b^2-bc+c^2 \geq \sqrt{\frac{b^4+c^4}{2}}$$

$$c^2 - ca + a^2 \geq \sqrt{\frac{c^4 + a^4}{2}}$$

$$\Rightarrow \sum \frac{a^2b^2}{(b^2 - bc + c^2)(c^2 - ca + a^2)} \leq \sum \left[\frac{2a^2b^2}{\sqrt{(b^4 + c^4)(c^4 + a^4)}}\right] =$$

$$= \sum \left[2\sqrt{\frac{a^4}{c^4 + a^4} \cdot \frac{b^4}{b^4 + c^4}}\right]$$

and

$$\sum \left[2\sqrt{\frac{a^4}{c^4 + a^4} \cdot \frac{b^4}{b^4 + c^4}}\right] \leq \sum \left[\frac{a^4}{c^4 + a^4} + \frac{b^4}{b^4 + c^4}\right] = 3$$

Inequality holds: $a = b = c$

46. If $x, y, z > 0$ then:

$$9\left(\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2}\right)^2 > 8\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{x^3}{y^3} + \frac{y^3}{z^3} + \frac{z^3}{x^3} - 3\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Saptak Bhatacharya-Kolkata-India

Put $\frac{x}{y} = a, \frac{y}{z} = b, \frac{z}{x} = c$

So, to show,

$$9(a^2 + b^2 + c^2)^2 > 8(a + b + c)(a^3 + b^3 + c^3 - 3abc)$$

Using $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

And rearranging

$$(a^2 + b^2 + c^2)^2 - 8(ab + bc + ca)(a^2 + b^2 + c^2) + 16(ab + bc + ca)^2 > 0$$

$$\Rightarrow (a^2+b^2+c^2-4ab-4bc-4ca)^2 > 0$$

which is true. (Proved)

Solution 2 by Nguyen Minh Triet-Quang Ngai-Vietnam

Let $a=\frac{x}{y}; b=\frac{y}{2}, c=\frac{z}{x}$ **then** ***abc=1***

The inequality becomes

$$9(a^2+b^2+c^2)^2 \ge 8(a+b+c)(a^3+b^3+c^3-3abc)$$

$$\Leftrightarrow 9(a^2+b^2+c^2) \ge 8(a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)$$

$$\Leftrightarrow 9(a^2+b^2+c^2)^2 \ge$$

$$\ge 8(a^2+b^2+c^2+2ab+2bc+2ca)(a^2+b^2+c^2-ab-bc-ca)$$

$$\Leftrightarrow 9t^2 \ge 8(t+29)(t-9)$$

(where $\begin{cases} t=a^2+b^2+c^2 \ge ab+bc+ca \\ 9=ab+bc=ca \end{cases}$**)**

$$\Leftrightarrow 9t^2 \ge 8t^2+89t-169^2$$

$\Leftrightarrow (t-49)^2 \ge 0$ **True** $\Rightarrow$ **q.e.d.**

The equality holds at $\begin{cases} t=49 \\ abc=1 \end{cases}$

$$\Leftrightarrow \begin{cases} a^2+b^2+c^2=4ab+bc+ca \\ abc=1 \end{cases} \Leftrightarrow \left[\begin{array}{l} a=\dfrac{4}{25+10\sqrt{21}}b=\dfrac{2}{5+\sqrt{21}}c \\ a=\dfrac{4}{25-10\sqrt{21}}b=\dfrac{2}{5-\sqrt{21}}c \end{array}\right.$$

$$\Leftrightarrow \frac{x}{y}=\frac{4}{25\pm10\sqrt{21}}\cdot\frac{y}{2}=\frac{2}{5\pm\sqrt{21}}\cdot\frac{z}{x}$$

$$\Leftrightarrow y=\frac{5\pm\sqrt{21}}{2}x=\frac{5\pm\sqrt{21}}{2}z$$

And permutations.

47. If $a,b,c>0, a^2+b^2+c^2=3$ then:

$$\frac{1}{(a+1)^3}+\frac{1}{(b+1)^3}+\frac{1}{(c+1)^3}\ge\frac{9}{8}-4\left(\frac{1}{(a+1)^4}+\frac{1}{(b+1)^4}+\frac{1}{(c+1)^4}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$a,b,c>0, a^2+b^2+c^2=3\Rightarrow\sum\frac{1}{(a+1)^3}\ge\frac{9}{8}-4\sum\frac{1}{(a+1)^4}$$

$$\frac{1}{(a+1)^3}+\frac{4}{(a+1)^4}\ge\frac{(1+2)^2}{(a+1)^3+(a+1)^4}\quad\text{(Bergstrom's inequality)}$$

$$=\frac{9}{(a+1)^3+(a+1)^4}$$

$$\therefore\sum\left\{\frac{1}{(a+1)^3}+\frac{4}{(a+1)^4}\right\}\ge\sum\frac{9}{(a+1)^3+(a+1)^4}$$

$\therefore$ **it suffices to prove:** $\sum\frac{1}{(a+1)^3+(a+1)^4}\ge\frac{1}{8}$ **(1)**

Now, let $f(x)=\frac{1}{(x+1)^3+(x+1)^4}\quad(x>0)$

$$f''(x)=\frac{2(10x^2+35x+31)}{(x+1)^5(x+2)^3}>0,\because x>0$$

$\therefore$ **applying Jensen's inequality**

$$\frac{1}{3}\sum f(a)\ge f\left(\frac{a+b+c}{3}\right)$$

$$\Rightarrow\sum\frac{1}{(a+1)^3+(a+1)^4}\ge\frac{3}{\left(\frac{a+b+c}{3}+1\right)^3+\left(\frac{a+b+c}{3}+1\right)^4}$$

$$=\frac{3^5}{3(a+b+c+3)^3+(a+b+c+3)^4}$$

Now, $\left(\sum a\right)^2\le3\sum a^2=9\Rightarrow\sum a\le3$

$$\therefore \sum \frac{1}{(a+1)^3+(a+1)^4} \geq \frac{3^5}{3(3+3)^3+(3+3)^4} = \frac{3^3}{6^3} = \frac{1}{8} \Rightarrow \text{(1) is true}$$

(Proved)

Solution 2 by Saptak Bhattacharya – Kolkata-India

$$\frac{\sum a}{3} \leq \sqrt{\frac{\sum a^2}{3}} \Rightarrow \sum a \leq 3 \Rightarrow \sum (a+1) \leq 6;$$

Let $a+1 = x$, $\sum x \leq 6 \Rightarrow (\sum x)^3 \leq 8 \cdot \frac{81}{3} \Rightarrow \frac{81}{(\sum x)^3} \geq \frac{3}{8}$

$$\Rightarrow \frac{\left(\sum \frac{1}{x}\right)^2}{\sum x} \geq \frac{3}{8} \quad \textit{(AM} \geq \textit{HM)} \Rightarrow \frac{2\left(\sum \frac{1}{x}\right)^2}{\sum x} + \frac{\left(\sum \frac{1}{x}\right)^2}{\sum x} \geq \frac{9}{8} \Rightarrow \sum \frac{4}{x^4} + \sum \frac{1}{x^3} \geq \frac{9}{8}$$

[$\because \sum x^2 = 2\sum x$; **then Titu's lemma]**

$$\Rightarrow \sum \frac{1}{x^3} \geq \frac{9}{8} - 4\sum \frac{1}{x^4} \Rightarrow \sum \frac{1}{(a+1)^3} \geq \frac{9}{8} - 4\sum \frac{1}{(a+1)^4}$$

(Proved)

Solution 3 by Soumitra Mandal – Chandar Nagore – India

$$\sqrt{3(a^2+b^2+c^2)} \geq a+b+c \Rightarrow 3 \geq a+b+c$$

we know,

$$\frac{1}{3}\sum_{cyc}(a+1)^{-3} \geq \left(\frac{a+1+b+1+c+1}{3}\right)^{-3} = \frac{27}{(a+b+c+3)^3} \geq \frac{27}{216}$$

$$\therefore \sum_{cyc} \frac{1}{(a+1)^3} \geq \frac{3}{8}$$

Similarly,

$$\frac{1}{3}\sum_{cyc}(a+1)^{-4} \geq \left(\frac{a+b+c+3}{3}\right)^{-4} = \frac{81}{(a+b+c+3)^4} \geq \frac{81}{1296}$$

$$\therefore 4\sum_{cyc}\frac{1}{(a+1)^4}\ge\frac{3}{4}$$

$$\therefore \sum_{cyc}\frac{1}{(a+1)^3}+4\sum_{cyc}\frac{1}{(a+1)^4}\ge\frac{9}{8}\Rightarrow\sum_{cyc}\frac{1}{(a+1)^3}\ge\frac{9}{8}-4\sum_{cyc}\frac{1}{(a+1)^4}$$

(Proved)

48. If $x,y,z>0$ then:

$$4\sum(x^2+y^2)z+4xyz\sum\frac{xy}{(x+y)^2}\ge 27xyz$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Stefan-Romania

$$4\sum(x^2+y^2)z+4xyz\sum\frac{xy}{(x+y)^2}\ge 27xyz|:xyz>0$$

$$4\sum\left(\frac{x}{y}+\frac{y}{x}\right)+4\sum\frac{1}{\frac{x}{y}+\frac{y}{x}+2}\ge 27$$

We'll prove: $4\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{4}{\frac{x}{y}+\frac{y}{x}+2}\ge 9$

$$\frac{x}{y}+\frac{y}{x}\overset{not}{=}\alpha\ge 2$$

$$4\alpha^2-\alpha-14\ge 0\Leftrightarrow(\alpha-2)(4\alpha+7)\ge 0,\text{ true }(\alpha\ge 2)$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$4\sum_{cyc}z(x^2+y^2)+4xyz\sum_{cyc}\frac{xy}{(x+y)^2}\ge 27xyz$$

$$\Leftrightarrow 4\left\{\sum_{cyc}z(x^2+y^2)-6xyz\right\}\ge xyz\left\{3-\sum_{cyc}\frac{4xy}{(x+y)^2}\right\}$$

$$\Leftrightarrow 4\sum_{cyc} x(y-z)^2 \geq xyz\sum_{cyc}\frac{(x-y)^2}{(x+y)^2} \Leftrightarrow 4\sum_{cyc}\frac{(x-y)^2}{xy} \geq \sum_{cyc}\left(\frac{x-y}{x+y}\right)^2$$

$$\Leftrightarrow \sum_{cyc}(x-y)^2\left\{\frac{4(x+y)^2-xy}{xy(x+y)^2}\right\} \geq 0, \text{ which is true}$$

$$\because 4(x+y)^2 - xy \geq 15xy > 0$$

Hence,

$$4\sum_{cyc} z\left(x^2+y^2\right) + 4xyz\sum_{cyc}\frac{xy}{(x+y)^2} \geq 27xyz$$

(proved)

Solution 3 by Redwane El Mellass-Morroco

The inequality $\Leftrightarrow 4\sum\left(\frac{x}{y}+\frac{y}{x}\right)+\sum\frac{xy}{\left(\frac{x+y}{2}\right)^2} \geq 27$

Let $f(x) = 4\left(x+\frac{1}{x}\right)+\frac{2}{x+\frac{1}{x}}$ for $x > 0$.

Since $f'(x) = 2\left(1-\frac{1}{x^2}\right)\left(2-\frac{1}{\left(x+\frac{1}{x}\right)^2}\right)$ with $x+\frac{1}{x} \geq 1$

we get $(\forall x > 0): f(x) \geq f(1) = 9$.

$\therefore LHS \geq \sum f\left(\frac{x}{y}\right) \geq 27$ with equality if

$$\frac{x}{y} = \frac{y}{x} = \frac{z}{x} = 1 \Rightarrow x = y = z > 0$$

49. If $a, b, c \in (1, \infty)$ then:

$$\log_{ab} e + \log_{bc} e + \log_{ca} e \leq \frac{1}{2}\left(\log_a e + \log_b e + \log_c e\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Myagmarsuren Yadamsuren – Darkhan – Mongolia

$$\log_{ab} e + \log_{bc} e + \log_{ca} e \le \frac{1}{2}\cdot(\log_a e + \log_b e + \log_c e)$$

$$\frac{1}{\ln ab} + \frac{1}{\ln bc} + \frac{1}{\ln ca} \le \frac{1}{2}\left(\frac{1}{\ln a} + \frac{1}{\ln b} + \frac{1}{\ln c}\right)$$

$$\frac{1}{\ln a + \ln b} + \frac{1}{\ln b + \ln c} + \frac{1}{\ln c + \ln a} \le \frac{1}{2}\cdot\sum\frac{1}{\ln a}$$

$$\frac{(1+1)^2}{4\cdot(\ln a + \ln b)} + \frac{(1+1)^2}{4\cdot(\ln b + \ln c)} + \frac{(1+1)^2}{4\cdot(\ln c + \ln a)} \overset{Cauchy}{\le}$$

$$\le \frac{1}{4}\cdot\left(\frac{2}{\ln a} + \frac{2}{\ln b} + \frac{2}{\ln c}\right) = \frac{1}{2}\left(\frac{1}{\ln a} + \frac{1}{\ln b} + \frac{1}{\ln c}\right)$$

Solution 2 by Ravi Prakash - New Delhi – India

Let $x = \log_e a$, $y = \log_e b$, $z = \log_e c$. As $a, b, c > 1, x, y, z > 0$

Now, $\log_{ab} e = \frac{\log e}{\log a + \log b} = \frac{1}{x+y}$, etc. [change of base]

We have: $\frac{2xy}{x+y} \le \frac{1}{2}(x+y) \Rightarrow \frac{1}{x+y} \le \frac{1}{4}(x+y)$

Thus, $\sum\frac{1}{x+y} \le \frac{1}{4}\sum(x+y) = \frac{1}{2}\sum x; \sum\log_{ab} e \le \frac{1}{2}\sum\log_a e$

50. If $a, b, c > 0$ then:

$$\left(\sum\frac{1}{(a^2 - ab + b^2)^6}\right)^2 \le 3\sum\left(\frac{a+b}{a^2+b^2}\right)^{24}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum\frac{1}{(a^2 - ab + b^2)^6}\right)^2 \le 3\sum\frac{1}{(a^2 - ab + b^2)^{12}}$$

$$\left(\because \left(\sum x\right)^2 \le 3\sum x^2\right)$$

Let's prove: $\frac{1}{(a^2-ab+b^2)^{12}} \le \left(\frac{a+b}{a^2+b^2}\right)^{24}$ **(1)**

$$\Leftrightarrow \frac{1}{a^2-ab+b^2} \le \frac{(a+b)^2}{(a^2+b^2)^2}$$

$$\Leftrightarrow a^4+b^4+2a^2b^2 \le (a+b)(a^3+b^3) = a^4+b^4+a^3b+ab^3$$

$$\Leftrightarrow a^2+b^2 \ge 2ab \Leftrightarrow (a-b)^2 \ge 0 \to \text{true}$$

Similarly, $\frac{1}{(b^2-bc+b^2)^{12}} \le \left(\frac{b+c}{b^2+c^2}\right)^{24}$ **(2) and**

$\frac{1}{(c^2-ca+a^2)^{12}} \le \left(\frac{c+a}{c^2+a^2}\right)^{24}$ **(3)**

(1) + (2) + (3) $\Rightarrow \sum \frac{1}{(a^2-ab+b^2)^{12}} \le \sum \left(\frac{a+b}{a^2+b^2}\right)^{24}$ **(4)**

$$\therefore \left(\sum \frac{1}{(a^2-ab+b^2)^6}\right)^2 \le 3\sum \frac{1}{(a^2-ab+b^2)^{12}} \le 3\sum \left(\frac{a+b}{a^2+b^2}\right)^{24}$$

(using (4))

(proved)

51. Let a, b, c be positive real numbers. Prove that:

$$4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+(a+b+c)^2\left(\frac{1}{\sqrt{a^2+ab+b^2}}+\frac{1}{\sqrt{b^2+bc+c^2}}+\frac{1}{\sqrt{c^2+ca+a^2}}\right)^2 \ge 33$$

Proposed by Do Chinh Quoc-Ho Chi Minh-Vietnam

Solution 1 by proposer

By the virtue of the AM-GM inequality, we have:

$$\frac{1}{\sqrt{a^2+ab+b^2}} = \frac{\sqrt{ab+bc+ca}}{\sqrt{(a^2+ab+b^2)(ab+bc+ca)}} \ge \frac{2\sqrt{ab+bc+ca}}{a^2+2ab+b^2+bc+ca} = \frac{2\sqrt{ab+bc+ca}}{(a+b)(a+b+c)}$$

Similarly, we get:

$$\frac{1}{\sqrt{a^2+ab+b^2}}+\frac{1}{\sqrt{b^2+bc+c^2}}+\frac{1}{\sqrt{c^2+ca+a^2}}\ge\frac{2\sqrt{ab+bc+ca}}{a+b+c}\cdot\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

$$\Leftrightarrow (a+b+c)^2\left(\frac{1}{\sqrt{a^2+ab+b^2}}+\frac{1}{\sqrt{b^2+bc+c^2}}+\frac{1}{\sqrt{c^2+ca+a^2}}\right)^2\ge 4(ab+bc+ca)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)^2$$

Using the Iran 1996 inequality, we have:

$$\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)^2=\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{4(a+b+c)}{(a+b)(b+c)(c+a)}$$

$$\ge\frac{9}{4(ab+bc+ca)}+\frac{4(a+b+c)}{(a+b)(b+c)(c+a)}$$

$$\Rightarrow 4(ab+bc+ca)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)^2\ge 9+\frac{16(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)}$$

$$=25+\frac{16abc}{(a+b)(b+c)(c+a)}$$

Therefore, we need to prove that:

$$4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+25+\frac{16abc}{(a+b)(b+c)(c+a)}\ge 33$$

$$\Leftrightarrow\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\ge 2$$

$$\Leftrightarrow a^3+b^3+c^3+3abc\ge ab(a+b)+bc(b+c)+ca(c+a)$$

Always true because this is Schur level 3 inequality.

The equality holds for $a=b=c$.

Solution 2 by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

Applying AM-GM inequality: $\sum\frac{1}{\sqrt{a^2+ab+b^2}}=\sqrt{ab+bc+ca}\cdot\sum\frac{1}{\sqrt{(a^2+ab+b^2)(ab+bc+ca)}}$

$$\ge\sum\frac{2\sqrt{ab+bc+ca}}{a^2+ab+b^2+(ab+bc+ca)}=\sum\frac{2\sqrt{ab+bc+ca}}{(a+b)(a+b+c)}$$

$$\Rightarrow (a+b+c)^2\left(\sum\frac{1}{\sqrt{a^2+ab+b^2}}\right)^2\ge(a+b+c)^2\cdot\frac{4(ab+bc+ca)}{(a+b+c)^2}\cdot\left(\sum\frac{1}{a+b}\right)^2=$$

$$=4(ab+bc+ca)\left(\sum\frac{1}{a+b}\right)^2$$

$$=4(ab+bc+ca)\cdot\sum\frac{1}{(a+b)^2}+8(ab+bc+ca)\sum\frac{1}{(a+b)(b+c)}\ge$$

$$\geq 9+\frac{16(ab+bc+ca)(a+b+c)}{(a+b)(b+c)(c+a)} \text{ (Iran 96)}= 9+16+\frac{16abc}{(a+b)(b+c)(c+a)}$$

So, we just need to prove that:

$$4\sum\frac{a}{b+c}+\frac{16abc}{(a+b)(b+c)(c+a)}\geq 8\text{, or }\sum\frac{a}{b+c}+\frac{4abc}{(a+b)(b+c)(c+a)}\geq 2$$

$$\Leftrightarrow a^3+b^3+c^3+3abc\geq\sum ab(a+b) \text{ (true because this is Schur } \deg 3) \Rightarrow \text{Q.E.D.}$$

52. If $a, b, c, d > 0$ then:

$$3(a+b+c+d) > 2\sum\sqrt[6]{a^3b(b+c)(b+c+d)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Anas Adlany - El Jadida – Morroco

$$2\sqrt[6]{a^3b(b+c)(b+c+d)}\leq 2\cdot\frac{3a+b+(b+c)+(b+c+d)}{6}=$$

$$=a+b+\frac{4c}{6}+\frac{2d}{6}$$

$$\Rightarrow\sum 2\sqrt[6]{a^3b(b+c)(b+c+d)}\leq\sum\left(a+b+\frac{4c}{6}+\frac{2d}{6}\right) < 3(a+b+c+d)$$

(true)

Solution 2 by Henry Ricardo - New York – USA

The AM – GM inequality gives us

$$2\sum_{cyclic}\sqrt[6]{a^3b(b+c)(b+c+d)} < 2\sum_{cyclic}\frac{a+a+a+b+(b+c)+(b+c+d)}{6}=$$

$$=2\sum_{cyclic}\frac{3a+3b+2c+d}{6}=\frac{1}{3}\left(3\sum_{cyclic}(a+b)+2\sum_{cyclic}c+\sum_{cyclic}d\right)=$$

$$=\frac{1}{3}\cdot 9(a+b+c+d)=3(a+b+c+d).$$

Solution 3 by Myagmarsuren Yadamsuren – Darkhan – Mongolia

$$a; b; c; d > 0$$

$$3 \cdot \sum a > 2 \cdot \sum \sqrt[6]{a^3 b(b+c) \cdot (b+c+d)}$$

$$3 \cdot (a+b+c+d) = 2 \cdot \frac{9 \cdot (a+b+c+d)}{6} =$$

$$= 2 \sum \frac{3a+3b+2c+d}{6} = 2 \cdot \sum \frac{3a+b+(b+c)+(b+c+d)}{6} \geq$$

$$\underset{Cauchy}{\geq} 2 \sum \sqrt[6]{a^3 \cdot b \cdot (b+c) \cdot (b+c+d)}$$

53. **If $a, b, c, d \in (0, \infty)$, $\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{1}{1+c^2} + \frac{1}{1+d^2} = 1$**

then: $a+b+c+d \geq 6$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren – Mongolia

1)

$$\left.\begin{array}{c} \sum_{i=1}^{n} \frac{1}{1+x_i^2} = 1 \\ \sum_{i=1}^{n} x_i \geq (n-1) \cdot \sum_{i=1}^{n} \frac{1}{x_i} \Leftrightarrow \sum \frac{x_i^2+1}{x_i} \geq n \cdot \sum \frac{1}{x_i} \end{array}\right\} Assume$$

$$\left.\begin{array}{c} x_1 \geq x_2 \geq \cdots \geq x_n \\ \frac{1}{x_1} \leq \frac{1}{x_2} \leq \cdots \leq \frac{1}{x_n} \\ \frac{x_1}{1+x_1^2} \leq \frac{x_2}{1+x_2^2} \leq \cdots \leq \frac{x_n}{1+x_n^2} \end{array}\right\} \textit{Chebyshev} \Rightarrow$$

$$\sum_{i=n}^{n} \frac{x_i}{x_i^2+1} \cdot \sum_{i=n}^{n} \frac{1}{x_i} \leq n \cdot \sum_{i=n}^{n} \frac{x_i}{x_i^2+1} \cdot \frac{1}{x_i} = n \cdot \sum_{i=n}^{n} \frac{1}{x_i^2+1} = n \quad (*)$$

$$\left(\sum_{i=n}^{n}\frac{x_i^2+1}{x_i}\right)\cdot\left(\sum_{i=n}^{n}\frac{x_i}{x_i^2+1}\right)\overset{Cauchy}{\geq} n\cdot\sqrt[n]{\prod\frac{x_i^2+1}{x_i}}\cdot n\cdot\sqrt{\prod\frac{x_i}{x_i^2+1}}=n^2\quad(**)$$

$$(*);(**)\Rightarrow\sum_{i=n}^{n}\frac{x_i^2+1}{x_i}\cdot\sum_{i=n}^{n}\frac{x_i}{x_i^2+1}\geq n\cdot\sum\frac{x_i}{x_i^2+1}\cdot\sum_{i=n}^{n}\frac{1}{x_i}\Rightarrow$$

$$\Rightarrow\sum_{i=n}^{n}\frac{x_i^2+1}{x_i}\geq n\cdot\sum_{i=n}^{n}\frac{1}{x_i}\quad(***)$$

2)

$$1=\sum_{i=n}^{n}\frac{1}{1+x_i^2}\overset{Cauchy}{\leq}\frac{1}{2}\cdot\sum\frac{1}{x_i}\Rightarrow\sum\frac{1}{x_i}\geq 2\quad(****)$$

$$\sum_{i=n}^{n}\frac{x_i^2+1}{x_i}\geq n\cdot\sum_{i=n}^{n}\frac{1}{x_i}\Leftrightarrow\sum_{i=n}^{n}x_i\geq(n-1)\cdot\sum_{i=n}^{n}\frac{1}{x_i}\geq$$

$$\overset{(****)}{\geq}(n-1)\cdot 2\Rightarrow\sum_{i=n}^{n}x_i\geq(n-1)\cdot 2$$

$$n=4\Rightarrow x_1+x_2+x_3+x_4\geq 3\cdot 2=6\Rightarrow a+b+c+d\geq 6$$

54. If $a,b,c,d>0, a+b+c+d=1$ then:

$$\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b}\geq\frac{1}{8}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Solution by Kevin Soto – Palacios – Huarmey – Peru

Si: $a,b,c,d>0. a+b+c+d=1$. Probar que:

$$\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b}\geq\frac{1}{8}$$

Por la desigualdad de Holder:

$$\left(\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b}\right)\left((b+c)+(c+d)+(d+a)+(a+b)\right)(1+1+1+)\geq$$

$$\geq(a+b+c+d)^3$$

$$\Rightarrow\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b}\geq\frac{(a+b+c+d)^3}{8(a+b+c+d)}=\frac{1}{8}$$

(LQQD)

55. If $x,y,z>0$ then:

$$\sqrt{2}\left(x\sqrt{x^2z^2+y^4}+y\sqrt{y^2x^2+z^4}+z\sqrt{z^2y^2+x^4}\right)>xy\sqrt{yz}+yx\sqrt{zx}+zy\sqrt{xy}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Myagmarsuren Yadamsuren – Mongolia

$$\sum\sqrt{2}\,x\cdot\sqrt{x^2z^2+y^4}=\sum\sqrt{(1^2+1^2)\cdot(x^4z^2+y^4\cdot x^2)}\geq$$

$$\geq\sum x^2z+y^2x=(x^2z+y^2x)+(y^2x+z^2y)+(z^2y+x^2z)=$$

$$=(x^2z+y^2x)+(y^2x+z^2y)+(z^2y+x^2z)\geq 2\cdot xy\cdot\sqrt{xz}+$$

$$+2\sqrt{xy}\cdot yz+2xz\cdot\sqrt{zy}>xy\sqrt{xz}+yx\sqrt{zx}+zy\sqrt{xy}$$

Solution 2 by Nguyen Minh Triet - Quang Ngai – Vietnam

By AM – GM inequality, we have:

$$x^2z^2+y^4\geq 2xzy^2\Rightarrow x\sqrt{2}\cdot\sqrt{x^2z^2+y^4}\geq 2xy\sqrt{xz}\quad\textbf{(1)}$$

Similarly, we get: $y\sqrt{2}\cdot\sqrt{y^2x^2+z^4}\geq yz\sqrt{xy}$ **(2)**

$$z\sqrt{2}\cdot\sqrt{x^2y^2+x^4}\geq 2xz\sqrt{xy}\quad\textbf{(3)}$$

(1) + (2) + (3) $\Rightarrow LHS\geq 2\,RHS>RHS$

q.e.d.

56. **If $a,b,c>0$ then the following relationship holds:**

$$a+b+c\geq\sum\frac{(a+b+2c)(a+c)(b+c)}{2(a+c)^2+2(b+c)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si: $a,b,c>0$. Probar la siguiente desigualdad:

$$a+b+c\geq\sum\frac{(a+b+2c)(a+c)(b+c)}{2(a+c)^2+2(b+c)^2}$$

Desde que: $a,b,c>0$. Por: *MA* $\geq$ *MG:*

$$2(a+c)^2+2(b+c)^2\geq 4(a+c)(b+c)\rightarrow$$

$$\rightarrow\sum\frac{(a+b+2c)(a+c)(b+c)}{2(a+c)^2+2(b+c)^2}\leq\frac{\sum(2c+a+b)}{4}=a+b+c\ \text{... (LQQD)}$$

Solution 2 by Anas Adlany-El Jadida-Morroco

We have $2(A^2+B^2)\geq 4AB$, so if we take

$A=a+c, B=b+c$, we get

$$\sum\frac{(a+b+2c)(a+c)(b+c)}{2((a+c)^2+(b+c)^2)}=$$

$$=\sum\frac{(A+B)AB}{2(A^2+B^2)}\leq\sum\frac{A+B}{4}=\sum a$$

Solution 3 by Sk Rejuan-West Bengal-India

We have to prove

$$\sum\frac{(a+b+2c)(a+c)(b+c)}{2(a+c)^2+2(b+c)^2}\leq a+b+c$$

$$LHS=\sum\frac{(a+b+c)(a+c)(b+c)}{2(a+c)^2+2(b+c)^2}\quad\text{(1)}$$

Now, $(a+c)^2+(b+c)^2\geq\frac{1}{2}(a+b+2c)^2$

$$\Rightarrow \frac{1}{(a+c)^2+(b+c)^2} \le \frac{2}{(a+b+c)^2}$$

$$\Rightarrow \frac{(a+b+2c)(a+c)(b+c)}{2(a+c)^2+2(b+c)^2} \le \frac{2(a+b+2c)(a+c)(b+c)}{2(a+b+2c)^2} =$$

$$= \frac{(a+c)(b+c)}{(a+b+2c)} \le \frac{1}{4}(a+b+2c)$$

[by G.M ≤ A.M]

Solution 4 by Nirapada Pal-India

$$\sum \frac{(a+b+2c)(a+c)(b+c)}{2(a+c)^2+2(b+c)^2} \le \sum \frac{(a+b+2c)(a+c)(b+c)}{(a+b+2c)^2} =$$

as $2(a^2+b^2) \ge (a+b)^2$

$$= \sum \frac{(a+c)(b+c)}{a+b+2c} = \frac{1}{2}\sum \frac{2}{\frac{1}{a+c}+\frac{1}{b+c}} \le \frac{1}{2}\sum \frac{(a+c)+(b+c)}{2} = \text{as } HM \le AM$$

$$= \frac{1}{4}\sum (a+b+2c) = a+b+c$$

Solution 5 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left.\begin{matrix} a+c=x \\ c+a=y \\ a+b=z \end{matrix}\right\} \quad a+b+c = \frac{x+y+z}{2}$$

$$\frac{x+y+z}{2} \ge \sum \frac{(x+y)\cdot xy}{2(x^2+y^2)}$$

$$x+y+z \ge \sum \frac{(x+y)\cdot xy}{x^2+y^2} \quad \text{(ASSURE)}$$

$$\left(\frac{x+y}{2}\right) + \left(\frac{y+z}{2}\right) + \left(\frac{z+x}{2}\right) \ge \sum \frac{(x+y)\cdot xy}{x^2+y^2}$$

$$\frac{x+y}{2} \ge \frac{(x+y)\cdot xy}{x^2+y^2} \quad \text{(ASSURE)}$$

$$\frac{1}{2} \ge \frac{xy}{x^2+y^2} \Rightarrow x^2+y^2 \ge 2xy \quad \text{(TRUE)}$$

Similarly

$$\frac{y+z}{2} \geq \frac{(y+z)zy}{y^2+z^2}$$

$$\frac{z+x}{2} \geq \frac{(z+x)\cdot zx}{z^2+x^2}$$

$$x+y+z \geq \sum \frac{(x+y)\cdot xy}{y^2+x^2}$$

57. If $a,b,c \in [2,\infty)$ then:

$$\sum \frac{ab-4}{c(a+b)} + \frac{12}{abc} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Proposed by Daniel Sitaru – Romania

Solution by Redwane El Mellass-Morroco

$$\therefore \frac{1}{a} + \frac{1}{b} \leq \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow a+b \leq ab$$

$$\Rightarrow \frac{1}{a+b} \geq \frac{1}{ab} \Rightarrow \frac{(ab-4)}{c}\left(\frac{1}{a+b} - \frac{1}{ab}\right) \geq 0$$

$$\Rightarrow \frac{ab-4}{c(a+b)} + \frac{4}{abc} - \frac{1}{c} \geq 0 \Rightarrow \sum \frac{ab-4}{c(a+b)} + \frac{12}{abc} \geq \sum \frac{1}{a}.$$

with equality if and only if $\frac{1}{a} + \frac{1}{b} = \frac{1}{b} + \frac{1}{c} = \frac{1}{c} + \frac{1}{a} = 1 \Rightarrow a = b = c = 2.$

58. If $a,b,c \in (0,2)$ then:

$$be^{\frac{1}{a-2}} + ce^{\frac{1}{b-2}} + ae^{\frac{1}{c-2}} \leq \frac{1}{e}\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Redwane El Mellass-Morrocco

Let $f(x) = xe^{\frac{1}{x-2}}, 0 < x < 2$. Since: $f'(x) = \frac{(x-1)(x-3)e^{\frac{1}{x-2}}}{(x-2)^2}$, we get $f(x) \leq f(1) = \frac{1}{e}$. So:

$$\sum_{cyc} \frac{b}{a} f(a) \leq \frac{1}{e} \sum_{cyc} \frac{b}{a}.$$

59. If $a, b, c > 0$ then:

$$\sum \frac{a^2 + b^2}{a + b} + 11 \sum \frac{ab}{a + b} > 6\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Ștefan – Romania

Let's prove: $\frac{a^2+b^2}{a+b} + 11\frac{ab}{a+b} \geq 6\sqrt{ab}$

$$a + b \overset{not}{=} s$$

$$ab \overset{not}{=} p$$

$$\Leftrightarrow s^2 - 6s\sqrt{p} + 9p > 9 \Leftrightarrow \left(s - 3\sqrt{p}\right)^2 \geq 0$$

$$\text{if} \begin{cases} a + b = 3\sqrt{ab} \Rightarrow \sqrt{a} = \frac{\sqrt{b}(3\pm\sqrt{5})}{2} \\ b + c = 3\sqrt{bc} \Rightarrow \sqrt{b} = \frac{\sqrt{c}(3\pm\sqrt{5})}{2} \\ c + a = 3\sqrt{ac} \Rightarrow \sqrt{c} = \frac{\sqrt{a}(3\pm\sqrt{5})}{2} \end{cases}$$

Anyway we will have the signs in $3 \pm \sqrt{5}$, it will result that a rational number is equal to an irrational number. Contradiction $\Rightarrow LHS > RHS$

Solution 2 by Ravi Prakash - New Delhi – India

Consider

$$\frac{a^2 + b^2}{a + b} + \frac{11ab}{a + b} - 6\sqrt{ab} =$$

$$= \frac{1}{a + b}\left[a^2 + b^2 + 2ab - 6\sqrt{ab}(a + b) + 9ab\right] =$$

$$=\frac{1}{a+b}\left[(a+b)^2-6\sqrt{ab}(a+b)+\left(3\sqrt{ab}\right)^2\right]=$$

$$=\frac{1}{a+b}\left[\left(a+b-3\sqrt{ab}\right)^2\right]\geq 0$$

$$\therefore \frac{a^2+b^2}{a+b}+\frac{11ab}{a+b}\geq 6\sqrt{ab}$$

Similarly for other two terms.

Solution 3 by Seyran Ibrahimov – Maasilli – Azerbaidjian

$$\frac{a^2+b^2}{a+b}+\frac{11ab}{a+b}=\frac{(a+b)^2}{a+b}+\frac{9ab}{a+b}\overset{AM-GM}{\geq} 6\sqrt{ab}$$

Solution 4 by Soumitra Mandal - Chandar Nagore – India

$$\sum_{cyc}\frac{a^2+b^2}{a+b}+11\sum_{cyc}\frac{ab}{a+b}>6\sum_{cyc}\sqrt{ab}$$

$$\Leftrightarrow \sum_{cyc}\frac{(a+b)^2+9ab-6(a+b)\sqrt{ab}}{a+b}>0$$

$$\Leftrightarrow \sum_{cyc}\frac{\left(a+b-3\sqrt{ab}\right)^2}{a+b}>0$$

which is true

$$\therefore \sum_{cyc}\frac{a^2+b^2}{a+b}+11\sum_{cyc}\frac{ab}{a+b}>6\sum_{cyc}\sqrt{ab}$$

(Proved)

Solution 5 by Abdallah El Farissi – Bechar – Algerie

$$\frac{a^2+b^2}{a+b}+11\frac{ab}{a+b}=(a+b)+9\frac{ab}{a+b}\geq 6\sqrt{ab}$$

and equality if and only if $a^2-7ab+b^2=0$, $a=\frac{3\sqrt{5}+7}{2}b$ then

$$\sum \frac{a^2+b^2}{a+b} + 11\sum \frac{ab}{a+b} \geq 6(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})$$

and equality if and only if $\begin{cases} a^2-7ab+b^2=0 \\ b^2-7bc+c^2=0 \\ c^2-7ca+a^2=0 \end{cases}$ it follow that

$\begin{cases} a = \frac{3\sqrt{5}+7}{2}b \\ b = \frac{3\sqrt{5}+7}{2}c \\ c = \frac{3\sqrt{5}+7}{2}a \end{cases}$ this is contradiction then

$$\sum \frac{a^2+b^2}{a+b} + 11\sum \frac{ab}{a+b} > 6(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})$$

60. If $a,b,c \geq 1$ then:

$$\left(\sum \log a \cdot \log b\right)^2 \geq \left(\sum \log^2 a\right)\left(\log^2 abc - 2\sum \log^2 a\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Ștefan – Romania

Take $\begin{cases} x = \ln a \geq 0 \\ y = \ln b \geq 0 \\ z = \ln c \geq 0 \end{cases}$

The inequality $\Leftrightarrow (\sum xy)^2 \geq (\sum x^2)(2\sum xy - \sum x^2)$

$$\Leftrightarrow g^2 \geq (p^2-2g)(4q-p^2)$$

$$\Leftrightarrow g^2 \geq 4gp^2 - p^4 - 8g^2 + 2gp^2 \geq 0$$

$$\Leftrightarrow (2g-p^2)^2 \geq 0 \text{ true}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

Let $\log a = u, \log b = v, \log c = w (u,v,w \geq 0)$

Given inequality $\Leftrightarrow (\sum uv)^2 \geq (\sum u^2)\left((\sum u)^2 - 2(\sum u^2)\right)$

$$\Leftrightarrow \left(\sum uv\right)^2 \geq \left(\sum u^2\right)\left(2\sum uv - \sum u^2\right)$$

Let $\sum uv = x$ and $\sum u^2 = y$

$\therefore$ given inequality $\Leftrightarrow x^2 \geq y(2x-y) \Leftrightarrow (x-y)^2 \geq 0$

which is true (Proved)

N.B.: Inequality is true $\forall\, a, b, c > 0$

Solution 3 by Abdallah El Farissi – Bechar – Algerie

$$\left(\sum \log a \log b - \sum \log^2 a\right)^2 \geq 0$$

then

$$\left(\sum \log a \log b\right)^2 \geq \left(\sum \log^2 a\right)\left(2\sum \log a \log b - \sum \log^2 a\right)$$

$$= \left(\sum \log^2 a\right)\left(\log^2 abc - 2\sum \log^2 a\right)$$

61. If $a, b, c > 0$ then:

$$\frac{(a^2+b^2)a}{a^3+b^3} + \frac{(b^2+c^2)b}{b^3+c^3} + \frac{(c^2+a^2)c}{c^3+a^3} \leq \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{a}}$$

Proposed by Daniel Sitaru – Romania

Solution by Redwane El Mellass – Morroco

Let $f(t) = \frac{t^2+t^6}{1+t^6} - t, t > 0$.

Since $f(t) = \frac{-t(t-1)^2(t^4+t^3+t^2+t+1)}{1+t^6} \leq 0$

$$\therefore \sum f\left(\sqrt{\frac{a}{b}}\right) \leq 0 \Rightarrow \sum \frac{(a^2+b^2)b}{a^3+b^3} \leq \sum \sqrt{\frac{a}{b}}$$

with equality if and only if $\sqrt{\frac{b}{a}} = \sqrt{\frac{c}{b}} = \sqrt{\frac{c}{a}} = 1 \Rightarrow a = b = c > 0.$

62. If $x, y, z > 0$ then:

$$\sum \frac{x^2 - xy + y^2}{y^2z^2} \geq \sum \frac{xy}{y^4 - y^3z + y^2z^2 - yz^3 + z^4}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Ștefan – Romania

$$LHS \geq \sum \frac{xy}{y^2z^2}$$

We'll prove: $\frac{xy}{y^2z^2} \geq \frac{xy}{y^4+z^4-y^3z-yz^3+y^2z^2}$

$\Leftrightarrow y^4 - y^3z + z^4 - yz^3 \geq 0 \Leftrightarrow (y-z)^2(y^2+yz+z^2) \geq 0$ **true**

Equality for $a = b = c$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\textit{1. } x^2 - xy + y^2 = (x^2 + y^2) - xy \underset{Cauchy}{\geq} 2xy - xy = xy$$

$$\textit{2. } (y^2 - z^2)^2 \geq 0$$

$$0 \leq \frac{1}{2}(y^4 + z^4) - y^2z^2 = y^4 + z^4 - \left(\frac{y^4 + z^4}{2}\right) - y^2z^2 =$$

$$= y^4 + z^4 - \frac{(y^2 + z^2)^2}{2} = y^4 + z^4 - \frac{(y^2 + z^2)}{2}(y^2 + z^2) \overset{Cauchy}{\leq}$$

$$\leq y^4 + z^4 - zy \cdot (y^2 + z^2)$$

$$y^4 + z^4 - y^3z - yz^3 \geq 0$$

$$y^4 + z^4 - y^3z + y^2z^2 - yz^3 \geq y^2z^2$$

$$\textit{3. } \sum \frac{x^2 - xy + y^2}{y^2z^2} \overset{(1);(2)}{\geq} \sum \frac{xy}{y^4 - y^3z + y^2z^2 - yz^3 + z^4}$$

Solution 3 by Soumitra Mandal - Chandar Nagore – India

$$\sum_{cyc}\frac{x^2-xy+y^2}{y^2z^2}\ge\sum_{cyc}\frac{xy}{y^4-y^3z+y^2z^2-yz^3+z^4}$$

$$\Leftrightarrow\sum_{cyc}\frac{x^3+y^3}{y^2z^2(x+y)}\ge\sum_{cyc}\frac{xy(y+z)}{y^5+z^5}\Leftrightarrow$$

$$\Leftrightarrow\sum\nolimits_{cyc}\frac{(x^3+y^3)(y^5+z^5)-xy^3z^2(y+z)(x+y)}{y^2z^2(x+y)(y^5+z^5)}\ge 0\ \dots\ (1)$$

Now, $(x^3+y^3)(y^5+z^5)-xy^3z^2(x+y)(y+z)$

$$\ge\frac{1}{4}(x+y)^3\cdot\frac{1}{16}(y+z)^5-xy^3z^2(x+y)(y+z)\ge 0$$

Hence, (1) is established.

$$\therefore\sum_{cyc}\frac{x^2-xy+y^2}{y^2z^2}\ge\sum_{cyc}\frac{xy}{y^4-y^3z+y^2z^2-yz^3+z^4}$$

(proved)

63. If $a,b,c\ge 1$ then:

$$\log(a^b\cdot b^c\cdot c^a)+6\sum\frac{b(1+2a)}{1+4a+a^2}\ge 3(a+b+c)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

Let $f(x)=\ln x+\frac{6(1+2x)}{1+4x+x^2}-3$ **for all** $x\in[1,\infty)$

$$\therefore f'(x)=\frac{1}{x}+\frac{12}{1+4x+x^2}-\frac{12(1+2x)(x+2)}{(1+4x+x^2)^2}$$

$$=\frac{1}{x}-\frac{12(1+x+x^2)}{(1+4x+x^2)^2}=\frac{(x-1)^4}{x(1+4x+x^2)^2}\ge 0 \text{ for all } x\in[1,\infty)$$

now f is continuous on $[1,\infty)$ and $f'(x)\geq 0$ for all $x\in[1,\infty)$

$$\therefore f(x)\geq f(1)=0$$

$\therefore$ for all $a,b,c\in[1,\infty)$ then $af(c)+bf(a)+cf(b)\geq 0$

$$\therefore \ln(a^b b^c c^a)+6\sum_{cyc}\frac{b(1+2a)}{1+4a+a^2}\geq 3(a+b+c)$$

(Proved)

64. If $a,b,c\in(0,\infty)$ then:

$$\sum\frac{(a^3+b^3)(1+ab)}{(a^2+b^4)(a^4+b^2)}\leq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

Proposed by Daniel Sitaru – Romania

Solution by Seyran Ibrahimov – Maasilli – Azerbaidjian

$$\sum\frac{(a^3+b^3)(1+ab)}{(a^2+b^4)(a^4+b^2)}\leq\sum\frac{1}{ab}$$

$$\frac{1}{ab}\geq\frac{(a^3+b^3)(1+ab)}{(a^2+b^4)(a^4+b^2)}$$

$$(a^2+b^4)(a^4+b^2)\geq(ab+a^2b^2)(a^3+b^3)$$

$$a^6+a^2b^2+a^4b^4+b^6\geq a^4b+ab^4+a^5b^2+a^2b^5$$

$$\left.\begin{array}{r}a^6+a^2b^2\geq 2a^4b\\ a^6+a^4b^4\geq 2a^5b^2\\ b^6+a^2b^2\geq 2b^4a\\ b^6+a^4b^4\geq 2b^5a^2\end{array}\right\rangle+\quad\text{proved(AM-GM)}$$

65. If $a,b,c,d>0, abcd=1$ then:

$$\left(\sum a-1\right)\left(\sum a-2\right)\left(\sum a-3\right)\geq 6$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Ștefan – Romania

Inequality (2) $(\sum a - 4)((\sum a)^2 - 2\sum a + 3) \geq 0$

true because $\sum a \geq 4\sqrt[4]{abcd} = 4$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidjian

$$x = a + b + c + d \geq 4$$

$$(x-1)(x-2)(x-3) \geq 6$$

$$(x-1)(x^2 - 5x + 6) \geq 6$$

$$x^3 - 5x^2 + 6x - x^2 + 5x - 6 \geq 6$$

$$x^3 - 6x^2 + 11x - 12 \geq 0$$

$$x^2(x-4) - 2x(x-4) + 3(x-4) \geq 0$$

$$(x-4)(x^2 - 2x + 3) \geq 0 \Rightarrow (+\cdot+) \geq 0$$

therefore $x \geq 4 \Rightarrow$ **Equality**

$$x^2 - 2x + 3x = (x-1)^2 + 2$$

66. If $a, b, c \in \mathbb{R}, abc = 1$ then:

$$\left(a - \frac{1}{a} + \frac{c}{b}\right)\left(b - \frac{1}{b} + \frac{a}{c}\right)\left(c - \frac{1}{c} + \frac{b}{a}\right) < 4$$

Proposed by Daniel Sitaru – Romania

Solution by Redwane El Mellass-Morroco

Let $\Delta(a, b, c \in \mathbb{R}^*) = a - \frac{1}{a} + \frac{c}{b}$.

$$\Delta\,(a, b, c) = c\left(\frac{a}{c} - \left(b - \frac{1}{b}\right)\right) \prod \Delta\,(a, b, c) = \left(\left(\frac{a}{c}\right)^2 - \left(b - \frac{1}{b}\right)^2\right)\left(c^2 + \frac{1}{a^2} - 1\right)$$

(1)

$$\therefore \Delta(b, c, a) = a\left(\frac{b}{a} - \left(c - \frac{1}{c}\right)\right) \Rightarrow \prod \Delta\,(a, b, c) = \left(\left(\frac{b}{a}\right)^2 - \left(c - \frac{1}{c}\right)^2\right)\left(a^2 + \frac{1}{b^2} - 1\right) \quad \textbf{(2)}$$

$$\Delta\,(c,a,b)=b\left(\frac{c}{b}-\left(a-\frac{1}{a}\right)\right)\ \prod\Delta(a,b,c)=\left(\left(\frac{c}{b}\right)^2-\left(a-\frac{1}{a}\right)^2\right)\left(b^2+\frac{1}{c^2}-1\right)\ \textbf{(3)}$$

Now lets study some cases about $|a|,|b|$ and $|c|$:

If $|a|\le|b|\le|c|:\therefore |a|\le 1\le|c|\Rightarrow c^2+\frac{1}{a^2}-1>0$

then (1) $\Rightarrow \prod\Delta(a,b,c)\le\left(\frac{a}{c}\right)^2\left(c^2+\frac{1}{a^2}-1\right)<\left(\frac{a}{c}\right)^2\left(c^2+\frac{1}{a^2}\right)=a^2+\frac{1}{c^2}\le 2.$

By the same idea, if $|b|\le|c|\le|a|$ we use (2) and if $|c|\le|a|\le|b|$ we use (3).

If $|c|\le|b|\le|a|:\therefore |c|\le 1\le|a|\Rightarrow \begin{matrix} a^2+\frac{1}{b^2}-1>0\\ b^2+\frac{1}{c^2}-1>0\end{matrix}$

If $|b|\le 1:\therefore$ ***(2)*** $\Rightarrow\prod\Delta(a,b,c)\le\left(\frac{b}{a}\right)^2\left(a^2+\frac{1}{b^2}-1\right)<\left(\frac{b}{a}\right)^2\left(a^2+\frac{1}{b^2}\right)=b^2+\frac{1}{a^2}\le 2$

If $|b|\ge 1:\therefore$ ***(3)*** $\Rightarrow\prod\Delta\,(a,b,c)\le\left(\frac{c}{b}\right)^2\left(b^2+\frac{1}{c^2}-1\right)<\left(\frac{c}{b}\right)^2\left(b^2+\frac{1}{c^2}\right)=c^2+\frac{1}{b^2}\le 2$

By the same idea, if $|a|\le|c|\le|b|$ we use (1) and (3) and if $|b|\le|a|\le|c|$ we use (1) and (2).

Finally $\prod\Delta\,(a,b,c\in\mathbb{R}^*|abc=1)<2<4.$

67. Let x,y,z be positive real numbers such that: $xyz=1$. Prove that:

$$\frac{x}{\sqrt{2(x^4+y^4)}+4xy}+\frac{y}{\sqrt{2(y^4+z^4)}+4yz}+\frac{z}{\sqrt{2(z^4+x^4)}+4zx}+\frac{2(x+y+z)}{3}\ge\frac{5}{2}\quad \textbf{(1)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Hoang Le Nhat Tung – Hanoi – Vietnam

*** Since Inequality Cauchy – Schwarz. We have:**

$$\left(\sqrt{2(x^4+y^4)}+2xy\right)^2\le(1^2+1^2)(2(x^4+y^4)+4x^2y^2)=$$

$$=4(x^4+2x^2y^2+y^4)=4(x^2+y^2)^2$$

$$\Leftrightarrow\sqrt{2(x^4+y^4)}+2xy\le 2(x^2+y^2)\Leftrightarrow\sqrt{2(x^4+y^4)}+4xy\le 2(x^2+xy+y^2)$$

$$\Leftrightarrow \frac{1}{\sqrt{2(x^4+y^4)}+4xy} \geq \frac{1}{2(x^2+xy+y^2)} \Leftrightarrow$$

$$\Leftrightarrow \frac{x}{\sqrt{2(x^4+y^4)}+4xy} \geq \frac{x}{2(x^2+xy+y^2)} \quad \textbf{(2)}$$

- Similar:

$$\frac{y}{\sqrt{2(x^4+y^4)}+4yz} \geq \frac{y}{2(y^2+yz+z^2)}; \frac{z}{\sqrt{2(z^4+x^4)}+4zx} \geq \frac{z}{2(z^2+zx+x^2)} \quad \textbf{(3)}$$

- Since (2), (3):

$$\Rightarrow \frac{x}{\sqrt{2(x^4+y^4)}+4xy} + \frac{y}{\sqrt{2(y^4+z^4)}+4yz} + \frac{z}{\sqrt{2(z^4+x^4)}+4zx} \geq$$

$$\geq \frac{x}{2(x^2+xy+y^2)} + \frac{y}{2(y^2+yz+z^2)} + \frac{z}{2(z^2+zx+x^2)} \quad \textbf{(4)}$$

- Other, Since inequality Cauchy – Schwarz:

$$\frac{x}{x^2+xy+y^2} + \frac{y}{y^2+yz+z^2} + \frac{z}{z^2+zx+x^2} =$$

$$= \frac{x^2}{x^3+x^2y+xy^2} + \frac{y^2}{y^3+y^2z+yz^2} + \frac{z^2}{z^3+z^2x+zx^2} \geq$$

$$\geq \frac{(x+y+z)^2}{(x^3+x^2y+xy^2)+(y^3+y^2z+yz^2)+(z^3+z^2x+zx^2)}$$

$$\Leftrightarrow \frac{x}{x^2+xy+y^2} + \frac{y}{y^2+yz+z^2} + \frac{z}{z^2+zx+x^2} \geq$$

$$\geq \frac{(x+y+z)^2}{x^2(x+y+z)+y^2(x+y+z)+z^2(x+y+z)}$$

$$\Leftrightarrow \frac{x}{x^2+xy+y^2} + \frac{y}{y^2+yz+z^2} + \frac{z}{z^2+zx+x^2} \geq \frac{(x+y+z)^2}{(x+y+z)(x^2+y^2+z^2)} = \frac{x+y+z}{x^2+y^2+z^2} \quad \textbf{(5)}$$

- Since (4), (5):

$$\Rightarrow \frac{x}{\sqrt{2(x^4+y^4)}+4xy} + \frac{y}{\sqrt{2(y^4+z^4)}+4yz} + \frac{z}{\sqrt{2(z^4+x^4)}+4zx} \geq \frac{x+y+z}{2(x^2+y^2+z^2)}$$

$$\Leftrightarrow \frac{x}{\sqrt{2(x^4+y^4)+4xy}} + \frac{y}{\sqrt{2(y^4+z^4)+4yz}} + \frac{z}{\sqrt{2(z^4+x^4)+4zx}} + \frac{2(x+y+z)}{3} \geq$$

$$\geq \frac{x+y+z}{2(x^2+y^2+z^2)} + \frac{2(x+y+z)}{3} \quad \textbf{(6)}$$

- Since Inequality AM-GM for 5 positive real numbers we have:

$$\frac{x+y+z}{2(x^2+y^2+z^2)} + \frac{2(x+y+z)}{3} =$$

$$= \frac{x+y+z}{2(x^2+y^2+z^2)} + \frac{x+y+z}{6} + \frac{x+y+z}{6} + \frac{x+y+z}{6} + \frac{x+y+z}{6} \geq$$

$$\geq 5 \cdot \sqrt[5]{\left(\frac{x+y+z}{2(x^2+y^2+z^2)}\right) \cdot \left(\frac{x+y+z}{6}\right) \cdot \left(\frac{x+y+z}{6}\right) \cdot \left(\frac{x+y+z}{6}\right) \cdot \left(\frac{x+y+z}{6}\right)}$$

$$\Leftrightarrow \frac{x+y+z}{2(x^2+y^2+z^2)} + \frac{2(x+y+z)}{3} \geq 5 \cdot \sqrt[5]{\frac{(x+y+z)^5}{6^4 \cdot 2(x^2+y^2+z^2)}}$$

- Since inequality: $(xy+yz+zx)^2 \geq 3xyz(x+y+z)$ **and supposed:**

$xyz = 1$**. We have:**

$$(x+y+z)(x^2+y^2+z^2) = 1 \cdot (x+y+z)(x^2+y^2+z^2) =$$

$$= xyz(x+y+z)(x^2+y^2+z^2) \leq \frac{(xy+yz+zx)^2}{3} \cdot (x^2+y^2+z^2) =$$

$$= \frac{(x^2+y^2+z^2)(xy+yz+zx)(xy+yz+zx)}{3}$$

$$\Leftrightarrow (x+y+z)(x^2+y^2+z^2) \leq \frac{(x^2+y^2+z^2)(xy+yz+zx)(xy+yz+zx)}{4} \quad \textbf{(8)}$$

- Other, Inequality AM-GM for 3 positive real numbers:

$$(x+y+z)^2 = (x^2+y^2+z^2) + (xy+yz+zx) + (xy+yz+zx) \geq$$

$$\geq 3 \cdot \sqrt[3]{(x^2+y^2+z^2) \cdot (xy+yz+zx) \cdot (xy+yz+zx)}$$

$$\Leftrightarrow (x+y+z)^6 \geq 27(x^2+y^2+z^2)(xy+yz+zx)(xy+yz+zx)$$

$$\Leftrightarrow (x^2+y^2+z^2)(xy+yz+zx)(xy+yz+zx) \le \frac{(x+y+z)^6}{27} \quad \textbf{(9)}$$

- Since (8), (9): $\Rightarrow (x+y+z)(x^2+y^2+z^2) \le \frac{(x+y+z)^6}{27\cdot 3} = \frac{(x+y+z)^6}{81}$

$$\Leftrightarrow (x^2+y^2+z^2) \le \frac{(x+y+z)^5}{81} \Leftrightarrow \frac{(x+y+z)^5}{x^2+y^2+z^2} \ge 81 \quad \textbf{(10)}$$

- Since (7), (10): $\Rightarrow \frac{x+y+z}{2(x^2+y^2+z^2)} + \frac{2(x+y+z)}{3} \ge 5\cdot\sqrt[5]{\frac{81}{6^4\cdot 2}} = 5\cdot\sqrt[5]{\frac{3^4}{2\cdot 2^4\cdot 3^4}} = 5\cdot\sqrt[5]{\frac{1}{2^5}} = \frac{5}{2}$ **(11)**

- Since (6), (11):

$$\Rightarrow \frac{x}{\sqrt{2(x^4+y^4)}+4xy} + \frac{y}{\sqrt{2(y^4+z^4)}+4yz} + \frac{z}{\sqrt{2(z^4+x^4)}+4zx} + \frac{2(x+y+z)}{3} \ge \frac{5}{2}$$

$\Rightarrow$ Inequality (1) True and we get the desired result.

+ Equality occurs if:

$$\Leftrightarrow \begin{cases} x,y,z>0;\ xyz=1 \\ \sqrt{2(x^4+y^4)}=2xy;\ \sqrt{2(y^4+z^4)}=2yz;\ \sqrt{2(z^4+x^4)}=2zx \\ \dfrac{1}{x^2+xy+y^2}=\dfrac{1}{y^2+yz+z^2}=\dfrac{1}{z^2+xz+x^2} \\ \dfrac{x+y+z}{2(x^2+y^2+z^2)}=\dfrac{x+y+z}{6} \\ xy=yz=zx \\ x^2+y^2+z^2=xy+yz+zx \end{cases} \Leftrightarrow x=y=z=1$$

68. If $x,y,z>0$ then:

$$\sum\frac{x}{7x+5(y+z)} \le \frac{3}{17}, \qquad \frac{3}{19} \le \sum\frac{x}{5x+7(y+z)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdallah El Farissi-Bechar-Algerie

Let $f(a) = \frac{a}{2a+5(x+y+z)}$, $a>0$ **we have**

$f''(a) = \frac{-20(x+y+z)}{(2a+5(x+y+z))^3} < 0$ **then f is concave function,**

$$\sum\frac{x}{7x+5(y+z)}=3\left(\frac{f(x)+f(y)+f(z)}{3}\right)\le 3f\left(\frac{x+y+z}{3}\right)=\frac{3}{17}$$

Let $g(a)=\frac{a}{7(x+y+z)-2a}$, $a<\frac{7}{2}(x+y+z)$, we have

$g''(a)=\frac{28(x+y+z)}{(7(x+y+z)-2a)^3}\ge 0$, then g is convex function,

$$\sum\frac{x}{5x+7(y+z)}=3\left(\frac{f(x)+f(y)+f(z)}{3}\right)\ge 3f\left(\frac{x+y+z}{3}\right)=\frac{3}{19}$$

Solution 2 by Ravi Prakash-New Delhi-India

Consider

$$E_1=\frac{x}{5(y+z)+7x}-\frac{1}{17}=\frac{17x-7x-5(y+z)}{17[7x+5(y+z)]}=\frac{5(x-y)+5(x-z)}{17(7x+5y+5z)}$$

Similarly,

$E_2=\frac{5(y-x)+5(y-z)}{17(7y+5x+5z)}$ and $E_2=\frac{5(z-x)+5(z-y)}{17(17z+5x+5y)}$

Now, $\frac{17}{5}(E_1+E_2+E_3)$

$$=(x-y)\left[\frac{1}{7x+5y+5z}-\frac{1}{7y+5x+5z}\right]+(y-z)\left[\frac{1}{7y+5x+5z}-\frac{1}{7z+5x+5y}\right]+$$

$$+(z-x)\left[\frac{1}{7z+5x+5y}-\frac{1}{7x+5y+5z}\right]=\frac{(-2)(x-y)^2}{(7x+5y+5z)(5x+7y+5z)}+$$

$$+\frac{(-2)(y-z)^2}{(5x+7y+5z)(5x+5y+7z)}+\frac{(-2)(z-x)^2}{(5x+5y+7z)(7x+5y+5z)}\le 0$$

$$\therefore \sum\frac{x}{7x+5(y+z)}\le\frac{3}{17}$$

Next, consider

$$F_1=\frac{1}{19}-\frac{x}{5x+7(y+z)}=\frac{7(y+z)+5x-19x}{19(5x+7y+7z)}=\frac{7(y-x)+7(z-x)}{19(5x+7y+7z)}$$

Similarly,

$$F_2=\frac{7(z-y)+7(x-y)}{19(5y+7x+7z)},F_3=\frac{7(x-z)+7(y-z)}{19(5z+7x+7y)}$$

$$\therefore\frac{19}{7}(F_1+F_2+F_3)$$

$$=(y-x)\left[\frac{1}{5x+7y+7z}-\frac{1}{5y+7x+7z}\right]+(x-z)\left[\frac{1}{5z+7x+7y}-\frac{1}{5x+7y+7z}\right]+$$

$$+(z-y)\left[\frac{1}{5y+7x+7z}-\frac{1}{5z+7x+7y}\right]=\frac{-2(y-x)^2}{(5x+7y+7z)(5y+7x+7z)}+$$

$$+\frac{(-2)(x-z)^2}{(5z+7x+7y)(5x+7y+7z)}+\frac{(-2)(z-y)^2}{(5y+7x+7z)(5z+7x+7y)}\le 0$$

$$\Rightarrow\sum\frac{x}{5x+7(y+z)}\ge\frac{3}{19}$$

Solution 3 by Imad Zak-Saida-Lebanon

Let $f(x)=\frac{x}{2x+15}$ for $x\in(0;3)$ we have:

$$f(x)-\left(\frac{15x}{289}+\frac{2}{289}\right)=\frac{-30(x-1)^2}{289(2x+5)}\le 0\Rightarrow f(x)\le\frac{15x}{289}+\frac{2}{289}\quad\dots\ (1)$$

Let $g(x)=\frac{x}{21-2x}$ for $x\in(0;3)$ we have:

$$g(x)-\left(\frac{21x}{361}-\frac{2}{361}\right)=\frac{42(x-1)^2}{361(21-2x)}\ge 0\Rightarrow g(x)\ge\frac{21x}{361}-\frac{2}{361}\quad\dots\ (2)$$

Both inqualities are homogeneous, so let $x+y+z=3$

Ineq (1) $\Leftrightarrow\sum f(x)\overset{?}{\le}\frac{3}{17}$ acc. to (1) we may write!

$$\sum f(x)\le\sum\left(\frac{15x}{289}+\frac{2}{289}\right)=\frac{15(x+y+3)}{289}+\frac{6}{289}=\frac{45+6}{289}=\frac{3}{17}\quad\text{Q.E.D.}$$

Ineq (2) $\Leftrightarrow\sum g(x)\overset{?!}{\ge}\frac{3}{19}$ acc. to (2) we may affirm

$$\sum g(x)\ge\sum\left(\frac{21x}{361}-\frac{2}{361}\right)=\frac{21(3)-6}{361}=\frac{57}{361}=\frac{3}{19}\quad\text{Q.E.D.}$$

$\ll=\gg$ when $x=y=z=1$ or $x=y=z$ in general

N.B.: $y = \frac{15x}{289} + \frac{2}{28}$ is the tangent, at $x = 1$ to (G_f)

$y = \frac{21x}{361} - \frac{2}{361}$ // // // // // // to (G_g)

69. Let a, b, c be positive real numbers such that: $a + b + c = 3$.

Prove that:

$$\frac{a^3}{b(2b^2 - bc + 2c^2)^2} + \frac{b^3}{c(2c^2 - ca + 2a^2)^2} + \frac{c^3}{a(2a^2 - ab + 2b^2)^2} \geq \frac{1}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by proposer

*** We have inequality:**

$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2)$ **(2)**

- *(2)* $a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2)$

$\Leftrightarrow a^4 + b^4 + c^4 + abc(a + b + c) - ab(a^2 + b^2) - bc(b^2 + c^2) - ca(c^2 + a^2) \geq 0$

$\Leftrightarrow a^2(a^2 - ab - ac + bc) + b^2(b^2 - bc - ba + ca) + c^2(c^2 - ca - cb + ab) \geq 0$

$\Leftrightarrow a^2(a - b)(a - c) + b^2(b - a)(b - c) + c^2(c - a)(c - b) \geq 0$ **(3)**

***** $a \geq b \geq c > 0$.

+ We have: $\begin{cases} c \leq a \\ c \leq b \end{cases} \Leftrightarrow \begin{cases} c - a \leq 0 \\ c - b \leq 0 \end{cases} \Rightarrow (c - a)(c - b) \geq 0 \Leftrightarrow c^2(c - a)(c - b) \geq 0$ **(4)**

+ We have: $a^2(a - b)(a - c) + b^2(b - a)(b - c) = (a - b)[a^2(a - c) - b^2(b - c)]$

$\Leftrightarrow a^2(a - b)(a - c) + b^2(b - a)(b - c) = (a - b)[(a^3 - b^3) - c(a^2 - b^2)]$

$= (a - b)[(a - b)(a^2 + ab + b^2) - c(a - b)(a + b)]$

$= (a - b) \cdot (a - b)(a^2 + ab + b^2 - ac - bc)$

$= (a - b)^2(a^2 + ab + b^2 - ac - bc)$ **(5)**

$a \geq b \geq c > 0 \Rightarrow a - c \geq 0; b - c \geq 0$

+ Therefore: $a^2 + ab + b^2 - ac - bc = a(a - c) + b(b - c) + ab \geq ab > 0$.

$(a - b)^2 \geq 0; \forall a, b \in R$

$$\Rightarrow (a-b)^2(a^2+ab+b^2-ac-bc) \geq 0 \quad \textbf{(5):}$$

$$\Rightarrow a^2(a-b)(a-c)+b^2(b-a)(b-c) \geq 0 \qquad \textbf{(6)}$$

- Since (4), (6): $\Rightarrow a^2(a-b)(a-c)+b^2(b-a)(b-c)+c^2(c-a)(c-b) \geq 0$

$\Rightarrow$ **Inequality (3) True** $\Rightarrow$ (2) **True.**

- Since inequality AM-GM for 2 positive real numbers:

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) \geq ab\cdot 2ab+bc\cdot 2bc+ca\cdot 2ca =$$
$$= 2(a^2b^2+b^2c^2+c^2a^2) \qquad \textbf{(7)}$$

*** Since (7):**

$$\Rightarrow a^4+b^4+c^4+abc(a+b+c) \geq 2(a^2b^2+b^2c^2+c^2a^2)$$
$$\Leftrightarrow a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2) \geq 4(a^2b^2+b^2c^2+c^2a^2)-abc(a+b+c)$$
$$\Leftrightarrow (a^2+b^2+c^2)^2 \geq 4(a^2b^2+b^2c^2+c^2a^2)-abc(a+b+c)$$
$$\Leftrightarrow \frac{(a^2+b^2+c^2)^2}{4(a^2b^2+b^2c^2+c^2a^2)-abc(a+b+c)} \geq 1 \qquad \textbf{(8)}$$

- Since inequality Cauchy – Schwarz. We have:

$$\frac{a^3}{b(2b^2-bc+2c^2)^2}+\frac{b^3}{c(2c^2-ca+2a^2)^2}+\frac{c^2}{a(2a^2-ab+2b^2)^2}$$
$$=\frac{\left(\frac{a^2}{2b^2-bc+2c^2}\right)^2}{ab}+\frac{\left(\frac{b^2}{2c^2-ca+2a^2}\right)^2}{bc}+\frac{\left(\frac{c^2}{2a^2-ab+2b^2}\right)^2}{ca} \geq$$
$$\geq \frac{\left(\frac{a^2}{2b^2-bc+2c^2}+\frac{b^2}{2c^2-ca+2a^2}+\frac{c^2}{2a^2-ab+2b^2}\right)^2}{ab+bc+ca} \qquad \textbf{(9)}$$

- Other, since inquality Cauchy – Schwarz:

$$\frac{a^2}{2b^2-bc+2c^2}+\frac{b^2}{2c^2-ca+2a^2}+\frac{c^2}{2a^2-ab+2b^2}$$
$$=\frac{a^4}{2a^2b^2-a^2bc+2c^2a^2}+\frac{b^4}{2b^2c^2-ab^2c+2a^2b^2}+\frac{c^4}{2c^2a^2-abc^2+2b^2c^2} \geq$$
$$\geq \frac{(a^2+b^2+c^2)^2}{(2a^2b^2-a^2bc+2c^2a^2)+(2b^2c^2-ab^2c+2a^2b^2)+(2c^2a^2-abc^2+2b^2c^2)}$$
$$\Leftrightarrow \frac{a^2}{2b^2-bc+2c^2}+\frac{b^2}{2c^2-ca+2a^2}+\frac{c^2}{2a^2-ab+2b^2} \geq$$

$$\geq \frac{(a^2+b^2+c^2)^2}{4(a^2b^2+b^2c^2+c^2a^2)-abc(a+b+c)} \qquad (10)$$

- Since (8), (9): $\Rightarrow \frac{a^2}{2b^2-bc+2c^2}+\frac{b^2}{2c^2-ca+2a^2}+\frac{c^2}{2a^2-ab+2b^2} \geq 1 \qquad (11)$

- Since (9), (11): $\Rightarrow \frac{a^3}{b(2b^2-bc+2c^2)^2}+\frac{b^3}{c(2c^2-ca+2a^2)^2}+\frac{c^3}{a(2a^2-ab+2b^2)^2} \geq \frac{1^2}{ab+bc+ca}$

$$\Leftrightarrow \frac{a^3}{b(2b^2-bc+2c^2)^2}+\frac{b^3}{c(2c^2-ca+2a^2)^2}+\frac{c^3}{a(2a^2-ab+2b^2)^2} \geq \frac{1}{ab+bc+ca} \quad (12)$$

- Since inequality AM-GM we have:

$$a^2+b^2+c^2=\frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2} \geq \frac{2ab}{2}+\frac{2bc}{2}+\frac{2ca}{2}=ab+bc+ca$$

$$\Leftrightarrow a^2+b^2+c^2 \geq ab+bc+ca \Leftrightarrow a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca)$$

$$\Leftrightarrow (a+b+c)^2 \geq 3(ab+bc+ca) \Leftrightarrow ab+bc+ca \leq \frac{(a+b+c)^2}{3}=\frac{3^2}{3}=3$$

(because $a+b+c=3$) (13)

- Since (12), (13): $\Rightarrow \frac{a^3}{b(2b^2-bc+2c^2)^2}+\frac{b^3}{c(2c^2-ca+2a^2)^2}+\frac{c^3}{a(2a^2-ab+2b^2)^2} \geq \frac{1}{3}$

$\Rightarrow$ Inequality (1) true and we get the desired result.

+ Equality occurs if: $\begin{cases} a+b+c=3 \\ a=b=c>0 \\ \dfrac{\frac{a^2}{2b^2-bc+2c^2}}{ab}=\dfrac{\frac{b^2}{2c^2-ca+2a^2}}{bc}=\dfrac{\frac{c^2}{2a^2-ab+2b^2}}{ca} \\ \frac{1}{2b^2-bc+2c^2}=\frac{1}{2c^2-ca+2a^2}=\frac{1}{2a^2-ab+2b^2} \end{cases} \Leftrightarrow a=b=c=1.$

Solution 2 by Anh Tai Tran – Hanoi – Vietnam

$$\sum \frac{a^3}{b(2b^2-bc+2c^2)^2}=\sum \frac{\left(\frac{a^2}{2b^2-bc+2c^2}\right)^2}{ab} \geq \frac{\left(\sum \frac{a^2}{2b^2-bc+2c^2}\right)^2}{\sum ab}$$

$$\geq \frac{\left(\sum \frac{a^2}{2b^2-bc+2c^2}\right)^2}{\frac{(a+b+c)^2}{3}} \geq \frac{T^2}{3}$$

We will prove:

$$T = \sum \frac{a^2}{2b^2 - bc + 2c^2} \ge 1$$

$$T \ge \frac{(\sum a^2)^2}{4\sum(bc)^2 - abc\sum a} \ge 1$$

$$\leftrightarrow \sum a^4 + abc\sum a \ge \sum (ab)^2$$

It's true by Shur and AM-GM

$$LHS \ge \sum ab(a^2 + b^2) \ge 2\sum (ab)^2$$

70. If $a, b, c > 0$ then:

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \left(\frac{a+b+c}{ab+bc+ca}\right)^2\right) \ge \frac{36}{abc\sqrt[3]{abc}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$LHS = \frac{(ab+bc+ca)^2}{a^2b^2c^2}\left\{\frac{\sum a^2b^2}{a^2b^2c^2} + \frac{(a+b+c)^2}{(ab+bc+ca)^2}\right\}$$

$$= \frac{(\sum ab)^2(\sum a^2b^2)}{a^4b^4c^4} + \frac{(a+b+c)^2}{a^2b^2c^2}$$

$$\overset{A-G}{\ge} \frac{\left(3\sqrt[3]{a^2b^2c^2}\right)^2 \cdot \left(3\sqrt[3]{a^4b^4c^4}\right)}{a^4b^4c^4} + \frac{\left(3\sqrt[3]{abc}\right)^2}{a^2b^2c^2}$$

$$= \frac{27\sqrt[3]{a^8b^8c^8}}{a^4b^4c^4} + \frac{9}{\sqrt[3]{a^4b^4c^4}} = \frac{27}{\sqrt[3]{a^4b^4c^4}} + \frac{9}{\sqrt[3]{a^4b^4c^4}}$$

$$= \frac{36}{\sqrt[3]{a^4b^4c^4}} = \frac{36}{abc\sqrt[3]{abc}}$$

(Proved)

Solution 2 by proposer

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\left(\frac{a+b+c}{ab+bc+ca}\right)^2=$$

$$=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2-2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+\left(\frac{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\right)^2=$$

$$=\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^4-2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2+\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)^2}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}=$$

$$=\left(\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2-\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\right)^2=\frac{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)^2}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}$$

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\left(\frac{a+b+c}{ab+bc+ca}\right)^2\right)=$$

$$=\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)^2\geq\left(6\sqrt[6]{\frac{1}{a^2}\cdot\frac{1}{b^2}\cdot\frac{1}{c^2}\cdot\frac{1}{ab}\cdot\frac{1}{bc}\cdot\frac{1}{ca}}\right)^2=$$

$$=36\sqrt[3]{\frac{1}{a^4b^4c^4}}=\frac{36}{abc\sqrt[3]{abc}}$$

71. Let a,b,c be positive real numbers such that $a+b+c=3$.

Prove that:

$$\frac{a^2}{\sqrt{b^4+4}}+\frac{b^2}{\sqrt{c^4+4}}+\frac{c^2}{\sqrt{a^4+4}}>\frac{3}{5}$$

Proposed by Anish Ray-Santaragachi-India

Solution 1 by Anish Ray-Santaragachi-India

Given that $a, b, c > 0$

We can write,

$$(b^2+2)^2 > b^4+4 \Rightarrow \frac{1}{b^4+4} > \frac{1}{(b^2+2)^2} \Rightarrow \frac{1}{\sqrt{b^4+4}} > \frac{1}{b^2+2}$$

So,

$$\sum_{cyc} \frac{a^2}{\sqrt{b^4+4}} > \sum_{cyc} \frac{a^2}{b^2+2}$$

Now, By Bergstorm's Lemma, we get that

$$\sum_{cyc} \frac{a^2}{b^2+2} \ge \frac{(a+b+c)^2}{(a^2+b^2+c^2+6)}$$

Now, for $a, b, c > 0$

$$(a+b+c)^2 > a^2+b^2+c^2 \Rightarrow (a+b+c)^2+6 > a^2+b^2+c^2+6$$

so,

$$\frac{1}{a^2+b^2+c^2+6} > \frac{1}{(a+b+c)^2+6} = \frac{1}{15}$$

which implies,

$$\frac{(a+b+c)^2}{a^2+b^2+c^2+6} > \frac{(a+b+c)^2}{(a+b+c)^2+6} = \frac{9}{15} = \frac{3}{5}$$

therefore,

$$\sum_{cyc} \frac{a^2}{b^2+2} \ge \frac{3}{5}$$

Thus,

$$\sum_{cyc}\frac{a^2}{\sqrt{b^4+4}}>\frac{3}{5}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $a,b,c\geq 0$ and $a+b+c=3$ then

$$\sum_{cyc}\frac{a^2}{\sqrt{b^4+4}}>\frac{3}{5}$$

$$\sum_{cyc}\frac{a^2}{\sqrt{b^4+4}}=\sum_{cyc}\frac{a^2}{\sqrt{(b^2-2b+2)(b^2+2b+2)}}$$

$$\geq\sum_{cyc}\frac{a^2}{b^2+2}=\sum_{cyc}\frac{a^4}{a^2b^2+2a^2}\geq\frac{(a^2+b^2+c^2)^2}{\sum_{cyc}a^2b^2+2\sum_{cyc}a^2}$$

$\therefore$ we need to prove,

$$5\left(\sum_{cyc}a^2\right)^2>3\sum_{cyc}a^2b^2+6\sum_{cyc}a^2$$

$$\Leftrightarrow 5\left(\sum_{cyc}a^2\right)^2>3\sum_{cyc}a^2b^2+\frac{2}{3}(a+b+c)^2\sum_{cyc}a^2$$

$$\Leftrightarrow 15\left(\sum_{cyc}a\right)^2>9\sum_{cyc}a^2b^2+2\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}a^2\right)$$

$$\Leftrightarrow 15\left(\sum_{cyc}a^4\right)+21\sum_{cyc}a^2b^2>2\left(\sum_{cyc}a^2\right)^2+4\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)$$

$$\Leftrightarrow 13\sum_{cyc}a^4+17\sum_{cyc}a^2b^2>4\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)\ldots(1)$$

we need to prove (1). Now we have

$$13\sum_{cyc} a^4 + 17\sum_{cyc} a^2b^2 = 11\sum_{cyc} a^4 + 11\sum_{cyc} a^2b^2 + 2\left(\sum_{cyc} a^2\right)^2 + 2\left(\sum_{cyc} ab\right)^2 - 12abc$$

$$\ge 22\sum_{cyc} a^2b^2 + 4\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right) - 12abc$$

since,

$$\sum_{cyc} a^4 \ge \sum_{cyc} a^2b^2$$

$$\ge 22abc(a+b+c) + 4\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right) - 12abc = 4\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right) + 54abc$$

$$> 4\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)$$

hence statement (1) is true.

$$\therefore \sum_{cyc} \frac{a^2}{\sqrt{b^2+4}} > \frac{3}{5}$$

(proved)

Solution 3 by Henry Ricardo - New York – USA

Without loss of generality we may assume that $a \ge b \ge c$. It follows that

$a^2 \ge b^2 \ge c^2$ and $\frac{1}{\sqrt{a^2+4}} \le \frac{1}{\sqrt{b^2+4}} \le \frac{1}{\sqrt{c^2+4}}$.

Now the Rearrangement Inequality give us

$$\frac{a^2}{\sqrt{b^2+4}} + \frac{b^2}{\sqrt{c^2+4}} + \frac{c^2}{\sqrt{a^2+4}} \ge \frac{a^2}{\sqrt{a^2+4}} + \frac{a^2}{\sqrt{a^2+4}} + \frac{b^2}{\sqrt{b^2+4}} + \frac{c^2}{\sqrt{c^2+4}}$$

It can be seen graphically (and proved with some tedious algebra/analysis) that the curve given by $y=\frac{x^2}{\sqrt{x^2+4}}$ lies on or above the tangent line to the curve at, $x=1, y=\frac{9\sqrt{5}}{25}(x-1)+\frac{\sqrt{5}}{5}$, on the interval $(0,3)$. Thus we have

$$\sum_{cyclic}\frac{a^2}{\sqrt{b^2+4}}\ge\sum_{cyclic}\left(\frac{9\sqrt{5}}{25}(a-1)+\frac{\sqrt{5}}{5}\right)$$

$$=\frac{9\sqrt{5}}{25}\sum_{cyclic}a-\frac{12\sqrt{5}}{25}$$

$$=\frac{27\sqrt{5}}{25}-\frac{12\sqrt{5}}{25}=\frac{3\sqrt{5}}{5}>\frac{3}{5}.$$

72. Prove that: $a+b+c+d\le\frac{a^5+b^5+c^5+d^5}{abcd}$, $(a,b,c,d>0)$

Proposed by Daniel Sitaru – Romania

Solution by Kunihiko Chikaya – Tokyo – Japan

$$\frac{a^5+a^5+b^5+c^5+d^5}{5}\ge\sqrt[5]{a^5a^5b^5c^5d^5}=a^2bcd$$

$$\frac{a^5+b^5+b^5+c^5+d^5}{5}\ge\sqrt[5]{a^5b^5b^5c^5d^5}=ab^2cd$$

$$\frac{a^5+b^5+c^5+c^5+d^5}{5}\ge\sqrt[5]{a^5b^5c^5c^5d^5}=abc^2d$$

$$\frac{a^5+b^5+c^5+d^5+d^5}{5}\ge\sqrt[5]{a^5b^5c^5d^5d^5}=abcd^2$$

$$\oplus\quad\frac{5(a^5+b^5+c^5+d^5)}{5}\ge abcd(a+b+c+d)$$

73. If $x, y, z > 0$ then:

$$2\left(\frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}\right)+18>3\left(\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)+3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$x,y,z>0\Rightarrow 2\sum\frac{x^3}{y^3}+18>3\sum\frac{x^2}{y^2}+3\sum\frac{x}{y}$$

$$\text{Let } f(t)=2t^3-3t^2-3t+6\quad\forall\, t>0$$

$$\text{Now, } t^3+t^3+\frac{24}{5}\overset{A-G}{\geq}3t^2\sqrt[3]{\frac{24}{5}}>5t^2$$

$$\therefore 2t^3+\frac{24}{5}-3t^2-3t+\frac{6}{5}>2t^2-3t+\frac{6}{5}$$

$$=2\left(t^2-\frac{3}{2}t+\frac{3}{5}\right)>2\left(t^2-\frac{3}{2}t+\frac{9}{16}\right)$$

$$=2\left(t-\frac{3}{4}\right)^2>0\Rightarrow 2t^3-3t^2-3+6>0\quad\forall t>0$$

$$\Rightarrow 2t^3+6>3t^2+3t\quad\forall\, t>0$$

$$\therefore 2\frac{x^3}{y^3}+6>3\frac{x^2}{y^2}+3\frac{x}{y}\quad (1)\quad \left(\text{putting } t=\frac{x}{y}\right)$$

$$2\frac{y^3}{z^3}+6>3\frac{y^2}{z^2}+3\frac{y}{z}\quad (2)\quad \left(\text{putting } t=\frac{y}{z}\right)$$

$$2\frac{z^3}{x^3}+6>3\frac{z^2}{x^2}+3\frac{z}{x}\quad (3)\quad \left(\text{putting } t=\frac{z}{x}\right)$$

$$(1)+(2)+(3)\Rightarrow 2\sum\frac{x^3}{y^3}+18>3\sum\frac{x^2}{y^2}+3\sum\frac{x}{y}$$

Solution 2 by Myagmarsuren Yadamsuren – Darkhan – Mongolia

$$\text{If } x,y,z>0$$

$$2\cdot\sum\frac{x^3}{y^2}+18>3\cdot\sum\frac{x^2}{y^2}+3\cdot\sum\frac{x}{y}$$

$$2t^3+6>3t^2+3t \quad \text{(ASSURE)}$$

$$2t^3+6=\left(\frac{1}{2}t^3+\frac{1}{2}t^3+4\right)+(t^3+1+1)\overset{Cauchy}{\geq}$$

$$\geq 3t^2+3t$$

$$t_1=\frac{x}{y};t_2=\frac{y}{z};t_3=\frac{z}{x}$$

$$2\cdot\sum_{i=1}^{3}t_i^3+18>3\cdot\sum_{i=1}^{3}t_i^2+3\cdot\sum_{i=1}^{3}t_i$$

Solution 3 by Aditya Narayan Sharma – Kanchrapara – India

$$2\left(\frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}\right)+18>3\left(\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}+\frac{x}{y}+\frac{y}{z}+\frac{y}{x}\right)$$

Let $\frac{x}{y}=a,\frac{y}{z}=b,\frac{z}{x}=c$

$$\therefore abc=1 \text{ with } a,b,c>0$$

To prove, $2(a^3+b^3+c^3+9)-3(a^2+b^2+c^2+a+b+c)>0$

Now define,

$$f(a,b,c)=2(a^3+b^3+c^3+z)-3(a^2+b^2+c^2+a+b+c)$$

Evaluating first partial derivatives,

$$\frac{\partial f}{\partial a}=6a^2-6a-3=0 \quad (1)$$

$$\frac{\partial f}{\partial b}=6b^2-6b-3=0 \quad (2)$$

$$\frac{\partial f}{\partial c}=6c^2-6c-3=0 \quad (3)$$

All of the equations are identical, thus we have same solution for a,b,c.

$$\therefore a=b=c$$

Now to justify that $a = b = c$ is a minimum we consider the hessian matrix,

$$H\big(f(a,b,c)\big) = \begin{bmatrix} 12a-b & 0 & 0 \\ 0 & 12b-6 & 0 \\ 0 & 0 & 12c-b \end{bmatrix}$$

$$= 6\begin{bmatrix} 2a-1 & 0 & 0 \\ 0 & 2b-1 & 0 \\ 0 & 0 & 2c-1 \end{bmatrix}$$

Now, from the solutions of (1), (2), (3) we know,

$$2a-1 > 0, 2b-1 > 0, 2c-1 > 0$$

because we neglect the negative roots because $a, b, c > 0$

Solution 4 by Seyran Ibrahimov – Maasilli – Azerbaidian

Equivalent to: $\frac{x}{y} = a; \frac{y}{z} = c; \frac{z}{x} = b$

$$\sum 2a^3 + 6 > \sum 3a^2 + 3a$$

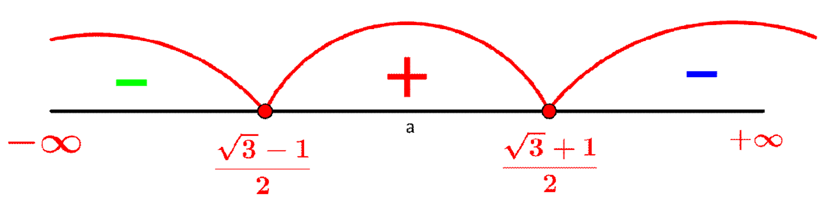
$$2a^3 + 6 > 3a^2 + 3a$$

$$3a^2 + 3a - 2a^3 < 6$$

$$f(a) = 3a^2 + 3a - 2a^3$$

$$f'(a) = -6a^2 + 6a + 3 = 0$$

$$a_1 = \frac{\sqrt{3}+1}{2}; a_2 = \frac{\sqrt{3}-1}{2}$$



$$\left(-\infty; \frac{\sqrt{3}-1}{2}\right) \downarrow \text{ and } \left(\frac{\sqrt{3}+1}{2}; +\infty\right) \downarrow$$

$$f(a) \Rightarrow \left[\frac{\sqrt{3}-1}{2};\frac{\sqrt{3}+1}{2}\right]\uparrow$$

$$f_{min}\left(\frac{\sqrt{3}-1}{2}\right) \Rightarrow f\left(\frac{\sqrt{3}-1}{2}\right) = \frac{8-3\sqrt{3}}{2}$$

$$f_{max}\left(\frac{\sqrt{3}+1}{2}\right) \Rightarrow f\left(\frac{\sqrt{3}+1}{2}\right) = \frac{4+3\sqrt{3}}{2} \overset{?}{<} 6$$

(Proved)

74. If $a, b, c \in (0, \infty)$ and $a + b + c = 1$ then

$$\sum_{cyc}\frac{1}{a} \geq (\lambda+3)\left(\sum_{cyc}\frac{1}{\lambda+a}\right), \lambda \in [1,3]$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Using Bergström inequality $\frac{x^2}{a}+\frac{y^2}{b} \geq \frac{(x+y)^2}{a+b}$, where $x, y, a, b > 0$ with equality if and only if

$\frac{x}{a}=\frac{y}{b}$ we obtain $\frac{1^2}{a}+\frac{3^2}{\lambda} \geq \frac{(1+3)^2}{a+\lambda} \Leftrightarrow \frac{1}{a} \geq \frac{16}{\lambda+a}-\frac{9}{\lambda}$.

It follows $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 16\left(\frac{1}{\lambda+a}+\frac{1}{\lambda+b}+\frac{1}{\lambda+c}\right)-\frac{27}{\lambda}$.

It is enough to prove that

$$16\left(\frac{1}{\lambda+a}+\frac{1}{\lambda+b}+\frac{1}{\lambda+c}\right)-\frac{27}{\lambda} \geq (3+\lambda)\left(\frac{1}{\lambda+a}+\frac{1}{\lambda+b}+\frac{1}{\lambda+c}\right)$$

$$\Leftrightarrow (13-\lambda)\left(\frac{1}{\lambda+a}+\frac{1}{\lambda+b}+\frac{1}{\lambda+c}\right) \geq \frac{27}{\lambda}$$

$\Leftrightarrow \frac{1}{\lambda+a}+\frac{1}{\lambda+b}+\frac{1}{\lambda+c} \geq \frac{27}{\lambda(13-\lambda)}$, which follows from means inequality $A_m \geq H_m$, or from Bergström inequality for three reports.

Indeed $\frac{1}{\lambda+a}+\frac{1}{\lambda+b}+\frac{1}{\lambda+c}\ge\frac{9}{3\lambda+a+b+c}=\frac{9}{3\lambda+1}\ge\frac{27}{\lambda(13-\lambda)}$, where the latter inequality is equivalent to $\frac{9}{3\lambda+1}\ge\frac{27}{\lambda(13-\lambda)}\Leftrightarrow\lambda(13-\lambda)\ge3(3\lambda+1)\Leftrightarrow\lambda^2-4\lambda+3\le0\Leftrightarrow\lambda\in[1,3]$.

The equality holds if and only if $\frac{1}{a}=\frac{3}{\lambda},\frac{1}{b}=\frac{3}{\lambda},\frac{1}{c}=\frac{3}{\lambda},a+b+c=1$, wherefrom we obtain $\lambda=1$ and $a=b=c=\frac{1}{3}$.

Solution 2 by Soumitra Mandal - Chandar Nagore – India

$$(\lambda+3)\left(\sum_{cyc}\frac{1}{a+\lambda}\right)=(\lambda+3)\left(\sum_{cyc}\frac{1}{\lambda(a+b+c)+a}\right)$$

$$=(\lambda+3)\left(\sum_{cyc}\frac{1}{(1+\lambda)a+\lambda b+\lambda c}\right)$$

$$\le\frac{\lambda+3}{(1+3\lambda)^2}\sum_{cyc}\left(\frac{1+\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c}\right)=\frac{\lambda+3}{1+3\lambda}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

we need to prove, $\frac{\lambda+3}{1+3\lambda}\left(\sum_{cyc}\frac{1}{a}\right)\le\sum_{cyc}\frac{1}{a}\Leftrightarrow1\le\lambda$, which is true $\because\ \lambda\in[1,3]$

$$\therefore\sum_{cyc}\frac{1}{a}\ge(\lambda+3)\left(\sum_{cyc}\frac{1}{\lambda+a}\right)$$

75. Let a,b and c be positive real numbers such that $abc=1$. Prove that:

$$\frac{1}{a^3+b^3+c^3}+\frac{1}{ab+bc+ca}\ge\frac{6}{(a^2+b^2+c^2)^2}$$

Proposed by Nguyen Phuc Tang - Dong Thap – Vietnam

Solution 1 by Khung Long Xanh - Da Nang – Vietnam

By Cauchy :

$$\frac{1}{a^3+b^3+c^3}+\frac{1}{ab+bc+ca}\geq 2\sqrt{\frac{1}{(a^3+b^3+c^3)(ab+bc+ca)}}$$

$$2\sqrt{\frac{1}{(a^3+b^3+c^3)(ab+bc+ca)}}\geq\frac{6}{(a^2+b^2+c^2)^2}$$

$$\Leftrightarrow 9(a^3+b^3+c^3)(ab+bc+ca)\leq(a^2+b^2+c^2)^4$$

$$\Leftrightarrow 81(a^3+b^3+c^3)^2(ab+bc+ca)^2\leq(a^2+b^2+c^2)^8 \quad (*)$$

By Cauchy – Schwarz :

$$(a^3+b^3+c^3)^2\leq(a^4+b^4+c^4)(a^2+b^2+c^2)$$

$$(ab+bc+ca)^2\leq 3(a^2b^2+b^2c^2+c^2a^2)\leq(a^2b^2+b^2c^2+c^2a^2)^2$$

$$(a^2b^2+b^2c^2+c^2a^2\geq 3)$$

$$\Rightarrow LHS\leq 81(a^4+b^4+c^4)(a^2b^2+b^2c^2+c^2a^2)^2\cdot(a^2+b^2+c^2)$$

$$27(a^4+b^4+c^4)(a^2b^2+b^2c^2+c^2a^2)^2\leq(a^2+b^2+c^2)^6 \text{ (AM-GM)}$$

$$\Rightarrow\ \leq 81(a^4+b^4+c^4)(a^2b^2+b^2c^2+c^2a^2)^2\cdot(a^2+b^2+c^2)\leq$$

$$\leq 3\cdot(a^2+b^2+c^2)^7\leq(a^2+b^2+c^2)^8$$

(*)

76. Let $x,y,z>0$ and $2\sqrt{xyz}+x+y+z=1$ then

1. $\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\geq\frac{3}{4}$
2. $\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\geq 12$

Proposed by Mihalcea Andrei Ștefan – Romania

Solution by Soumitra Mandal – Chandar Nagore – India

Let $x+y+z=3p^2$ where $p>0$ then

$$x+y+z+2\sqrt{xyz}=1\Rightarrow 1\le x+y+z+2\sqrt{\left(\frac{x+y+z}{3}\right)^2}=3p^2+2p^3$$

$$\Rightarrow 2(p^3+1)+3(p^2-1)\ge 0\Rightarrow (p+1)(2p^2+p-1)\ge 0$$

$$\Rightarrow (p+1)^2(2p-1)\ge 0\Rightarrow p\ge\frac{1}{2}\Rightarrow x+y+z\ge\frac{3}{4}$$

Applying *A.M* ≥ G.M, $\frac{xy}{z}+\frac{zx}{y}\ge 2x$, $\frac{yz}{x}+\frac{zx}{y}\ge 2z$ ***and*** $\frac{xy}{z}+\frac{yz}{x}\ge 2y$

$$\therefore\sum_{cyc}\frac{xy}{z}\ge x+y+z\ge\frac{3}{4}$$

(Proved)

Let $xyz=a^6$ **where** $a>0$ **now,** $x+y+z+2\sqrt{xyz}=1\Rightarrow$

$$\Rightarrow 1\ge 3\sqrt[3]{xyz}+2\sqrt{xyz}$$

$$\Rightarrow 2a^3+3a^2\le 1\Rightarrow 0\ge 2a^3+3a^2-1\Rightarrow (a+1)^2(2a-1)\le 0\Rightarrow a\le\frac{1}{2}$$

$\Rightarrow xyz\le\frac{1}{64}$**. Now applying *A.M* ≥ *G.M*,**

$$\frac{x}{yz}+\frac{y}{zx}\ge\frac{2}{z},\ \frac{y}{zx}+\frac{z}{xy}\ge\frac{2}{x}$$

and $\frac{z}{xy}+\frac{x}{yz}\ge\frac{2}{y}$ **so,**

$$\sum_{cyc}\frac{x}{yz}\ge\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge\frac{3}{\sqrt[3]{xyz}}\ge 12$$

77. If $a,b,c,d,e\in(0,\infty)$ then:

$$\frac{a-c}{b+c}+\frac{b-d}{c+d}+\frac{c-e}{d+e}+\frac{d-a}{e+a}+\frac{e-b}{a+b}\ge 0$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

Si: $a,b,c,d,e \in <0,\infty>$ **. Probar que:**

$$\frac{a-c}{b+c}+\frac{b-d}{c+d}+\frac{c-e}{d+e}+\frac{d-a}{e+a}+\frac{e-b}{a+b}\geq 0$$

$$\Rightarrow\left(\frac{a-c}{b+c}+1\right)+\left(\frac{b-d}{c+d}+1\right)+\left(\frac{c-e}{d+3}+1\right)+\left(\frac{d-a}{e+a}+1\right)+\left(\frac{e-b}{a+b}+1\right)\geq 5$$

$$\Rightarrow\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+e}+\frac{d+e}{e+a}+\frac{e+a}{a+b}\geq 5$$

Desde que: $a,b,c>0$**. Por las desigualdades entre las medias:**

$$MA\geq MG$$

$$\Rightarrow\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+e}+\frac{d+e}{e+a}+\frac{e+a}{a+b}\geq 5\sqrt[5]{\left(\frac{a+b}{b+c}\right)\left(\frac{b+c}{c+d}\right)\left(\frac{c+d}{d+e}\right)\left(\frac{d+e}{e+a}\right)\left(\frac{e+a}{a+b}\right)}$$

$$\Rightarrow\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+e}+\frac{d+e}{e+a}+\frac{e+a}{a+b}\geq 5$$

$\Rightarrow$ **La igualdad se alcanza cuando:** $a=b=c=d=e$

78. Prove that for all positive real numbers a,b,c,d **such that:**

$$\left(\frac{a}{b+c+d}+\frac{2}{3}\right)\left(\frac{b}{c+d+a}+\frac{2}{3}\right)\left(\frac{c}{d+a+b}+\frac{2}{3}\right)\left(\frac{d}{a+b+c}+\frac{2}{3}\right)\geq 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: $a,b,c,d\ \mathbb{R}^+$**. Probar la siguiente desigualdad:**

$$\left(\frac{a}{b+c+d}+\frac{2}{3}\right)\left(\frac{b}{a+c+d}+\frac{2}{3}\right)\left(\frac{c}{a+b+d}+\frac{2}{3}\right)\left(\frac{d}{a+b+c}+\frac{2}{3}\right)\geq 1$$

Sean: $b+c+d=3x\geq 0$, $a+c+d=3y\geq 0$, $a+b+d=3z\geq 0$,

$$a+b+c=3w\geq 0$$

Por lo tanto:

$$a = y + z + w - 2x \geq 0, b = z + x + w - 2y \geq 0, c = x + y + w - 2z \geq 0,$$

$$d = x + y + z - 2w \geq 0$$

La desigualdad es equivalente:

$$\left(\frac{y+z+w-2x}{3x}+\frac{2}{3}\right)\left(\frac{z+x+w-2y}{3y}+\frac{2}{3}\right)\left(\frac{x+y+w-2z}{3z}+\frac{2}{3}\right)\left(\frac{x+y+z-2w}{3w}+\frac{2}{3}\right) \geq 1$$

$$\Rightarrow \left(\frac{y+z+w}{3x}\right)\left(\frac{z+x+w}{3y}\right)\left(\frac{x+y+w}{3z}\right)\left(\frac{x+y+z}{3w}\right) \geq 1$$

$$\Rightarrow (y+z+w)(z+x+w)(x+y+w)(x+y+z) \geq 81xyzw \Leftrightarrow$$

$\Leftrightarrow$ **(Válido por:** $MA \geq MG$**)**

79. If $a, b, c, d > 0, \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} = 1$ then:

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d} \leq \sqrt[3]{abcd}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$1 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} =$$

$$= \frac{1}{3}\cdot\sum\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \overset{AM\geq GM}{\geq} \frac{1}{3}\cdot 3\cdot\sum\frac{1}{\sqrt[3]{abc}} =$$

$$= \sum\frac{1}{\sqrt[3]{abc}} \Rightarrow 1 \geq \sum\frac{1}{\sqrt[3]{abc}} \Big| \cdot \sqrt[3]{abca} \Leftrightarrow \sqrt[3]{abcd} \geq \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}$$

$$a = b = c = d = 4$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $a, b, c, d > 0$ ***and*** $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} = 1$

We get that $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^3 \geq 4^2\left(\frac{1}{abc}+\frac{1}{bcd}+\frac{1}{cda}+\frac{1}{dab}\right)$

Hence $\frac{1}{abc}+\frac{1}{bcd}+\frac{1}{cda}+\frac{1}{dab}\le\frac{1}{4^2}$. *Hence* $(a+b+c+d)\le\frac{abcd}{4^2}$

Hence $4^2(a+b+c+d)\le abcd$. *Hence* $\sqrt[3]{4^2(a+b+c+d)}\le\sqrt[3]{abcd}$

But $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}\le\sqrt[3]{4^2(a+b+c+d)}$

Therefore $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}\le\sqrt[3]{abcd}$

80. Let a,b,c,d be non-negative real numbers. Prove that:

$$\sqrt{(a^2+b^2)(c^2+d^2)}+2\sqrt{abcd}\ge(a+b)\sqrt{cd}+(c+d)\sqrt{ab}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sean: a,b,c,d números $\mathbb{R}$ no negativos. Probar la siguiente desigualdad:

$$\sqrt{(a^2+b^2)(c^2+d^2)}+2abcd\ge(a+b)\sqrt{cd}+(c+d)\sqrt{ab}$$

Por: $MC\ge MA$

$$\sqrt{(a^2+b^2)(c^2+d^2)}+2abcd\ge\frac{(a+b)(c+d)}{2}+2abcd\ge(a+b)\sqrt{cd}+(c+d)\sqrt{ab}$$

Sean: $a=x^2\ge0,\ b=y^2\ge0,\ c=z^2\ge0,\ d=w^2\ge0$

La desigualdad es equivalente:

$$(a+b)(c+d)+4abcd\ge2(a+b)\sqrt{cd}+2(c+d)\sqrt{ab}$$

$$(x^2+y^2)(z^2+w^2)+4xyzw\ge2x^2zw+2y^2zw+2z^2xy+2w^2xy$$

$$(xz)^2+(xw)^2+(yw)^2+(yz)^2+4xyzw\ge$$

$$\ge2x^2zw+2y^2zw+2z^2xy+2w^2xy$$

$$\Rightarrow x^2(z^2+w^2-2wz)+y^2(z^2+w^2-2wz)-2xy(z^2+w^2-2wz)\ge0$$

$\Leftrightarrow(z-w)^2(x^2+y^2-2xy)=(z-w)^2(x-y)^2\ge0\rightarrow$ **(Lo cual es cierto)**

81. If $a, b, c, d \in (0, \infty), \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{d}, a \neq d, b \neq d, c \neq d$ then:

$$\frac{(a+b)(b+c)(c+a)}{(a-d)(b-d)(c-d)} \geq 27$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

Del dato: $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{d} \to \frac{a+b}{ab} = \frac{c-d}{cd} \to \frac{a+b}{c-d} = \frac{ab}{cd}$ (A)

De forma análoga: $\frac{b+c}{bc} = \frac{a-d}{ad} \to \frac{b+c}{a-d} = \frac{bc}{ad}$ $(B) \wedge \frac{a+c}{ac} = \frac{b-d}{bd} \to \frac{a+c}{b-d} = \frac{ac}{bd}$ (C)

Multiplicando: $(A)(B)(C)$

$$\frac{(a+b)(b+c)(c+a)}{(a-d)(b-d)(c-d)} = \frac{ab}{cd} \cdot \frac{bc}{ad} \cdot \frac{ac}{bd} = \frac{abc}{d^3}$$

Desde que: $a, b, c, d \in < 0, \infty >$. Aplicamos: $MA \geq MG$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3\sqrt[3]{\frac{1}{abc}} \to \frac{1}{d} \geq 3\sqrt[3]{\frac{1}{abc}} \to \frac{1}{d^3} \geq \frac{27}{abc} \to \frac{abc}{d^3} \geq 27$$

82. If $a, b, c, d \in (0, \infty), abcd = 1$ then:

$$\frac{(a+b+c)^5}{(b+c+d)^4} + \frac{(b+c+d)^5}{(c+d+a)^4} + \frac{(c+d+a)^5}{(d+a+b)^4} + \frac{(d+a+b)^5}{(a+b+c)^4} \geq 12$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

Si: $a, b, c, d, e \in < 0, \infty >, abcd = 1$. Probar que:

$$\frac{(a+b+c)^5}{(b+c+d)^4} + \frac{(b+c+d)^5}{(c+d+a)^4} + \frac{(c+d+a)^5}{(d+a+b)^4} + \frac{(d+a+b)^5}{(a+b+c)^4} \geq 12$$

Desde que: $a, b, c, d, e \in < 0, \infty >$. Por: $MA \geq MG$

$$\frac{(a+b+c)^5}{(b+c+d)^4}+\frac{(b+c+d)^5}{(c+d+a)^4}+\frac{(c+d+a)^5}{(d+a+b)^4}+\frac{(d+a+b)^5}{(a+b+c)^4}\geq$$

$$\geq 4\sqrt[4]{(a+b+c)(b+c+d)(c+d+a)(d+a+b)}$$

Pero por: $MA \geq MG: a+b+c \geq 3\sqrt[3]{abc}\quad (A)$

$$b+c+d \geq 3\sqrt[3]{bcd}\quad (B)$$

$$c+d+a \geq 3\sqrt[3]{cda}\quad (C)$$

$$d+a+b \geq 3\sqrt[3]{dab}\quad (D)$$

→ **Multiplicando:** $(A)(B)(C)(D)$

$$(a+b+c)(b+c+d)(c+d+a)(d+b+a) \geq 81abcd = 81 \Leftrightarrow$$

$$\Leftrightarrow \sqrt[4]{(a+b+c)(b+c+d)(c+d+a)(d+a+b)} \geq 3$$

Por transitividad:

$$\frac{(a+b+c)^5}{(b+c+d)^4}+\frac{(b+c+d)^5}{(c+d+a)^4}+\frac{(c+d+a)^5}{(d+a+b)^4}+\frac{(d+a+b)^5}{(a+b+c)^4}\geq$$

$$\geq 4\sqrt[4]{(a+b+c)(b+c+d)(c+d+a)(d+a+b)} = 12$$

83. If $a, b, c, d \in (0,\infty)$ then:

$$(ab+cd)^2 \leq \left(b\sqrt[5]{ab^4}+d\sqrt[5]{cd^4}\right)\left(a\sqrt[5]{a^4b}+c\sqrt[5]{c^4d}\right)$$

Proposed by Daniel Sitaru – Romania

Proposed by Kevin Soto Palacios

Si: $a, b, c, d \in <0,\infty>$**. Probar que:**

$$(ab+cd)^2 \leq \left(b\sqrt[5]{ab^4}+d\sqrt[5]{cd^4}\right)\left(a\sqrt[5]{a^4b}+c\sqrt[5]{c^4d}\right)$$

Por la desigualdad entre las medias:

Usando: Cauchy – Schwarz

Sean: $x_1, x_2, \dots, x_n,\ y_1, y_2, \dots, y_n$ **números reales, se cumple:**

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \le (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \qquad (A)$$

La igualdad se alacanza cuando: $\frac{x_1}{y_1} = \frac{x_2}{y_2}$

Sea: $x_1 = \sqrt{b}\sqrt[10]{ab^4}, y_1 = \sqrt{a}\sqrt[10]{a^4b}, x_2 = \sqrt{d}\sqrt[10]{cd^4}, y_2 = \sqrt{c}\sqrt[10]{c^4d}$

Por la tanto, reemplazando en (A)**:**

$$(ab + cd)^2 \le \left(b\sqrt[5]{ab^4} + d\sqrt[5]{cd^4}\right)\left(a\sqrt[5]{a^4b} + c\sqrt[5]{c^4d}\right)$$

La igualdad se alcanza cuando: $\frac{b\sqrt[10]{ab^4}}{a\sqrt[10]{ba^4}} = \frac{d\sqrt[10]{cd^4}}{c\sqrt[10]{c^4d}} \Leftrightarrow bc = ad$

84. Let a, b, c, d be positive real numbers such that:

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} \ge 3.$$

Prove that:

$$\sqrt{a+b+c+d+4} \ge \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sean a, b, c, d números: $\mathbb{R}^+$**, de tal manera se cumple que:**

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} \ge 3$$

Probar que: $\sqrt{a+b+c+d+4} \ge \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}$

Siendo: $a, b, c, d > 0$. **Por la desigualdad de Cauchy Schwarz:**

$$\left((a+1) + (b+1) + (c+1) + (d+1)\right)\left(\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} + \frac{d}{d+1}\right) \ge \left(\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}\right)^2 \dots (A)$$

Ahora bien:

$$\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} + \frac{d}{d+1} = \left(1 - \frac{1}{1+a}\right) + \left(1 - \frac{1}{1+b}\right) + \left(1 - \frac{1}{1+c}\right) + \left(1 - \frac{1}{1+d}\right) \le 4 - 3 = 1 \dots (B)$$

De (B) ∧ (A) ...

$$\Rightarrow (a+b+c+d+4) \geq \frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2}{\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+\frac{d}{d+1}} \geq$$

$$\geq \frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2}{1} = \left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2$$

Por transitividad: ⇒

$$\Rightarrow (a+b+c+d+4) \geq \left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{a+b+c+4} \geq \sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d} \quad \text{... (LQQD)}$$

85. If $a,b,c,x,y,z \in (0,\infty)$ then:

$$(a+b)\left(x^a y^b\right)^{\frac{1}{a+b}} + cz \geq (a+b+c)\left(x^a y^b z^c\right)^{\frac{1}{a+b+c}}$$

Proposed by Daniel Sitaru – Romania

Solution by Hamza Mahmood-Lahore-Pakistan

Let $w_1 = a+b, w_2 = c, X_1 = \left(x^a y^b\right)^{\frac{1}{a+b}}, X_2 = z$, **Now** $w_1, w_2, X_1, X_2 \in (0,\infty)$

By Weighted AM-GM inequality: $\frac{w_1X_1+w_2X_2}{w_1+w_2} \geq \left(X_1^{w_1} \cdot X_2^{w_2}\right)^{\frac{1}{w_1+w_2}}$

$$\Rightarrow \frac{(a+b)\left(x^a y^b\right)^{\frac{1}{a+b}} + cz}{a+b+c} \geq \left(\left(x^a y^b\right)^{\frac{1}{a+b}\cdot(a+b)} \cdot z^c\right)^{\frac{1}{a+b+c}}$$

$$\Rightarrow (a+b)\left(x^a y^b\right)^{\frac{1}{a+b}} + cz \geq (a+b+c)\left(x^a y^b z^c\right)^{\frac{1}{a+b+c}}$$

Equality holds when $X_1 = X_2 \Rightarrow \left(x^a y^b\right)^{\frac{1}{a+b}} = z \Rightarrow x^a y^b = z^{a+b}$

86. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^3+8abc}+\frac{1}{b^3+8abc}+\frac{1}{c^3+8abc}\le\frac{1}{3abc}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Vietnam

Siendo: a, b, c números $\mathbb{R}^+$. Probar que:

$$\frac{1}{a^3+8abc}+\frac{1}{b^3+8abc}+\frac{1}{c^3+8abc}\le\frac{1}{3abc}$$

$$\Rightarrow\frac{bc}{a^2+8bc}+\frac{ac}{b^2+8ac}+\frac{ab}{c^2+8ab}\le\frac{1}{3}$$

$$\Rightarrow\frac{bc}{a^2+8bc}=\frac{1}{8}-\frac{a^2}{8(a^2+8bc)}$$

$$\left(\frac{1}{8}-\frac{a^2}{8(a^2+8bc)}\right)+\left(\frac{1}{8}-\frac{b^2}{8(b^2+8ac)}\right)+\left(\frac{1}{8}-\frac{c^2}{8(c^2+8ab)}\right)\le\frac{1}{3}$$

Demostraremos que: $\frac{a^2}{(a^2+8bc)}+\frac{b^2}{(b^2+8ac)}+\frac{c^2}{(c^2+8ab)}\ge\frac{1}{3}$

Por desigualdad de Cauchy:

$$\frac{a^2}{(a^2+8bc)}+\frac{b^2}{(b^2+8ac)}+\frac{c^2}{(c^2+8ab)}\ge\frac{(a+b+c)^2}{\sum a^2+8\sum ab}\ge\frac{1}{3}\Rightarrow$$

$$\Rightarrow 3(a+b+c)^2\ge\sum a^2+8\sum ab$$

$\Leftrightarrow\sum a^2\ge\sum ab$ **(LQQD). Por lo tanto:**

$$\left(\frac{1}{8}-\frac{a^2}{8(a^2+8bc)}\right)+\left(\frac{1}{8}-\frac{b^2}{8(b^2+8ac)}\right)+\left(\frac{1}{8}-\frac{c^2}{8(c^2+8ab)}\right)\le\frac{3}{8}-\frac{1}{24}\le\frac{1}{3}$$

(LQQD)

Solution 2 by Soumitra Moukherjee - Chandar Nagore – India

$$\frac{1}{3abc} \geq \sum_{cyc} \frac{1}{a^3 + 8abc} \Leftrightarrow \sum_{cyc} \left(\frac{1}{9abc} - \frac{1}{a^3 + 8abc}\right) \geq 0$$

$$\Leftrightarrow \sum_{cyc} \frac{a^2 - bc}{9abc(a^2 + 8bc)} \geq 0 \Leftrightarrow \sum_{cyc} \frac{(a-b)(a+c) + (a-b)(a+c)}{18abc(a^2 + 8bc)} \geq 0$$

$$\Leftrightarrow \sum_{cyc} \frac{1}{18abc}\left(\frac{(a-b)(a+c)}{a^2 + 8bc} + \frac{(a+b)(a-c)}{a^2 + 8bc}\right) \geq 0$$

$$\Leftrightarrow \sum_{cyc} \frac{1}{18abc}\left\{\frac{(a-b)(a+c)}{a^2 + 8bc} + \frac{(b-a)(b+c)}{b^2 + 8ac}\right\} \geq 0$$

$$\Leftrightarrow \sum_{cyc} \frac{(a-b)}{18abc}\left\{\frac{7c(a^2 - b^2) + 8c^2(a-b) - ab(a-b)}{(a^2 + 8bc)(a^2 + 8ac)}\right\} \geq 0$$

$\Leftrightarrow \sum_{cyc} \frac{(a-b)^2}{18abc}(7ac + 7bc + 8c^2 - ab) \geq 0,$ **which is true**

so, $\frac{1}{3abc} \geq \frac{1}{a^3+8abc}$ **(proved)**

87. If $a, b, c \in (0, \infty), a^2 + b^2 + c^2 = 4 - abc$ then:

$$\frac{(2+a)(2+b)(2+c)}{(2-a)(2-b)(2-c)} \geq 3\sqrt{3}$$

Proposed by Daniel Sitaru, Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Si: $a, b, c \in < 0, \infty >$**, de tal manera que:** $a^2 + b^2 + c^2 + abc = 4$**. Probar que:**

$\frac{(2+a)(2+b)(2-c)}{(2-a)(2-b)(2+c)} \geq 3\sqrt{3}$ **... (A)**

Desde que: $a, b, c > 0$**. Si:** $A + B + C = \pi$

$$4\cos^2 A + 4\cos^2 B + 4\cos^2 C + 8\cos A \cos B \cos C = 4$$

Sean: $\mathbf{a = 2\cos A > 0, b = 2\cos B > 0, c = 2\cos C > 0}$

Reemplazando en ... (A)

$$\frac{2(1+\cos A)2(1+\cos B)2(1+\cos C)}{2(1-\cos A)2(1-\cos B)2(1-\cos C)} = \frac{4\cos^2\frac{A}{2}4\cos^2\frac{B}{2}4\,\text{sen}^2\frac{C}{2}}{4\,\text{sen}^2\frac{A}{2}4\,\text{sen}^2\frac{B}{2}4\cos^2\frac{C}{2}} =$$

$$= \left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^2 \geq 27 > 3\sqrt{3} \text{ ... (LQQD)}$$

88. Prove that for all positive real numbers a, b, c:

$$\frac{2}{3} + \frac{abc}{a^3+b^3+c^3} \geq \frac{ab+bc+ca}{a^2+b^2+c^2}$$

Proposed by Adil Abdullayev – Baku – Azerbaidian

Solution by Nguyen Viet Hung – Hanoi – Vietnam

The required inequality may be written as:

$$1 - \frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{1}{3} - \frac{abc}{a^3+b^3+c^3}$$

which is equivalent to:

$$\frac{a^2+b^2+c^2-ab-bc-ca}{a^2+b^2+c^2} \geq \frac{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}{3(a^3+b^3+c^3)}$$

It suffices to show that:

$$\frac{1}{a^2+b^2+c^2} \geq \frac{a+b+c}{3(a^3+b^3+c^3)}$$

Or

$$3(a^3+b^3+c^3) \geq (a^2+b^2+c^2)(a+b+c)$$

But this is true by Tchebyshev's inequality and we are done.

89. If $a, b, c \in (0, \infty)$ then:

$$\sqrt[3]{(2a+5)(2b+5)(2c+5)} \geq \frac{6abc}{ab+bc+ca} + 5$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si: $a, b, c \in < 0, \infty >$. Probar que:

$$\sqrt[3]{(2a+5)(2b+5)(2c+5)} \geq \frac{6abc}{ab+bc+ac} + 5$$

Por la desigualdad de Holder:

$$(2a+5)(2b+5)(2c+5) \geq \left(\sqrt[3]{8abc} + \sqrt[3]{125}\right)^3$$

$$\Rightarrow \sqrt[3]{(2a+5)(2b+5)(2c+5)} \geq 2\sqrt[3]{abc} + 5 \geq \frac{6abc}{ab+bc+ac} + 5$$

Por lo cual solo quenda demostrar:

$$2\sqrt[3]{abc} \geq \frac{6abc}{ab+bc+ac} \Leftrightarrow 2\sqrt[3]{(ab+bc+ac)} \geq$$

$$\geq 2\sqrt[3]{abc}\left(3\sqrt[3]{(abc)^2}\right) = 6abc \text{ ... (LQQD)}$$

Solution 2 by Pham Quy –Quang Ngai- Vietnam

If $a, b, c \in (0, \infty)$ then

$$\sqrt[3]{(2a+5)(2b+5)(2c+5)} \geq \frac{6abc}{ab+bc+ca} + 5$$

We have: $ab + bc + ca \geq 3\sqrt[3]{(abc)^2}$ ***(AM-GM)***$\Rightarrow 2\sqrt[3]{abc} \geq \frac{6abc}{ab+bc+ca}$

$$(2a+5)(2b+5)(2c+5) \geq \left(2\sqrt[3]{abc} + 5\right)^3 \text{ (Holder inequality)}$$

$$\Rightarrow \sqrt[3]{(2a+5)(2b+5)(2c+5)} \geq 2\sqrt[3]{abc} + 5 \geq \frac{6abc}{ab+bc+ca} + 5 \text{ (q.e.d)}$$

The equality holds at $a = b = c$

90. If $a, b, c \in (0, \infty)$ then:

$$\sum c\left(\frac{4a}{b^2}+\frac{3b}{a^2}\right) \geq 12 + 3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

$$\sum c\left(\frac{4a}{b^2}+\frac{3b}{a^2}\right) = c\left(\frac{4a}{b^2}+\frac{3b}{a^2}\right) + a\left(\frac{4b}{c^2}+\frac{3c}{b^2}\right) + b\left(\frac{4c}{a^2}+\frac{3a}{c^2}\right)$$

$$= \left(\frac{4ac}{b^2}+\frac{4abc}{a^2}+\frac{4ab}{c^2}\right) + \left(\frac{3ab}{c^2}+\frac{3bc}{a^2}+\frac{3ca}{b^2}\right) = 7\sum\left(\frac{ab}{c^2}\right)$$

$$AM \geq GM \Rightarrow \frac{4ac}{b^2}+\frac{4bc}{a^2}+\frac{4ab}{c^2} \geq 3\sqrt[3]{4^3} = 12 \quad (1)$$

Again, $AM \geq GM \Rightarrow \frac{ab}{c^2}+\frac{ab}{c^2}+\frac{ca}{b^2} \geq 3\sqrt[3]{\frac{a^3}{c^3}} = 3\left(\frac{a}{c}\right)$ (2)

$$AM \geq GM \Rightarrow \frac{bc}{a^2}+\frac{bc}{a^2}+\frac{ab}{c^2} \geq 3\sqrt[3]{\frac{b^3}{a^3}} = 3\left(\frac{b}{a}\right) \quad (3)$$

$$AM \geq GM \Rightarrow \frac{ca}{b^2}+\frac{ca}{b^2}+\frac{bc}{a^2} \geq 3\sqrt{\frac{c^3}{b^3}} = 3\left(\frac{c}{b}\right) \quad (4)$$

$$(1)+(2)+(3)+(4) \Rightarrow 7\left(\sum\left(\frac{ab}{c^2}\right)\right) \geq 12 + 3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$$

91. Let $a, b, c > 0$ and sum of any two is not zero, then:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{3\sqrt[3]{abc}} > \frac{3\left(a+b+c+\sqrt[3]{abc}\right)^2}{(a+b+c)^3}$$

Proposed by Soumitra Mandal - Chandar Nagore – India

Solution by Hung Nguyen Viet – Hanoi – Vietnam

By Chauchy – Schwarz inequality we have

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{3\sqrt[3]{abc}}=\frac{c^2}{c^2(a+b)}+\frac{a^2}{a^2(b+c)}+\frac{b^2}{b^2(c+a)}+\frac{\left(\sqrt[3]{abc}\right)^2}{3abc}$$

$$\geq\frac{\left(a+b+c+\sqrt[3]{abc}\right)^2}{a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc}$$

It's suffices to show that $\frac{1}{a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc}\geq\frac{3}{(a+b+c)^3}$

which is equivalent to

$$a^3+b^3+c^3+3(a+b)(b+c)(c+a)\geq 3[ab(a+b)+bc(b+c)+ca(c+a)+3abc]$$

or $a^3+b^3+c^3\geq 3abc$

But this result is clearly true by AM-GM inequality.

Therefore the proof is complete. Note that the equality can't occur.

92. If $a,b,c\in\mathbb{N}-\{0,1,2\}$ then:

$$8\log_a(a+1)\log_b(b+1)\log_c(c+1)<\frac{(a+b+2)(b+c+2)(c+a+2)}{abc}$$

Proposed by Daniel Sitaru – Romania

Solution by Safal Das Biswas – Chinsurah – India

Take a function $f(x)=x^{\frac{1}{x-1}}$. We will Check the The maximum value of this function $f(x)$ within the domain $x\in(1,2]$.

Now $f(x)=x^{\frac{1}{x-1}}$. So $f'(x)=x^{\frac{1}{x-1}}\left(\frac{1}{x(x-1)}-\frac{\ln x}{(x-1)^2}\right)$

Now at maximum $f'(x)=0$ then $x^{\frac{1}{x-1}}\left(\frac{1}{x(x-1)}-\frac{\ln x}{(x-1)^2}\right)=0$.

As $x>1$ then, $\frac{1}{x}=\frac{\ln x}{(x-1)}$ implies $\frac{x-1}{x}=\ln x$, or $e^{\frac{x-1}{x}}=x$, or $e^{1-\frac{1}{x}}=x$ or, $\frac{e}{x}=e^{\frac{1}{x}}$,

or $e=xe^{\frac{1}{x}}$, this is only possible for $x=1$ but since $x>1$ we can claim that the

$f'(x) = 0$ when $\lim x \to 1$. Thus we need to find

$\lim_{x\to 1} f(x)$ for the max value of f within the domain $x \in (1, 2]$,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} x^{\frac{1}{x-1}} = e^{\lim_{x\to 1}\frac{\ln x}{x-1}} = e^{\lim_{x\to 1}\frac{1}{x}} = e$$

Thus $f(x) < 3\ \forall\ 2 \geq x > 1$.

Now assume $p, q, r \in \mathbb{R}$ satisfying $a^p = a + 1$,

$b^q = b + 1$ and $c^r = c + 1$.

As $a^p > a\ \forall\ a \in \mathbb{N}$ then $p > 1$ likely $q > 1$ and also $r > 1$.

As, $a, b, c \in \mathbb{N} - \{1, 2\}$ then $a^2 > a + 1 = a^p$ so $p < 2$, likely for q and r.

Thus we have, $1 > (p, q, r) < 2$.

Since we know that $a, b, c \in \mathbb{N} - \{1, 2\}$ then $(a, b, c) \geq 3$.

This implies $(a, b, c) > f(x)\ \ \forall\ x \in (1, 2]$ as $(p, q, r) \in (1, 2)$ then $a > f(p)$ or,

$a > p^{\frac{1}{p-1}}$, or $a^{p-1} > p$ then $a^p > ap$, likely $b^q > bq$ and $c^r > cr$

$$\prod_{a,b,c\ cyclic} (a + b + 2) = \prod_{a^p, b^q, c^r\ cyclic} (a^p + b^q) \geq 8a^p b^q c^r > 8abcpqr$$

So we have the following

$$\prod_{a,b,c\ cyclic} \frac{(a + b + 2)}{c} > 8pqr = 8 \prod_{a,b,c\ cyclic} \log_a(a + 1)$$

93. If $x, y, z \in \mathbb{N} - \{0, 1\}$ then:

$$2^x + 2^y + 2^z + 2^{x+y+z} > \sqrt[x+y]{16^{xy}} + \sqrt[y+z]{16^{yz}} + \sqrt[z+x]{16^{zx}} + 1$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash – New Delhi – India

$$2^x + 2^y + 2^z + 2^{x+y+z} - 2^{x+y} - 2^{y+z} - 2^{z+x} - 1$$

$$= (2^x - 1)(2^y - 1)(2^z - 1) > 0$$

$$\Rightarrow 2^x + 2^y + 2^z + 2^{x+y+z} > 2^{x+y} + 2^{y+z} + 2^{z+x} + 1$$

But: $2^{x+y} = 4^{\frac{x+y}{2}} \geq 4^{\frac{2xy}{x+y}} = \sqrt[x+y]{16^{xy}}$. Hence:

$$2^x + 2^y + 2^z + 2^{x+y+z} > (16^{xy})^{\frac{1}{x+y}} + (16^{yz})^{\frac{1}{y+z}} + (16^{zx})^{\frac{1}{z+x}} + 1$$

94. Given $a, b, c \in [1, +\infty)$ prove that: $\frac{(1+a)(1+b)(1+c)(abc+\sqrt{abc})}{(a+\sqrt{a})(b+\sqrt{b})(c+\sqrt{c})} \geq 2$.

Proposed by Daniel Sitaru-Romania

Solutions by Ngô Minh Ngọc Bảo-Vietnam

Solution 1

We have: $\frac{(1+a)(1+b)(1+c)(abc+\sqrt{abc})}{(a+\sqrt{a})(b+\sqrt{b})(c+\sqrt{c})} \geq 2 \Leftrightarrow$

$$\left(\sum \ln(a+1) + \frac{1}{2}\sum \ln a - \sum(a+\sqrt{a})\right) + \ln\left(\sqrt{abc}+1\right) \geq \ln 2$$

Considering function: $f(t) = \ln(t+1) + \frac{1}{2}\ln t - \ln\left(t+\sqrt{t}\right), \forall\, t \in [1, +\infty)$

$$\Rightarrow f'(t) = \frac{1}{2t} + \frac{1}{1+t} - \frac{2\sqrt{t}+1}{2t\left(\sqrt{t}+1\right)} =$$

$$= \frac{(t+1)\left(\sqrt{t}+1\right) + 2t\left(\sqrt{t}+1\right) - \left(2\sqrt{t}+1\right)(t+1)}{2t(t+1)\left(\sqrt{t+1}\right)} =$$

$$= \frac{t+2\sqrt{t}-1}{2\sqrt{t}(t+1)\left(\sqrt{t}+1\right)} > 0$$

$$\Rightarrow \left(\sum \ln(a+1) + \frac{1}{2}\sum \ln a - \sum \ln\left(a+\sqrt{a}\right)\right) + \ln\left(\sqrt{abc}+1\right) \geq$$

$$\geq 3\ln 2 - 3\ln 2 + \ln 2 = \ln 2 \left(\sqrt{abc}+1 \geq 2\right)$$

Solution 2

We have: $f(a,b,c)=\frac{(1+a)(1+b)(1+c)(abc+\sqrt{abc})}{(a+\sqrt{a})(b+\sqrt{b})(c+\sqrt{c})}$**, we need to prove** $f(a,b,c)\geq 2$.

Use: $(1+x^3)(1+y^3)(1+z^3)\geq(1+xyz)^3,\ (x,y,z>0)$ **we have:**

$$(1+a)(1+b)(1+c)\geq\left(1+\sqrt[3]{abc}\right)^3$$

And $(a+\sqrt{a})(b+\sqrt{b})(c+\sqrt{c})\leq(a+a)(b+b)(c+c)=8abc$

$$\Rightarrow f(a,b,c)\geq\frac{\left(1+\sqrt[3]{abc}\right)^3(abc+\sqrt{abc})}{8abc}=\frac{1}{8}\left(1+\sqrt[3]{abc}\right)^3\left(1+\frac{1}{\sqrt{abc}}\right)=$$

$$=\frac{1}{8}\left[\left(1+\sqrt[3]{abc}\right)^3+\left(\frac{1+\sqrt[3]{abc}}{\sqrt[6]{abc}}\right)^3\right]=\frac{1}{8}\left[\left(1+\sqrt[3]{abc}\right)^3+\left(\frac{1}{\sqrt[6]{abc}}+\sqrt[6]{abc}\right)^3\right]\geq$$

$$\geq\frac{1}{8}[(1+1)^3+(2)^3]=2$$

95. **Prove that for all positive real numbers** x,y,z

$$\frac{x^3+y^3+z^3}{x(y+z)^2}\geq\frac{3}{4}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los números $\mathrm{R}^+\ \mathrm{x.y.z}$:

$$\frac{x^3+y^3+z^3}{x(y+z)^2}\geq\frac{3}{4}$$

Supongamos sin pérdida de generalidad que:

$$x+y+z=3\Leftrightarrow 0<x,y,z<3$$

Por: MP ≥ MA

$$9(x^3+y^3+z^3)\geq(x+y+z)^3\Leftrightarrow$$

$$\Leftrightarrow 9(x^3 + y^3 + z^3) \geq 27 \Leftrightarrow (x^3 + y^3 + z^3) \geq 3$$

Lo cual solo nos falta probar que:

$$4 \geq x(y + z)^2 \Leftrightarrow 4 \geq x(3 - x)^2 \Leftrightarrow 4 \geq 9x - 6x^2 + x^3$$

$$\Rightarrow -x^3 + 6x^2 - 9x + 4 \geq 0 \Leftrightarrow -x^3 + 6x^2 - 9x + 4 = (x - 1)^2(4 - x) \geq 0 \ldots$$

(LQQD)

La igualdad se alcanza cuando: $x = y = z$

Solution 2 by Marian Dincă – Romania

$$\frac{x^3 + y^3 + z^3}{x(y + z)^2} \geq \frac{3}{4}$$

$y^3 + z^3 \geq 2\left(\frac{y+z}{2}\right)^3$ – **proof elementary of Jensen inequality and use AM – GM**

$$x^3 + y^3 + z^3 \geq x^3 + 2\left(\frac{y + z}{2}\right)^3 = x^3 + \left(\frac{y + z}{2}\right)^3 + \left(\frac{y + z}{2}\right)^3$$

$$3\sqrt[3]{x^3 \cdot \left(\frac{y + z}{2}\right)^3 \cdot \left(\frac{y + z}{2}\right)^3} = 3x\left(\frac{y + z}{2}\right)^2 = \frac{3}{4}x(y + z)^2$$

96. Let x, y, z be positive real numbers such that $xyz = 1$. Prove that:

$$\frac{1}{1 + 8x} + \frac{1}{1 + 8y} + \frac{1}{1 + 8z} \geq \frac{1}{3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Sean: x, y, z **números** $\mathbb{R}^+$**, de tal manera que** $xyz = 1$**. Probar que:**

$$\frac{1}{1 + 8x} + \frac{1}{1 + 8y} + \frac{1}{1 + 8z} \geq \frac{1}{3}$$

Sea: $x = \frac{a}{b} > 0, y = \frac{b}{c} > 0, z = \frac{c}{a} > 0 \Leftrightarrow (a, b, c) > 0$

La desigualdad es equivalente:

$$\Rightarrow \frac{1}{1+\frac{8a}{b}}+\frac{1}{1+\frac{8b}{c}}+\frac{1}{1+\frac{8c}{a}}\ge\frac{1}{3}$$

$$\Rightarrow \frac{b}{b+8a}+\frac{c}{c+8b}+\frac{a}{a+8c}\ge\frac{1}{3}$$

$$\Rightarrow \frac{b^2}{b^2+8ab}+\frac{c^2}{c^2+8cb}+\frac{a}{a^2+8ac}\ge\frac{1}{3}$$

Por desigualdad de Bergstrom's (Cauchy):

$$\Rightarrow \frac{(a+b+c)^2}{a^2+b^2+c^2+8ab+8bc+8ac}\ge\frac{1}{3}\Leftrightarrow$$

$$\Leftrightarrow 3(a+b+c)^2\ge a^2+b^2+c^2+8ab+8bc+8ac$$

$$\Rightarrow 3a^2+3b^2+3c^2+6ab+6bc+6ac\ge a^2+b^2+c^2+8ab+8c+8ac$$

$$\Rightarrow a^2+b^2+c^2\ge ab+bc+ac \ldots \text{(LQQD)}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Sean x, y, z números $\mathbb{R}^+$, de tal manera que: $xyz = 1$. Probar que:

$$\frac{1}{1+8x}+\frac{1}{1+8y}+\frac{1}{1+8z}\ge\frac{1}{3}$$

Multiplicando $(1+8x)(1+8y)(1+8z)$:

$$3\big((1+8y)(1+8z)+(1+8x)(1+8z)+(1+8x)(1+8y)\big)\ge$$

$$\ge(1+8x)(1+8y)(1+8z)$$

$$\Rightarrow 3(3+16(x+y+z)+64xy+64yz+64xz)\ge$$

$$\ge 1^3+1^2x8(x+y+z)+1x64(xy+yz+xz)+512xyz$$

$$\Rightarrow 9+48(x+y+z)+192(xy+yz+zx)\ge$$

$$\ge 1+8(x+y+z)+64(xy+yz+xz)+512$$

$$\Rightarrow 40(x+y+z)+128(xy+yz+xz)\ge 504$$

Desde que: $x, y, z > 0$. **Por:** $MA \geq MG$

$$\Rightarrow 40(x+y+z)+128(xy+yz+xz) \geq 40x3\sqrt[3]{xyz}+128x3\sqrt[3]{(xyz)^2} =$$

$$= 120 + 384 = 504 \ldots \textit{(LQQD)}$$

97. If $a, b, c \in (0, \infty)$ then:

$$\sum c\sqrt{8(a^2+b^2)} \leq 4(ab+bc+ca)+\sum \frac{c(a-b)^2}{a+b}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash - New Delhi – India

Consider $[2(a+b)^2+(a-b)^2]^2-8(a^2+b^2)(a+b)^2 =$

$$= [3a^2+3b^2+2ab]^2-8(a^2+b^2)(a^2+b^2+2ab) =$$

$$= 9a^4+ab^4+4a^2b^2+12a^3b+12ab^3+18a^2b^2-$$

$$-8(a^4+b^4+2a^2b^2+2a^3b+2ab^3) =$$

$$= a^4+b^4+6a^2b^2-4a^3b-4ab^3 =$$

$$= (a^2+b^2-2ab)^2 = (a-b)^4 \geq 0$$

$$\Rightarrow 2(a+b)^2+(a-b)^2 \geq 8(a^2+b^2)(a+b)^2$$

$$\Rightarrow \sqrt{8(a^2+b^2)} \leq \frac{(a-b)^2}{a+b}+2(a+b)$$

$$\Rightarrow c\sqrt{8(a^2+b^2)} \leq 2(ac+bc)+\frac{c(a-b)^2}{a+b}$$

$$\Rightarrow \sum c\sqrt{8(a^2+b^2)} \leq \sum \frac{c(a-b)^2}{a+b}+4(ab+bc+ca)$$

Solution 2 by Richdad Phuc – Hanoi – Vietnam

We have:

$$\sum c\sqrt{8(a^2+b^2)}-4(ab+bc+ca)=\sum 2c\left[\sqrt{2(a^2+b^2)}-(a+b)\right]=$$

$$=\sum\frac{2c(a-b)^2}{\sqrt{2(a^2+b^2)}+(a+b)}$$

By Cauchy – Schwarz

$$\sqrt{2(a^2+b^2)}\geq a+b$$

$$\Rightarrow\sum c\sqrt{8(a^2+b^2)}-4(ab+bc+ca)\leq\sum\frac{c(a-b)^2}{(a+b)}$$

Done. Equality for hold $a=b=c$

Solution 3 by Soumitra Mandal - Chandar Nagore – India

$$\sum_{cyc}\frac{c(a-b)^2}{a+b}+4(ab+bc+ca)=\sum_{cyc}\left\{\frac{c(a-b)^2}{a+b}+c(a+b)\right\}+2(ab+bc+ca)=$$

$$=\sum_{cyc}\frac{2c\left(a^2+b^2\right)}{a+b}+2(ab+bc+ca)=\sum_{cyc}\left\{\frac{2c\left(a^2+b^2\right)}{a+b}+c(a+b)\right\}\geq$$

$$\geq 2\sum_{cyc}\sqrt{\left\{\frac{2c\left(a^2+b^2\right)}{a+b}\right\}\{c(a+b)\}}=\sum_{cyc}c\sqrt{8\left(a^2+b^2\right)}$$

(proved)

98. If $a,b,c\in(0,\infty)$ then:

$$(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)\geq 2\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

$$(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)=3+\left(\frac{a^4}{b^4}+\frac{b^4}{a^4}\right)+\left(\frac{a^4}{c^4}+\frac{c^4}{a^4}\right)+\left(\frac{b^4}{c^4}+\frac{c^4}{b^4}\right)$$

If $\frac{a}{b} < 1, \frac{a^4}{b^4} + \frac{b^4}{a^4} > 2 > 2\frac{a}{b}$

Similarly for any expression < 1 on RHS.

If $\frac{a}{b} \geq 1, \frac{a^4}{b^4} + \frac{b^4}{a^4} + 1 > \frac{a^4}{b^4} + 1$

and $\frac{a^4}{b^4} + 1 \geq 2\frac{a^2}{b^2} \geq 2\frac{a}{b}$.

Similarly for other expression ≥ 1 on RHS.

In any case we get

$$(a^4 + b^4 + c^4)\left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}\right) > 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)$$

99. Prove that for all positive real numbers a, b, c

$$\frac{(a+b+c)^3}{abc} \geq \frac{7}{2}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) + 6$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los números R^+ a, b, c:

$$\frac{(a+b+c)^3}{abc} \geq \frac{7}{2}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) + 6$$

Recordar la siguiente identidad:

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3ab(a+b) + 3bc(b+c) + 3ac(a+c) + 6abc$$

Reemplazando en la desigualdad:

$$\frac{a^3 + b^3 + c^3}{abc} + 3\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) + 6 \geq \frac{7}{2}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) + 6$$

$$\frac{a^3 + b^3 + c^3}{abc} \geq \frac{1}{2}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) \Leftrightarrow$$

$$\Leftrightarrow 2a^3+2b^3+2c^3 \geq ab(a+b)+bc(b+c)+ac(a+c)$$

Desde que: $a, b, c > 0$. **Por: MA ≥ MG**

$$a^3+a^3+b^3 \geq 3a^2b \ \dots \text{(I)}$$

$$b^3+b^3+a^3 \geq 3b^2a \ \dots \text{(II)}$$

Sumando (I) + (II): $a^3+b^3 \geq ab(a+b)$ **... (III)**

Por lo tanto: $b^3+c^3 \geq bc(b+c)$ **... (IV)** $\wedge$ $a^3+c^3 \geq ac(a+c)$ **... (V)**

Sumando: (III) + (IV) + (V):

$$2a^3+2b^3+2c^3 \geq ab(a+b)+bc(b+c)+ac(a+c) \ \dots \text{(LQQD)}$$

100. Prove that for all real numbers a, b, c

$$(a+b+c)^2(ab+bc+ca)^2+18(a+b+c)(ab+bc+ca)abc$$
$$\geq 4(a+b+c)^3abc+4(ab+bc+ca)^3+27(abc)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

$$\rightarrow RHS = 4\left(\sum a^3+3\left(\sum a\right)\left(\sum ab\right)-3abc\right)+$$
$$+4\left(\sum (ab)^3+3\left(\sum ab\right)\left(\sum a\right)abc-3(abc)^2\right)+27(abc)^2$$

$$\rightarrow RHS = 4\left(\sum a^3\right)abc+4\sum (ab)^3+24\left(\sum a\right)\left(\sum ab\right)abc+3(abc)^2$$

$$\Rightarrow LHS = (ab(a+b)+bc(b+c)+ac(a+c)+3abc)^2+$$
$$+18(a+b+c)(ab+bc+ac)abc$$

$$\rightarrow LHS = \sum (ab)^2(a+b)^2+9(abc)^2+2abc\sum a(a+b)(a+c)+$$
$$+6abc\sum ab(a+b)+18\left(\sum a\right)\left(\sum ab\right)abc$$

$$\to LHS - RHS:$$

$$\to \sum (ab)^2[(a+b)^2 - 4ab] - 6\left(\sum a\right)\left(\sum ab\right)abc + 6abc\sum ab(a+b) +$$

$$+2abc\sum a(a+b)(a+c) - 4\left(\sum a^3\right)abc + 6(abc)^2$$

$$\to \sum (ab)^2 (a-b)^2 - 6abc\left(\sum ab(a+b) + 3abc\right)abc +$$

$$+6abc\sum ab(a+b) + 2abc\sum a(a+b)(a+c) - 4\left(\sum a^3\right)abc + 6(abc)^2$$

$$\to \sum (ab)^2 (a-b)^2 + 2abc\sum a(a+b)(a+c) - 4\left(\sum a^3\right)abc - 12(abc)^2$$

$$\to \sum (ab)^2 (a-b)^2 + 2abc\sum (a^3 + a^2b + a^2c + abc) -$$

$$-4\left(\sum a^3\right)abc - 12(abc)^2$$

$$\Rightarrow \sum (ab)^2(a-b)^2 - 2\left(\sum a^3\right)abc + 2abc\left(\sum a^2b + \sum a^2c\right) - 6(abc)^2$$

$$\Rightarrow \sum (ab)^2(a-b)^2 + 2\sum abc(-a^3 + a^2b + a^2c - abc)$$

$$\Rightarrow \sum (ab)^2(a-b)^2 + 2\sum abc\left(-a^2(a-b) + ac(a-b)\right) =$$

$$= \sum (ab)^2(a-b)^2 + 2\sum (ab)(ac)(a-b)(c-a)$$

$$(ab)^2(a-b)^2 + (bc)^2(b-c)^2 + (ac)^2(c-a)^2 +$$

$$+2abc\left(a(a-b)(c-a) + b(b-c)(a-b) + c(c-a)(b-c)\right)$$

$$\Rightarrow \left(ab(a-b) + bc(b-c) + ca(c-a)\right)^2 =$$

$$= [(a-b)(a-c)(b-c)]^2 = (a-b)^2(a-c)^2(b-c)^2 \geq 0$$

(LQQD)

La igualdad se alcanza cuando: $a = b \vee b = c, c = a$

Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru