RMM - Triangle Marathon 1 - 100

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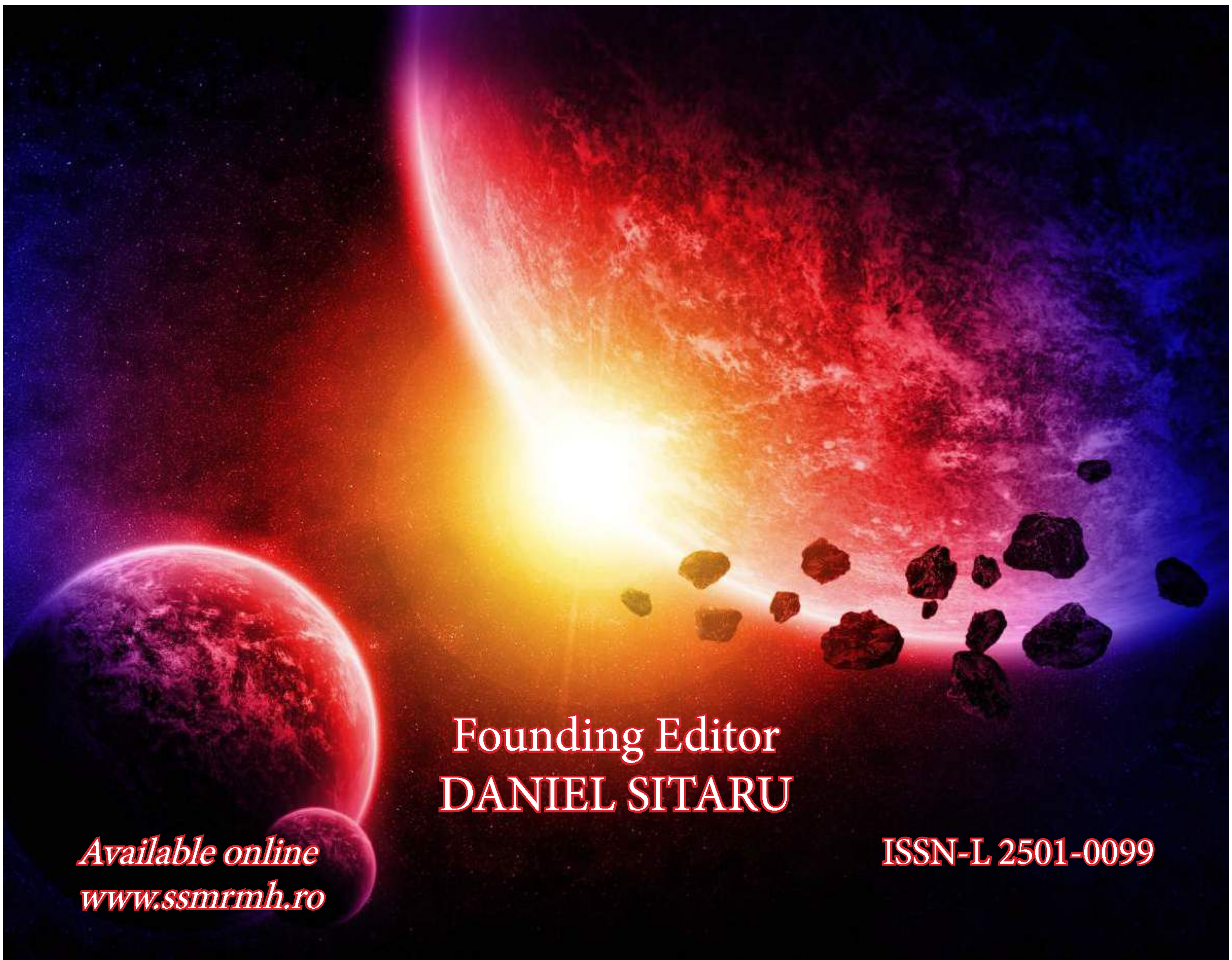
ROMANIAN MATHEMATICAL MAGAZINE



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1. Prove that in any triangle ABC the following relationship holds:

$$\frac{\cot\frac{B}{2}\cot\frac{C}{2}}{\cot\frac{A}{2}}+\frac{\cot\frac{C}{2}\cot\frac{A}{2}}{\cot\frac{B}{2}}+\frac{\cot\frac{A}{2}\cot\frac{B}{2}}{\cot\frac{C}{2}}\ge 2\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$$

Proposed by Hung Nguyen Viet-Vietnam

Solution by Rozeta Atanasova- Skopje

$$LHS=\frac{\cot\frac{B}{2}\cot\frac{C}{2}}{\cot\frac{A}{2}}+\frac{\cot\frac{C}{2}\cot\frac{A}{2}}{\cot\frac{B}{2}}+\frac{\cot\frac{A}{2}\cot\frac{B}{2}}{\cot\frac{C}{2}}=$$

$$=\frac{1}{2}\left(\cot\frac{A}{2}\left(\frac{\cot\frac{B}{2}}{\cot\frac{C}{2}}+\frac{\cot\frac{C}{2}}{\cot\frac{B}{2}}\right)+\cot\frac{B}{2}\left(\frac{\cot\frac{A}{2}}{\cot\frac{C}{2}}+\frac{\cot\frac{C}{2}}{\cot\frac{A}{2}}\right)+\cot\frac{C}{2}\left(\frac{\cot\frac{B}{2}}{\cot\frac{A}{2}}+\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}}\right)\right)=$$

$$\ge\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\ge 3\cot\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}=3\cot\frac{\pi}{6}=3\sqrt{3}=$$

$$=2\cdot 3\cos\frac{\pi}{6}=2\cdot 3\cos\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\ge 2\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)=RHS$$

2. Let ABC be a triangle with $a=BC, b=CA,$ and $c=AB$. Let $A'B'C'$ be another triangle with $B'C'=\sqrt{a}, C'A'=\sqrt{b}$, and $A'B'=\sqrt{c}$. Prove that:

$$\sin\left(\frac{1}{2}A\right)\sin\left(\frac{1}{2}B\right)\sin\left(\frac{1}{2}C\right)=\cos A'\cos B'\cos C'.$$

Proposed by Mehmet Sahin- Ankara - Turkey

Solution by Daniel Sitaru – Romania

$$\prod\cos A'=\prod\frac{a+b-c}{2\sqrt{ab}}=\frac{1}{8abc}\prod(2s-2c)=$$

$$= \frac{1}{abc}\prod(s-c) = \prod\sqrt{\frac{(s-b)(s-c)}{bc}} = \prod \sin\frac{A}{2}$$

3. In acute-angled $\Delta\ ABC$ the following relationship holds:

$$\sum(\sin 2A + \sin 2B)\left(\frac{1}{\sin 2A} + \frac{1}{\sin 2B}\right) \le \sum(\tan A + \tan B)(\cot A + \cot B)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Chakraborty-India:

$$A, B, C \ne \frac{\pi}{2}$$

$$\sum(\tan A + \tan B)(\cot A + \cot B) \ge \sum(\sin 2A + \sin 2B)\left(\frac{1}{\sin 2A} + \frac{1}{\sin 2B}\right)$$

$$\Leftrightarrow \sum\left\{\left(\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}\right)\left(\frac{\cos A}{\sin A} + \frac{\cos B}{\sin B}\right) - \frac{(\sin 2A + \sin 2B)^2}{\sin 2A \sin 2B}\right\} \ge 0$$

$$\Leftrightarrow \sum\left\{\frac{4\sin^2(A+B)}{4\sin A\cos A\sin B\cos B} - \frac{4\sin^2(A+B)\cos^2(A-B)}{\sin 2A\sin 2B}\right\} \ge 0$$

$$\Leftrightarrow \sum\left\{\frac{\sin^2(A+B)}{\sin 2A\sin 2B}(1-\cos^2(A-B))\right\} \ge 0$$

$$\Leftrightarrow \sum\left\{\frac{\sin^2 C\sin^2(A-B)}{\sin 2A\sin 2B}\right\} \ge 0 \quad \text{(1)}$$

$$0 < A, B, C < \frac{\pi}{2}, 0 < 2A, 2B, 2C < \pi \Rightarrow \sin 2A, \sin 2B, \sin 2C > 0$$

(1) always holds true, equality at $A = B = C = \frac{\pi}{3}$

Solution 2 by Kevin Soto Palacios-Peru:

En un triángulo acutángulo ABC. Probar que:

$$\sum(\operatorname{sen}2A+\operatorname{sen}2B)\left(\frac{1}{\operatorname{sen}2A}+\frac{1}{\operatorname{sen}2B}\right)\le\sum(\tan A+\tan B)(\cot A+\cot B)$$

A lo que es equivalente:

$$2\cdot 3+\frac{\operatorname{sen}2A+\operatorname{sen}2B}{\operatorname{sen}2C}+\frac{\operatorname{sen}2B+\operatorname{sen}2C}{\operatorname{sen}2A}+\frac{\operatorname{sen}2A+\operatorname{sen}2C}{\operatorname{sen}2B}\le$$

$$\le 2\cdot 3+\frac{\tan A+\tan B}{\tan C}+\frac{\tan B+\tan C}{\tan A}+\frac{\tan A+\tan C}{\tan B}$$

$$\Rightarrow(\operatorname{sen}2A+\operatorname{sen}2B+\operatorname{sen}2C)\left(\frac{1}{\operatorname{sen}2A}+\frac{1}{\operatorname{sen}2B}+\frac{1}{\operatorname{sen}2C}\right)\le$$

$$\le(\tan A+\tan B+\tan C)(\cot A+\cot B+\cot C)$$

$$\Rightarrow(\operatorname{sen}2A+\operatorname{sen}2B+\operatorname{sen}2C)\left(\frac{\operatorname{sen}2B\operatorname{sen}2C+\operatorname{sen}2A\operatorname{sen}2C+\operatorname{sen}2A\operatorname{sen}2B}{\operatorname{sen}2A\operatorname{sen}2B\operatorname{sen}2C}\right)\le$$

$$\le(\tan A\tan B\tan C)(\cot A+\cot B+\cot C)$$

$$\Rightarrow 4\operatorname{sen}A\operatorname{sen}B\operatorname{sen}C\left(\frac{\operatorname{sen}2B\operatorname{sen}2C+\operatorname{sen}2A\operatorname{sen}2C+\operatorname{sen}2A\operatorname{sen}2B}{(2\operatorname{sen}A\operatorname{sen}B\operatorname{sen}C)(4\cos A\cos B\cos C)}\right)\le$$

$$\le\left(\frac{\operatorname{sen}A\operatorname{sen}B\operatorname{sen}C}{\cos A\cos B\cos C}\right)\left(\frac{\operatorname{sen}^2A+\operatorname{sen}^2B+\operatorname{sen}^2C}{2\operatorname{sen}A\operatorname{sen}B\operatorname{sen}C}\right)$$

Simplificando la expresión, se obtiene que:

$$\operatorname{sen}2B\operatorname{sen}2C+\operatorname{sen}2A\operatorname{sen}2C+\operatorname{sen}2A\operatorname{sen}2B\le$$

$$\le\operatorname{sen}^2A+\operatorname{sen}^2B+\operatorname{sen}^2C\rightarrow\textbf{(Multiplicando}\times 2\textbf{)}$$

$$-\cos(2B+2C)+\cos(2B-2C)-\cos(2A+2C)+\cos(2A-2C)-\cos(2A-2B)+\cos(2A+2B)\le$$

$$\le 3-\cos 2A-\cos 2B-\cos 2C$$

$$-\cos 2A+\cos(2B-2C)-\cos 2B+\cos(2A-2C)-\cos 2C+\cos(2A+2B)$$

$$\le 3-\cos 2A-\cos 2B-\cos 2C$$

$$\cos(2B-2C)+\cos(2A-2C)+\cos(2A-2B)\le 3\ \textbf{(LQQD)}$$

Solution 3 by Soumitra Mandal-India:

$$\sum_{cyc}(\sin 2A+\sin 2B)\left(\frac{1}{\sin 2A}+\frac{1}{\sin 2B}\right)\le\sum_{cyc}(\tan A+\tan B)(\cot A+\cot B)$$

$$\Leftrightarrow 0 \le \sum_{cyc} \left(\frac{\sin 2A}{\sin 2B} - \tan A \cot B + \frac{\sin 2B}{\sin 2A} - \tan B \cot A \right)$$

$$\Leftrightarrow 0 \le \sum_{cyc} \left(\frac{\sin A}{\sin B} - \frac{\sin B}{\sin A} \right) \left(\frac{\cos B}{\cos A} - \frac{\cos A}{\cos B} \right)$$

$$\Leftrightarrow 0 \le \sum_{cyc} \left(\frac{\sin^2 A - \sin^2 B}{\sin A \sin B} \right) \left(\frac{\cos^2 B - \cos^2 A}{\cos A \cos B} \right)$$

$$\Leftrightarrow \le \sum_{cyc} \frac{(\sin^2 A - \sin^2 B)^2}{\sin A \sin B \cos A \cos B}, \text{ wich is true}$$

$$\sum_{cyc} (\sin 2A + \sin 2B) \left(\frac{1}{\sin 2A} + \frac{1}{\sin 2B} \right) \le$$

$$\le \sum_{cyc} (\tan A + \tan B)(\cot A + \cot B)$$

4. If $n \in \mathbb{N}$ then in $\Delta\ ABC$ the following relationship holds:

$$27 \sum \sin(2^n A) \cos(2^n B) \cos(2^n C) \le \left(\sum \sin(2^n A) \right)^3$$

Proposed by Daniel Sitaru-Romania

Solution by Kevin Soto Palacios-Huarmey-Peru:

Si: $n \ge 0$. Probar en un triángulo ABC:

$$27 \sum \operatorname{sen}(2^n A) \cos(2^n B) \cos(2^n C) \le \left(\sum \operatorname{sen}(2^n A) \right)^3$$

Sea: $A = \sum \operatorname{sen} kA \operatorname{sen} kB \cos kC \ \wedge B = \sum \cos kA \cos kB \operatorname{sen} kC \Leftrightarrow k \in \mathbb{Z}^+$

Se puede observar que:

$$(\cos A + i \operatorname{sen} A)(\cos B + i \operatorname{sen} B)(\cos C + i \operatorname{sen} C) = \cos(A + B + C) + i \operatorname{sen}(A + B + C)$$

$$(\cos A + i \operatorname{sen} A)(\cos B + i \operatorname{sen} B)(\cos C + i \operatorname{sen} C) = -1$$

$\Rightarrow$ **Por la fórmula de Moivre:**

$$(\cos kA + i\,\text{sen}\,kA)(\cos kB + i\,\text{sen}\,kB)(\cos kC + i\,\text{sen}\,kC) = (-1)^n$$

Desarrollando tenemos:

$$-A + iB + \cos kA \cos kB \cos kC - i\,\text{sen}\,kA\,\text{sen}\,kB\,\text{sen}\,kC = (-1)^n$$

Por lo tanto:

$$\to B = \text{sen}\,kA\,\text{sen}\,kB\,\text{sen}\,kC \wedge A = (-1)^{n+1} + \cos kA \cos kB \cos kC$$

Desde que: $B = \text{sen}\,kC\,\text{sen}\,kB\,\text{sen}\,kC$, **entonces:**

$$\sum \text{sen}(2^n A)\cos(2^n B)\cos(2^n C) = \text{sen}(2^n A)\,\text{sen}(2^n B)\,\text{sen}(2^n C)$$

La desigualdad es equivalente:

$$27\,\text{sen}(2^n A)\,\text{sen}(2^n B)\,\text{sen}(2^n C) \le \left(\sum \text{sen}(2^n A)\right)^3 \to \textbf{Válido } (MA \ge MG)$$

5. In acute - angled $\Delta\, ABC$ the following relationship holds:

$$\cos\left(\frac{\pi}{4} - A\right) + \cos\left(\frac{\pi}{4} - B\right) + \cos\left(\frac{\pi}{4} - C\right) > \frac{2S}{R^2}$$

S – area, R – circumradius

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

In acute – angled $\Delta\, ABC$: $\sum \cos\left(\frac{\pi}{4} - A\right) > \frac{2S}{R^2}$

$$\frac{2S}{R^2} = \frac{2}{R^2}\left(\frac{abc}{4R}\right) = \frac{abc}{2R^3} = \frac{(2R\sin A)(2R\sin B)(2R\sin C)}{2R^3}$$

$$= \frac{8R^3 \sin A \sin B \sin C}{2R^3} = 4\sin A \sin B \sin C$$

$$= 2\sin C\,(2\sin A \sin B) = 2\sin C\,(\cos(A-B) - \cos(A+B))$$

$$= 2\sin(A+B)\cos(A-B) - 2\sin C\,(-\cos C)$$

$$= \sin 2A + \sin 2B + \sin 2C$$

$$= ((\sin A + \cos A)^2 - 1) + ((\sin B + \cos B)^2 - 1) + ((\sin C + \cos C)^2 - 1)$$

$$= \left\{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos A + \frac{1}{\sqrt{A}}\sin A\right)\right\}^2 + \left\{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin B + \frac{1}{\sqrt{2}}\cos B\right)\right\}^2 +$$

$$+ \left\{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos C + \frac{1}{\sqrt{2}}\sin C\right)\right\}^2 - 3 =$$

$$= 2\sum\left(\cos\frac{\pi}{4}\cos A + \sin\frac{\pi}{4}\sin A\right)^2 - 3 = 2\sum\cos^2\left(\frac{\pi}{4} - A\right) - 3$$

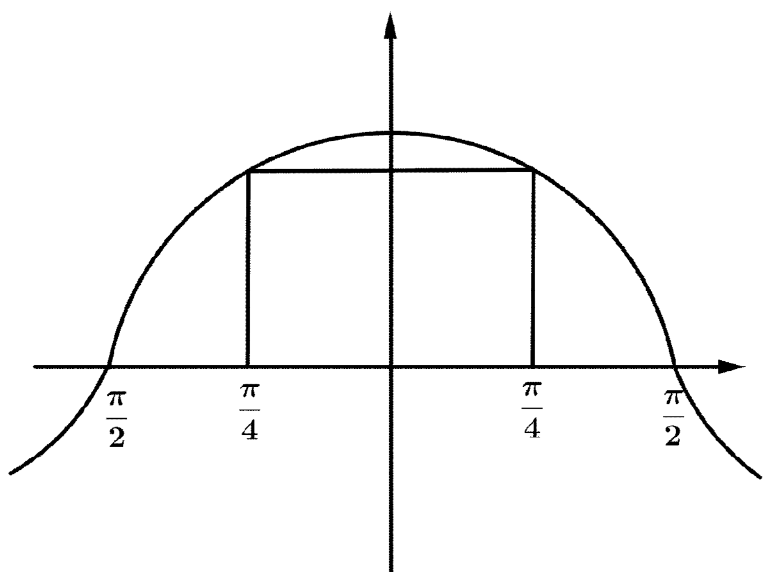
$$= 2\cos^2\left(\frac{\pi}{4} - A\right) + 2\cos^2\left(\frac{\pi}{4} - B\right) + 2\cos^2\left(\frac{\pi}{4} - C\right) - 3$$

Let $\cos\left(\frac{\pi}{4} - A\right) = x, \cos\left(\frac{\pi}{4} - B\right) = y, \cos\left(\frac{\pi}{4} - C\right) = z$

The proposed inequality $\Rightarrow x + y + z > 2x^2 + 2y^2 + 2z^2 - 3$

$\Leftrightarrow \sum(2x^2 - x - 1) < 0 \Leftrightarrow \sum(x - 1)(2x + 1) < 0$ **(1)**

$\Delta\ ABC$ is acute, $0 < A, B, C < \frac{\pi}{2}$



$$\Rightarrow -\frac{\pi}{2} < -A, -B, -C < 0 \Rightarrow -\frac{\pi}{4} < \frac{\pi}{4} - A, \frac{\pi}{4} - B, \frac{\pi}{4} - C < \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{\sqrt{2}} < \cos\left(\frac{\pi}{4} - A\right), \cos\left(\frac{\pi}{4} - B\right), \cos\left(\frac{\pi}{4} - C\right) < 1$$

$\Rightarrow \frac{1}{\sqrt{2}} < x, y, z < 1, x - 1 < 0$ and $(2x + 1) > 0$

$$\Rightarrow (x - 1)(2x + 1) < 0$$

Similarly, $(y - 1)(2y + 1) < 0$ and $(z - 1)(2z + 1) < 0$

$\sum(x - 1)(2x + 1) < 0 \Rightarrow$ (1) is proved, which proves the proposed inequality

6. En un triángulo ABC: Probar que si es (V) O (F)

$$a^4c + b^4a + c^4b \geq 24\sqrt[3]{2}(RS)^{\frac{5}{3}}$$

Proposed by Daniel Sitaru-Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

Desarrollando el lado derecho tenemos:

$$24\sqrt[3]{2}(RS)^{\frac{5}{3}} = 24\sqrt[3]{2}\left(\frac{abc}{4}\right)^{\frac{5}{3}} = 24\sqrt[3]{2}^3\sqrt{\frac{1}{2^{10}}(abc)^5} = 24\sqrt[3]{\frac{1}{2^9}(abc)^5} = 3\sqrt[3]{(abc)^5}$$

Por las desigualdades entre las medias:

Siendo: $a, b, c > 0$ (lados de un triángulo ABC)

$$a^4c + b^4a + c^4b \geq 3\sqrt[3]{(abc)^5} \quad \text{(LQQD)}$$

7. In $\Delta\ ABC$:

$$\sum(2a^2b^2 - c^4) \geq 16S^2$$

$$S - \text{area}$$

Proposed by Daniel Sitaru-Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

Tener presente en un triángulo ABC que:

$$a = 2R\,\mathrm{sen}\,A\,,\, b = 2R\,\mathrm{sen}\,B\,,\, c = 2R\,\mathrm{sen}\,C$$

$$S = 2R^2\,\mathrm{sen}\,A\,\mathrm{sen}\,B\,\mathrm{sen}\,C \ \wedge \cot A\cot B + \cot B\cot C + \cot A\cot C = 1$$

Reemplazando en la desigualdad:

$$2(16R^4\,\mathrm{sen}^2 A\,\mathrm{sen}^2 B) + 2(16R^4\,\mathrm{sen}^2 B\,\mathrm{sen}^2 C) + 2(16R^4\,\mathrm{sen}^2 A\,\mathrm{sen}^2 C)$$
$$- 16R^4(\mathrm{sen}^4 A + \mathrm{sen}^4 B + \mathrm{sen}^4 C) \geq 16S^2$$

Dividiendo a la expresión: $(16R^4\,\mathrm{sen}^2 A\,\mathrm{sen}^2 B\,\mathrm{sen}^2 C) \Leftrightarrow$ **(La desigualdad no se altera)**

$$\Rightarrow 2\csc^2 C + 2\csc^2 A + 2\csc^2 B - \left(\frac{\mathrm{sen}\,A}{\mathrm{sen}\,B\,\mathrm{sen}\,C}\right)^2 - \left(\frac{\mathrm{sen}\,B}{\mathrm{sen}\,A\,\mathrm{sen}\,C}\right)^2 - \left(\frac{\mathrm{sen}\,C}{\mathrm{sen}\,A\,\mathrm{sen}\,B}\right)^2 \geq 4$$

$$\Rightarrow 2(3 + \cot^2 A + \cot^2 B + \cot^2 C) - (\cot B + \cot C)^2 - (\cot A + \cot C)^2 - (\cot A + \cot B)^2 \geq 4$$

$$\Rightarrow 6 - 2(\cot A\cot B + \cot B\cot C + \cot A\cot C) \geq 4 \Leftrightarrow 4 \geq 4 \text{ (Lo cual si es cierto) (V)}$$

8. Prove that if $x, y, z \in [1, 2]$ then in $\Delta\, ABC$:

$$\sqrt[3]{abcxyz}\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)^2 \leq \frac{16s^3}{3}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

$$x \leq 2; y \leq 2; z \leq 2 \Rightarrow xyz \leq 8 \Rightarrow \sqrt[3]{xyz} \leq 2$$

$$\sqrt[3]{abcxyz} = \sqrt[3]{abc} \cdot \sqrt[3]{xyz} \leq \frac{a + b + c}{3} \cdot 2 = \frac{2(a + b + c)}{3}$$

$$x, y, z \geq 1 \Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \leq 1 \Rightarrow \frac{a}{x} + \frac{b}{y} + \frac{c}{z} \leq a + b + c$$

$$\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)^2 \le (a+b+c)^2$$

$$\sqrt[3]{abcxyz}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)^2 \le \frac{2(a+b+c)}{3}\cdot(a+b+c)^3 =$$

$$=\frac{2}{3}\cdot(2s)^3=\frac{16s^3}{3}$$

9. In $\Delta\, ABC$ the following relationship holds:

$$\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(\sqrt{a}-\sqrt{b}+\sqrt{c}\right)\left(-\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \ge \sqrt{(a+b-c)(a-b+c)(-a+b+c)}$$

Proposed by Lucian Stamate-Romania

Solution by Nguyen Viet Hung-Hanoi-Vietnam

From the condition $(b+c-a)(c+a-b)(a+b-c)\ge 0$, we deduce that

$$b+c-a\ge 0,\, c+a-b\ge 0,\, a+b-c\ge 0$$

The desired inequality is equivalent to:

$$\left(\sqrt{b}+\sqrt{c}-\sqrt{a}\right)^2\left(\sqrt{c}+\sqrt{a}-\sqrt{b}\right)^2\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)^2 \ge (b+c-a)(c+a-b)(a+b-c)$$

or

$$\left(a-\left(\sqrt{b}-\sqrt{c}\right)^2\right)\left(b-\left(\sqrt{c}-\sqrt{a}\right)^2\right)\left(c-\left(\sqrt{a}-\sqrt{b}\right)^2\right) \ge (b+c-a)(c+a-b)(a+b-c)$$

Now we will show that: $\left(a-\left(\sqrt{b}-\sqrt{c}\right)^2\right)\ge (c+a-b)(a+b-c)$

Indeed, this inequality is equivalent to: $\left(\sqrt{b}-\sqrt{c}\right)^2(b+c-a)\ge 0$,

which is true. Writing two similar relations and multiplying them up we obtain the desired result.

10. In $\Delta\ ABC, n \in \mathbb{N}^*$:

$$\prod_{k=1}^{n}\left(m_a + km_b + \frac{m_c}{k}\right)\left(m_b + km_c + \frac{m_a}{k}\right)\left(m_c + km_a + \frac{m_b}{k}\right) \geq (27w_aw_bw_c)^n$$

m_a – median's length, w_a – bisector's length

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

En un triángulo $ABC, n \in N$. Probar que:

$$\prod_{k=1}^{n}\left(m_a + km_b + \frac{m_c}{k}\right)\left(m_b + km_c + \frac{m_a}{k}\right)\left(m_c + km_a + \frac{m_b}{k}\right) \geq (27w_aw_bw_c)^n$$

Recodar lo siguiente: 1.

$$\prod_{k=1}^{n}[f(i)g(i)h(i)] = \left[\prod_{k=1}^{n} f(i)\right]\left[\prod_{k=1}^{n} g(i)\right]\left[\prod_{k=1}^{n} h(i)\right]$$

2. $w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{p(p-a)} \leq \sqrt{p(p-a)} \leq m_a$, **De forma análoga:**

$w_b \leq m_b \wedge w_c \leq m_c$. **Por la tanto: Por:** $MA \geq MG$

$$\prod_{k=1}^{n} f(i) = \left[(m_a + m_b + m_c)\left(m_a + 2m_b + \frac{m_c}{2}\right)\dots\left(m_a + nm_b + \frac{m_c}{n}\right)\right] \geq \left(3\sqrt[3]{m_am_bm_c}\right)^n \quad (A)$$

$$\prod_{k=1}^{n} g(i) = (m_a + m_b + m_c)\left(m_b + 2m_c + \frac{m_a}{2}\right)\dots\left(m_b + nm_c + \frac{m_a}{n}\right) \geq \left(3\sqrt[3]{m_am_bm_c}\right)^n \quad (B)$$

$$\prod_{k=1}^{n} h(i) = (m_a + m_b + m_c)\left(m_c + 2m_a + \frac{m_b}{2}\right)\dots\left(m_c + nm_a + \frac{m_b}{n}\right) \geq \left(3\sqrt[3]{m_am_bm_c}\right)^n \quad (C)$$

Multiplicando $(A)(B)(C) \rightarrow$

$$\rightarrow \prod_{k=1}^{n}\left(m_a + km_b + \frac{m_c}{k}\right)\left(m_b + km_c + \frac{m_a}{k}\right)\left(m_c + km_a + \frac{m_b}{k}\right) \geq (27m_am_bm_c)^n \geq (27w_aw_bw_c)^n$$

11. **In $\Delta\ ABC$ the following relationship holds:**

$$a^3(b+c)+b^3(c+a)+c^3(a+b)\ge 16RSp$$

$R-$ **circumscribed radius,** $S-$ **area,** $p-$ **semiperimeter**

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo si es (V) ó (F):

$$a^3(b+c)+b^3(c+a)+c^3(a+b)\ge 16RSp$$

En un triángulo ABC tener en cuenta, lo siguiente:

$$4RS=abc\ \wedge\ 2p=a+b+c$$

Desde que: a,b,c son lados de un triángulo, por la tanto: $a,b,c>0$

Aplicando: $MA\ge MG$

$$a^3b+a^3b+c^3b+a^3c+a^3c+b^3c\ge 6a^2bc \quad \text{(A)}$$

$$b^3c+b^3c+a^3c+b^3a+b^3a+c^3a\ge 6b^2ac \quad \text{(B)}$$

$$c^3b+c^3b+a^3b+c^3a+c^3a+b^3a\ge 6c^2ab \quad \text{(C)}$$

Sumando: $(A)+(B)+(C)$

$$a^3(b+c)+b^3(c+a)+c^3(a+b)\ge 2abc(a+b+c)$$

$$a^3(b+c)+b^3(c+a)+c^3(a+b)\ge 16RSp$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$a^3(b+c)+b^3(c+a)+c^3(a+b)\ge 16RSp$$

$$\Leftrightarrow ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\ge 16R\left(\frac{abc}{4R}\right)\left(\frac{a+b+c}{2}\right)$$

$$\ge 2ab \qquad\qquad \ge 2bc \qquad\qquad \ge 2ca$$

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)=2abc(a+b+c)$$

$$\ge 2a^2b^2+2b^2c^2+2c^2a^2\ (AM\ge GM)$$

$$= 2(x^2 + y^2 + z^2) \geq 2(xy + yz + zx) = 2abc(a + b + c)$$

(where $x = ab, y = bc, z = ca$)

12. In acute triangle ABC the following relationship holds:

$$\frac{27m_a^2m_b^2m_c^2}{S^6} \leq \left(\frac{1}{(p-a)^2} + \frac{1}{(p-b)^2} + \frac{1}{(p-c)^2}\right)^3$$

m_a − median length, S − area

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\Delta = S$$

$$\frac{27m_a^2m_b^2m_c^2}{\Delta^6} \leq \left\{\frac{1}{(p-a)^2} + \frac{1}{(p-b)^2} + \frac{1}{(p-c)^2}\right\}^3 \quad \textbf{(1)}$$

$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4};\ m_b^2 = \frac{27c^2 + 2a^2 - b^2}{4};\ m_c^2 = \frac{2a^2 + 2b^2 - c^2}{4}$$

$$\textbf{(1)} \Leftrightarrow \frac{27}{64}(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)$$

$$\leq \left\{\frac{\Delta^2}{(p-a)^2} + \frac{\Delta^2}{(p-b)^2} + \frac{\Delta^2}{(p-c)^2}\right\}^3$$

$$\Leftrightarrow \frac{3}{4}\sqrt[3]{(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)}$$

$$\leq \frac{\Delta^2}{(p-a)^2} + \frac{\Delta^2}{(p-b)^2} + \frac{\Delta^2}{(p-c)^2}$$

$$GM \leq AM \Rightarrow \frac{3}{4}\sqrt[3]{(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)}$$

$$\leq \frac{3}{4}\frac{(2b^2 + 2c^2 - a^2) + (2c^2 + 2a^2 - b^2) + (2a^2 + 2b^2 - c^2)}{3}$$

$= \frac{3}{4}(a^2+b^2+c^2)$; Δ is acute; $b^2+c^2-a^2>0, 2b^2+2c^2-a^2>0$....

similarly I shall show that, $\frac{3}{4}(a^2+b^2+c^2) \le \frac{\Delta^2}{(p-a)^2}+\frac{\Delta^2}{(p-b)^2}+\frac{\Delta^2}{(p-c)^2}$ **(2)**

which will prove (1)

$$RHS \text{ of (2)} = \frac{p(p-a)(p-b)(p-c)}{(p-a)^2}+\frac{p(p-a)(p-b)(p-c)}{(p-b)^2}+\frac{p(p-a)(p-b)(p-c)}{(p-c)^2}$$

$$= p\left\{\frac{(p-b)(p-c)}{p-a}+\frac{(p-c)(p-a)}{p-b}+\frac{(p-a)(p-b)}{p-c}\right\}$$

$$= \frac{(a+b+c)}{4}\left\{\frac{(c+a-b)(a+b-c)}{b+c-a}+\frac{(a+b-c)(b+c-a)}{c+a-b}+\frac{(b+c-a)(c+a-b)}{a+b+c}\right\}$$

$$(2) \Leftrightarrow 3(a^2+b^2+c^2) \le (a+b+c)\left\{\frac{(c+a-b)(a+b-c)}{b+c-a}+\frac{(a+b-c)(b+c-a)}{c+a-b}+\frac{(b+c-a)(c+a-b)}{a+b-c}\right\} \quad (3)$$

Let $a+b-c=x, b+c-a=y, c+a-b=z$

$$a+b+c=x+y+z, a=\frac{z+x}{2}, b=\frac{x+y}{2}, c=\frac{y+z}{2}$$

$$(3) \Leftrightarrow \frac{3}{4}\{(z+x)^2+(x+y)^2+(y+z)^2\} \le (x+y+z)\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)$$

$$\Leftrightarrow 6\left(\sum x^2+\sum xy\right)xyz \le 4\left(\sum x\right)\left(\sum x^2y^2\right)$$

$$\Leftrightarrow 2(x+y+z)(x^2y^2+y^2z^2+z^2x^2)$$

$$\ge 3(x^2+y^2+z^2+xy+yz+zx)xyz$$

$$\Leftrightarrow 2x^3y^2+2x^2y^3+2y^2z^2+2y^2z^3+2z^3x^2+2z^2x^3$$

$$+2xyz(xy+yz+zx) \ge 3xyz\left(\sum x^2\right)+3xyz(3xyz+2x)$$

$$\Leftrightarrow 2x^3y^2+2x^2y^3+2y^3z^2+2y^2z^3+2z^3x^2+2z^2x^3 \ge$$

$$\ge 3x^3yz+3y^3zx+3z^3xy+x^2y^2z+y^2z^2x+z^2x^2y \quad (4)$$

$$AM \ge GM \Rightarrow \frac{3}{2}(x^3y^2+x^3z^2) \ge 3x^3yz \quad \text{(i)}$$

$$\frac{3}{2}(y^3x^2 + y^3z^2) \ge 3y^3zx \quad \text{(ii)}$$

$$\frac{3}{2}(z^3x^2 + z^3y^2) \ge 3z^3xy \quad \text{(iii)}$$

$$(i) + (ii) + (iii) \Rightarrow \frac{3}{2}(x^3y^2 + x^2y^3 + y^3z^2 + y^2z^3 + z^3x^2 + z^2x^3) \ge$$

$$\ge 3x^3yz + 3y^3zx + 3z^3xy \quad \text{(A)}$$

$$\left.\begin{cases} x^3y^2 + y^2z^3 + y^2z^3 \ge 3xy^2z^2 \\ y^3x^2 + x^2z^3 + x^2z^3 \ge 3yx^2z^2 \\ y^3z^2 + z^2x^3 + z^2x^3 \ge 3yz^2x^2 \\ z^2y^2 + y^2x^3 + y^2x^3 \ge 3zy^2x^2 \\ z^3x^2 + x^2y^3 + x^2y^3 \ge 3zx^2y^2 \\ x^3z^2 + z^2y^3 + z^2y^3 \ge 3xy^2z^2 \end{cases}\right\} AM \ge GM$$

Adding,

$$3(x^3y^2 + x^2y^3 + y^3z^2 + y^2z^3 + z^3x^2 + z^2x^3) \ge 6(x^2y^2z + y^2z^2x + z^2x^2y)$$

$$\Rightarrow \frac{1}{2}(x^3y^2 + x^2y^3 + y^3z^2 + y^2z^3 + z^3x^2 + z^2x^3) \ge x^2y^2z + y^2z^2x + z^2x^2y$$

(B)

$$A + B \Rightarrow 2(x^3y^2 + x^2y^3 + y^3z^2 + y^2z^3 + z^3x^2 + z^2x^3) \ge$$

$$\ge 3x^3yz + 3y^3zx + 3z^3xy + x^2y^2z + y^2z^2x + z^2x^2y$$

which proves (4) and thus, the inequality.

13. In $\Delta\ ABC$ the following relationship holds:

$$\sum_{cyc} \frac{1}{a^2}\left(\frac{b}{a^2} + \frac{1}{b}\right) \ge \frac{3}{2RS}$$

$S -$ area, $R -$ circumscribed radius

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios –Huarmey- Peru

Probar en un triángulo ABC si es (V) o (F): $\sum \frac{1}{a^2}\left(\frac{b}{a^2}+\frac{1}{b}\right) \geq \frac{3}{2RS}$

Tener presente en un triángulo ABC: $4RS = abc$

La desigualdad es equivalente: $\frac{1}{a^2}\left(\frac{b}{a^2}+\frac{1}{b}\right)+\frac{1}{b^2}\left(\frac{c}{b^2}+\frac{1}{c}\right)+\frac{1}{c^2}\left(\frac{a}{c^2}+\frac{1}{a}\right) \geq 6abc$

Por desigualdades entre las medias: $MA \geq MG$, Siendo: $a, b, c > 0$

$$\frac{b}{a^2}+\frac{1}{b} \geq \frac{2}{a} \Leftrightarrow \frac{1}{a^2}\left(\frac{b}{a^2}+\frac{1}{b}\right) \geq \frac{2}{a^3} \quad \textbf{(A)}$$

$$\frac{1}{b^2}\left(\frac{c}{b^2}+\frac{1}{c}\right) \geq \frac{2}{b^3} \quad \textbf{(B)}$$

$$\frac{1}{c^2}\left(\frac{a}{c^2}+\frac{1}{a}\right) \geq \frac{2}{c^3} \quad \textbf{(C)}$$

Sumando: $(A)+(B)+(C)$ y aplicando: $MA \geq MG$

$$\frac{1}{a^2}\left(\frac{b}{a^2}+\frac{1}{b}\right)+\frac{1}{b^2}\left(\frac{c}{b^2}+\frac{1}{c}\right)+\frac{1}{c^2}\left(\frac{a}{c^2}+\frac{1}{a}\right) \geq \frac{2}{a^3}+\frac{2}{b^3}+\frac{2}{c^3} \geq \frac{6}{abc} \quad \textbf{(LQQD)} \rightarrow$$

La desigualdad es (V).

14. In $\Delta\ ABC$ the following inequality holds:

$$\sum \frac{\left(a\sqrt{a}+b\sqrt{b}\right)\sin C}{\sqrt{a}+\sqrt{b}} \leq \sum \frac{\left(a^3\sqrt{a}+b^3\sqrt{b}\right)\sin C}{a^2\sqrt{a}+b^2\sqrt{b}}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru:

En un triángulo ABC. Probar que: $\sum \frac{\left(a\sqrt{a}+b\sqrt{b}\right)\sin C}{\sqrt{a}+\sqrt{b}} \leq \sum \frac{\left(a^3\sqrt{a}+b^3\sqrt{b}\right)\sin C}{a^2\sqrt{a}+b^2\sqrt{b}} \Leftrightarrow$

$$\Leftrightarrow \sum \frac{\left(a^3\sqrt{a}+b^3\sqrt{b}\right)\sin C}{a^2\sqrt{a}+b^2\sqrt{b}} - \sum \frac{\left(a\sqrt{a}+b\sqrt{b}\right)\sin C}{\sqrt{a}+\sqrt{b}} \geq 0$$

Realizamos los siguientes cambios de variables:

$$\sqrt{a}=x^2,\sqrt{b}=y^2,\sqrt{c}=z^2 \Leftrightarrow a,b,c>0, \text{ por lo tanto: } x,y,z>0$$

$$\Rightarrow \frac{(a\sqrt{a}+b\sqrt{b})\sin C}{\sqrt{a}+\sqrt{b}} \le \frac{(a^3\sqrt{a}+b^3\sqrt{b})\sin C}{a^2\sqrt{a}+b^2\sqrt{b}}$$

$$\Rightarrow \sin C\left(\frac{x^7+y^7}{x^5+y^5}-\frac{x^3+y^3}{x+y}\right)\ge 0 \to \sin C\left(\frac{(x^7+y^7)(x+y)-(x^3+y^3)(x^5+y^5)}{(x+y)(x^5+y^5)}\right)\ge 0$$

$$\Rightarrow \sin C\frac{xy(x^6+y^6)-x^3y^3(x^2+y^2)}{(x+y)(x^5+y^5)}\ge 0 \to \sin C\frac{xy[x^6+y^6-x^2y^2(x^2+y^2)]}{(x+y)(x^5+y^5)}\ge 0$$

$$\Rightarrow \sin C\frac{xy[(x^2+y^2)^3-4x^2y^2(x^2+y^2)]}{(x+y)(x^5+y^5)} \to \sin C\frac{xy(x^2+y^2)[(x^2+y^2)^2-4x^2y^2]}{(x+y)(x^5+y^5)}\ge 0$$

$$\Rightarrow \sin C\frac{xy(x^2-y^2)^2}{(x+y)(x^5+y^5)}\ge 0 \quad \text{(A)} \quad \text{Por la tanto: } \sin A\frac{yz(y^2-z^2)^2}{(y+z)(y^5+z^5)}\ge 0 \quad \text{(B)}$$

$$\Rightarrow \sin B\frac{xz(z^2-x^2)^2}{(x+z)(x^5+z^5)}\ge 0 \quad \text{(C)} \to \text{Sumando } (A)+(B)+(C):$$

$$\sum\frac{(a^3\sqrt{a}+b^3\sqrt{b})\sin C}{a^2\sqrt{a}+b^2\sqrt{b}}-\sum\frac{(a\sqrt{a}+b\sqrt{b})\sin C}{\sqrt{a}+\sqrt{b}}\ge 0$$

15. In $\Delta\, ABC$ the following relationship holds:

$$\frac{1}{r^3}\sum a^3\cos B\cos C\ge 16\left(\sum\sin A\right)\left(\sum\cos^2 A\right)$$

r – inradius

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$a^3\cos B\cos C=R^3(8\sin^3 A\cos B\cos C)$$

$$8\sin^3 A\cos B\cos C=(2\sin^2 A)(4\sin A\cos B\cos C)=$$

$$=(1-\cos 2A)(2\cos C)(2\sin A\cos B)=$$

$$=(1-\cos 2A)(2\cos C)\{\sin(A+B)+\sin(A-B)\}=$$

$$=(1-\cos 2A)\{2\cos C\sin C-2\cos(A+B)\sin(A-B)\}=$$

$$= (1-\cos 2A)\{\sin 2C - (\sin 2A - \sin 2B)\} =$$

$$= (1-\cos A)(\sin 2B + \sin 2C - \sin 2A) =$$

$$= \sin 2B + \sin 2C - \sin 2A - \cos 2A \sin 2B - \cos 2A \sin 2C + \cos 2A \sin 2A$$

(1)

Similarly, $8\sin^3 B\cos C\cos A = (1-\cos 2B)(\sin 2C + \sin 2A - \sin 2B)$

$$= \sin 2C + \sin 2A - \sin 2B - \cos 2B \sin 2C - \cos 2B \sin 2A + \sin 2B \cos 2B$$

(2)

and $8\sin^3 C\cos A\cos B = (1-\cos 2C)(\sin 2A + \sin 2B - \sin 2C)$

$$= \sin 2A + \sin 2B - \sin 2C - \cos 2C \sin 2A - \cos 2C \sin 2B + \sin 2C \cos 2C$$

(3)

$$(1)+(2)+(3) \Rightarrow \frac{1}{r^3}\sum a^3\cos B\cos C$$

$$= \frac{R^3}{r^3}\big((\sin 2A + 2\sin 2B + \sin 2C) - \sin 2C\,(\cos 2A + \cos 2B) - \sin 2B(\cos 2C + \cos 2A) - \sin 2A\,(\cos 2B + \cos 2C) + \cos 2A \sin 2A + \cos 2B \sin 2B + \cos 2C \sin 2C\big)$$

$$-\sin 2C\,(\cos 2A + \cos 2B) = -\sin 2C\,\{2\cos(A+B)\cos(A-B)\}$$

$$= -2\sin C\cos C\,(-2\cos C\cos(A-B))$$

$$= 4\cos^2 C\sin(A+B)\cos(A-B) = 2\cos^2 C\,(\sin 2A + \sin 2B)$$

$$= (1+\cos 2C)(\sin 2A + \sin 2B)$$

$$= (\sin 2A + \sin 2B) + \cos 2C\,(\sin 2A + \sin 2B) \quad \text{(4)}$$

Similarly, $-\sin 2B\,(\cos 2C + \cos 2A)$

$$= (\sin 2C + \sin 2A) + \cos 2B\,(\sin 2C + \sin 2A) \quad \text{(5)}$$

and $-\sin 2A\,(\cos 2B + \cos 2C)$

$$= (\sin 2B + \sin 2C) + \cos 2A\,(\sin 2B + \sin 2C) \quad \text{(6)}$$

$$\frac{1}{r^3}\sum a^3\cos B\cos C$$

$$=\frac{R^3}{r^3}\left\{3\sum\sin 2A+(\cos 2A+\cos 2B+\cos 2C)\left(\sum\sin 2A\right)\right\}$$

$$=\frac{R^3}{r^3}\left(\sum\sin 2A\right)\{(1+\cos 2A)+(1+\cos 2B)+(1+\cos 2C)\}$$

$$=\frac{R^3}{r^3}(\sum\sin 2A)(2)(\sum\cos^2 A)=\frac{2R^3}{r^3}(\sum\sin 2A)(\sum\cos^2 A)\quad \textbf{(A)}$$

$$\sum\sin 2A=\sin 2A+\sin 2B+\sin 2C=$$

$$=2\sin(A+B)\cdot\cos(A-B)+2\sin C\cos C=$$

$$=2\sin C\{\cos(A-B)-\cos(A+B)\}=$$

$$=2\sin C\cdot 2\sin A\sin B=4\sin A\sin B\sin C=$$

$$=4\cdot 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=$$

$$=\left(8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\quad \textbf{(7)}$$

Now, $\sin A+\sin B+\sin C=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

(7) $\Rightarrow\sum\sin 2A=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}(\sum\sin A)$

$$=8\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}\left(\sum\sin A\right)$$

$$=\frac{8s(s-a)(s-b)(s-b)}{sabc}\left(\sum\sin A\right)=\left(\frac{8\Delta^2}{sabc}\right)\left(\sum\sin A\right)$$

$\Delta=\frac{abc}{4R}$ and $\Delta=rs$, $\Delta^2=\frac{(sabc)r}{4R}$

$$\sum \sin 2A = \frac{8(sabc)r}{4R(sabc)}\left(\sum \sin A\right) = \frac{2r}{R}\left(\sum \sin A\right)$$

(A) $\Rightarrow \frac{1}{r^3}(\sum a^3 \cos B \cos C) = \left(\frac{2R^3}{r^3}\right)\left(\frac{2r}{R}(\sum \sin A)\right)(\sum \cos^2 A)$

(B)

$$= 4\left(\frac{R^2}{r^2}\right)(\sum \sin A)(\sum \cos^2 A) \geq 4(2^2)(\sum \sin A)(\sum \cos^2 A) \quad (R \geq 2r)$$

$$= 16\left(\sum \sin A\right)\left(\sum \cos^2 A\right)$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC. Probar que:

$\frac{1}{r^3}\sum a^3 \cos B \cos C \geq 16(\sum \operatorname{sen} A)(\cos^2 A) \rightarrow r$ **(Inradio)**

$$\frac{R^3}{r^3}(8\operatorname{sen}^3 A \cos B \cos C + 8\operatorname{sen}^3 B \cos A \cos C + 8\operatorname{sen}^3 C \cos A \cos B) \geq$$

$$\geq 16(\operatorname{sen} A + \operatorname{sen} B + \operatorname{sen} C)(\cos^2 A + \cos^2 B + \cos^2 C)$$

Tener presente en un triángulo ABC:

1. $\operatorname{sen} 4A + \operatorname{sen} 4B + \operatorname{sen} 4C = -4\operatorname{sen} 2A \operatorname{sen} 2B \operatorname{sen} 2C$

2. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

3. $\frac{r}{R} = 4\operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}$

4. $\operatorname{sen}(B + C) = \operatorname{sen} A \wedge$ **5.** $\operatorname{Sen}(2B + 2C) = -\operatorname{sen} 2A$

$$T_1 = 8\operatorname{sen}^3 A \cos B \cos C \rightarrow T_1 = (2\operatorname{sen}^2 A)(2\operatorname{sen} A)(\cos(B + C) + \cos(B - C))$$

$$T_1 = (1 - \cos 2A)(2\operatorname{sen}(B + C)(\cos(B + C) + \cos(B - C))$$

$$T_1 = (1 - \cos 2A)(\operatorname{sen}(2B + 2C) + \operatorname{sen} 2B + \operatorname{sen} 2C)$$

$$T_1 = (-\operatorname{sen} 2A + \operatorname{sen} 2B + \operatorname{sen} 2C) - \operatorname{sen} 2B \cos 2A - \operatorname{sen} 2C \cos 2A + (0.5)2\operatorname{sen} 2A \cos 2A$$

$$T_2 = 8\operatorname{sen}^3 B \cos A \cos C \rightarrow T_2$$

$$= (-\operatorname{sen} 2B + \operatorname{sen} 2A + \operatorname{sen} 2C) - \operatorname{sen} 2A \cos 2B - \operatorname{sen} 2C \cos 2B + (0,5)2\operatorname{sen} 2B \cos 2B$$

$$T_3 = 8\,\text{sen}^3 C \cos A \cos B \to T_3$$
$$= (-\,\text{sen}\,2C + \text{sen}\,2A + \text{sen}\,2B) - \text{sen}\,2A\cos 2C - \text{sen}\,2B\cos 2C + (0,5)2\,\text{sen}\,2C\cos 2C$$

$$T_1 + T_2 + T_3 = 2(\text{sen}\,2A + \text{sen}\,2B + \text{sen}\,2C) - 2\,\text{sen}\,2A\,\text{sen}\,2B\,\text{sen}\,2C$$

$$T_1 + T_2 + T_3 = 2(4\,\text{sen}\,A\,\text{sen}\,B\,\text{sen}\,C)(1 - 2\cos A\cos B\cos C) \geq$$

$$\geq 16(\text{sen}\,A + \text{sen}\,B + \text{sen}\,C)(\cos^2 A + \cos^2 B + \cos^2 C)\left(4\,\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}\right)^3$$

$$8\,\text{sen}\,A\,\text{sen}\,B\,\text{sen}\,C \geq 16\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\left(2\,\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}\right)8x4\left(\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}\right)^2 \to$$

$$\to \frac{1}{64} \geq \left(\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}\right)^2 \to \frac{1}{8} \geq \text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}$$

16. In $\Delta\ ABC$ the following relationship holds:

$$\sum\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}} \leq \cos\frac{\pi - A}{4} + \cos\frac{\pi - B}{4} + \cos\frac{\pi - C}{4}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

En un triángulo ABC. Probar si es (V) o (F):

$$\sum\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}} \leq \cos\left(\frac{\pi - A}{4}\right) + \cos\left(\frac{\pi - B}{4}\right) + \cos\left(\frac{\pi - C}{4}\right)$$

Sea: $T_1 = \sqrt{\cos\frac{A}{2}\cos\frac{B}{2}} = \sqrt{\frac{\cos\left(\frac{A+B}{2}\right) + \left(\cos\frac{A-B}{2}\right)}{2}} \Leftrightarrow \cos\left(\frac{A-B}{2}\right) \leq 1$

$$\Rightarrow T_1 = \sqrt{\frac{\cos\left(\frac{A+B}{2}\right) + \left(\cos\frac{A-B}{2}\right)}{2}} \leq \sqrt{\frac{\text{sen}\frac{C}{2} + 1}{2}} = \sqrt{\frac{1 + \cos\left(\frac{\pi - C}{2}\right)}{2}} =$$

$$= \sqrt{\cos^2\left(\frac{\pi - C}{4}\right)} = \cos\left(\frac{\pi - C}{4}\right) \quad \text{(A)}$$

Análogamente para los demás términos:

$$\Rightarrow T_2=\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}\le\cos\left(\frac{\pi-A}{4}\right)\quad \text{(B)}$$

$$T_3=\sqrt{\cos\frac{A}{2}\cos\frac{C}{2}}\le\cos\left(\frac{\pi-B}{4}\right)\quad \text{(C)}$$

Sumando: $(A)+(B)+(C)$:

$$\sum\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}}\le\cos\left(\frac{\pi-A}{4}\right)+\cos\left(\frac{\pi-B}{4}\right)+\cos\left(\frac{\pi-C}{4}\right);\ \text{(LQQD)}\rightarrow\text{es (V)}$$

17. In $\Delta\ ABC$ the following relationship holds:

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\ge\frac{\sqrt{S}}{2R}$$

$R-$ circumradius, $S-$ area

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios –Huarmey- Peru

En un triángulo ABC: Probar que si es (V) o (F) lo siguiente:

$$\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}+\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\ge\frac{\sqrt{2}}{2R}$$

Tener en cuenta lo siguiente: $S=2R^2\,\text{sen}\,A\,\text{sen}\,B\,\text{sen}\,C$

$\text{sen}\,2x=2\,\text{sen}\,x\cos x$. Por desigualdades entre las medias:

$$MA\ge MG$$

$$\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}+\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\ge 2\sqrt[2]{\frac{2R^2\,\text{sen}\,A\,\text{sen}\,B\,\text{sen}\,C}{8x2R^2}}$$

$$\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}+\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\ge\frac{\sqrt{S}}{2R}\quad \text{(LQQD)}\rightarrow\text{(V)}$$

18. **In $\Delta\ ABC$ the following relationship holds:**

$$(m_a+r_a)(m_b+r_b)(m_c+r_c)\ge 8w_aw_bw_c$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

En un triángulo ABC. Probar que: $(m_a+r_a)(m_b+r_b)(m_c+r_c)\ge 8w_aw_bw_c$

Considerar lo siguiente en un triángulo ABC:

$$m_a=\frac{\sqrt{2b^2+2c^2-a^2}}{2},m_b=\frac{\sqrt{2a^2+2c^2-b^2}}{2},m_c=\frac{\sqrt{2a^2+2b^2-c^2}}{2}$$

$$r_ar_br_c=S_p,S=\sqrt{p(p-a)(p-b)(p-c)}$$

$\Leftrightarrow S=$ Área de región triangular, $p=$ semiperímetro

$$w_a=\frac{2\sqrt{bc}\sqrt{p(p-a)}}{b+c},w_b=\frac{2\sqrt{ac}\sqrt{p(p-b)}}{a+c},w_c=\frac{2\sqrt{ab}\sqrt{p(p-c)}}{a+b}$$

Por: $MA\ge MG$: $(m_a+r_a)(m_b+r_b)(m_c+r_c)\ge 8\sqrt{m_am_bm_cr_ar_br_c}$

Ahora demostraremos que: $m_am_bm_c\ge r_ar_br_c$

Por: $MP\ge MA\to 2(b^2+c^2)\ge(b+c)^2\to 2(b^2+c^2)-a^2\ge(b+c)^2-a^2$

$\to\frac{\sqrt{2(b^2+c^2)-a^2}}{2}\ge\sqrt{(b+c+a)(b+c-a)}(0,5)\to m_a\ge\sqrt{p(p-a)}$ (A)

Por lo tanto: $m_b\ge\sqrt{p(p-b)}$ (B); $m_c\ge\sqrt{p(p-c)}$ (C)

Multiplicando $(A)(B)(C)\to m_am_bm_c\ge r_ar_br_c$. Por lo tanto:

$$(m_a+r_a)(m_b+r_b)(m_c+r_c)\ge 8\sqrt{m_am_bm_cr_ar_br_c}\ge$$

$$\ge 8r_ar_br_c=8Sp\frac{(a+b)(b+c)(a+c)}{(a+b)(b+c)(a+c)}\ge$$

$$\ge\frac{64abcp\sqrt{p(p-a)(p-b)(p-c)}}{(a+b)(b+c)(a+c)}=8w_aw_bw_c$$

19. In ΔABC:

$$\frac{\sin A\sin B}{m_a m_b}+\frac{\sin B\sin C}{m_b m_c}+\frac{\sin C\sin A}{m_c m_a}\geq\frac{1}{R^2}$$

m_a – median's length, R – circumradius

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

Tener en cuenta las siguientes desigualdades:

$$4m_a m_b\leq(2c^2+ab),4m_b m_c\leq(2a^2+bc),4m_a m_c\leq(2b^2+ac)$$

La desigualdad es equialente: $\frac{1}{R^2}\cdot\frac{ab}{4m_a m_b}+\frac{1}{R^2}\cdot\frac{bc}{4m_b m_c}+\frac{1}{R^2}\cdot\frac{ac}{4m_a m_c}\geq\frac{1}{R^2}$

$$\Rightarrow\frac{ab}{2c^2+ab}+\frac{bc}{2a^2+bc}+\frac{ac}{2b^2+ac}\geq 1$$

$$\Rightarrow ab(2b^2+ac)(2a^2+bc)+bc(2c^2+ab)(2b^2+ac)+$$

$$+ac(2c^2+ab)(2a^2+bc)\geq(2c^2+ab)(2a^2+bc)(2b^2+ac)$$

$$\Rightarrow A=ab(4a^2b^2+2b^3c+2a^2c+b^2ac)+bc(4b^2c^2+2b^3a+2c^3a+a^2bc)+$$

$$+ac(4a^2c^2+2a^3b+2c^3b+b^2ac)$$

$$\Rightarrow A=(4a^3b^3+2b^2ac+2a^4cb+a^2b^2c^2)+(4b^3c^3+2b^4ca+2c^4ab+a^2b^2c^2)+$$

$$+(4a^3c^3+2a^4cb+2c^4ab+a^2b^2c^2)$$

$$\Rightarrow A=4a^3b^3+4b^3c^3+4a^3c^3+4a^2cb+4b^4ac+4c^4ab+3a^2b^2c^2$$

$$\Rightarrow B=(2c^2+ab)(2a^2+bc)(2b^2+ac)=(4a^2c^2+2a^3b+2c^3b+b^2ac)(2b^2+ac)$$

$$\Rightarrow B=(2c^2+ab)(2a^2+bc)(2b^2+ac)=$$

$$=(8a^2b^2c^2+4a^3b^3+4b^3c^3+2b^4ca+4a^3c^3+2a^4cb+2c^4ba+b^2a^2c^2)$$

$$\Rightarrow B=(2c^2+ab)(2a^2+bc)(2b^2+ac)=$$

$$=4a^3b^3+4b^3c^3+4a^3c^3+4a^3c^3+2a^4cb+2b^4ac+2c^4ab+9b^2a^2c^2$$

$\Rightarrow$ **Por la tanto nos queda:**

$$4a^3b^3+4b^3c^3+4a^3c^3+4a^4cb+4b^4ac+4c^4ab+3a^2b^2c^2\geq$$

$$\geq 4a^3b^3+4b^3c^3+4a^3c^3+4a^3c^3+2a^4cb+2b^4ac+2c^4ab+9b^2a^2c^2$$

$$\Rightarrow 2a^4cb+2b^4ca+2c^4ab\geq 6a^2b^2c^2 \Leftrightarrow$$ (Válido por: $MA\geq MG$)

20. In acute triangle ABC the following relationship holds:

$$2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\tan B\tan C\geq 9\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

En un triángulo acutángulo ABC:

$$2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\tan B\tan C\geq 9\sqrt{3}.$$ Si: $A+B+C=\pi$

$$\to \tan(A+B)=\tan(\pi-C)\to\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$

$$\Rightarrow \tan A+\tan B+\tan C=\tan A\tan B\tan C$$

Desde que es un triángulo acutángulo: $\tan A, \tan B, \tan C>0$.

Por: $MA\geq MG$: $\tan A+\tan B+\tan C\geq 3\sqrt[3]{\tan A\tan B\tan C}\to$

$$\to(\tan A\tan B\tan C)^3\geq 27\tan A\tan B\tan C$$

$$\Rightarrow \tan A\tan B\tan C\geq 3\sqrt{3}\quad \text{(A)}$$

Si: $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\to\cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)\to\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

Por: $MA\geq MG$: $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\geq 3\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}\to$

$$\to\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^3\geq 27\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

$$2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\geq 6\sqrt{3}\quad \text{(B)}\to$$

→ **Sumando:** $(A)+(B)\rightarrow 2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\tan B\tan C\geq 9\sqrt{3}$

21. In ΔABC the following relationship holds:

$$\frac{r_a^2}{a}+\frac{r_b^2}{b}+\frac{r_c^2}{c}\geq\frac{81r^2}{2p}$$

r_a – exinscribed radius, r – inscribed radius

p – semiperimeter

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$\frac{r_a^2}{a}+\frac{r_b^2}{b}+\frac{r_c^2}{c}\geq\frac{81r^2}{2p};\ p=s\Leftrightarrow\frac{\Delta^2}{a(s-a)^2}+\frac{\Delta^2}{b(s-b)^2}+\frac{\Delta^2}{c(s-c)^2}\geq\frac{81r^2}{2s^3}$$

$\Delta=rs$. **Let** $a+b-c=x, b+c-a=y, c+a-b=z$

$$a=\frac{z+x}{2}, b=\frac{x+y}{2}, c=\frac{y+z}{2}, a+b+c=x+y+z$$

$$\Leftrightarrow\frac{8}{y^2(z+x)}+\frac{8}{z^2(x+y)}+\frac{8}{x^2(y+z)}\geq\frac{8\cdot 8}{2(x+y+z)^3}$$

$$\Leftrightarrow\frac{y^2z^2(x+y)(z+x)+z^2x^2(x+y)(y+z)+x^2y^2(y+z)(z+x)}{x^2y^2z^2(x+y)(y+z)(z+x)}\geq$$

$$\geq\frac{8}{2(x+y+z)^3}$$

$$\Leftrightarrow\frac{(xy+yz+zx)(x^2y^2+y^2z^2+z^2x^2)+3x^2y^2z^2}{x^2y^2z^2(x+y)(y+z)(z+x)}\geq\frac{8}{2(x+y+z)^3}$$

$$AM\geq GM\Rightarrow\sqrt[3]{xyz}\leq\frac{x+y+z}{3}\Rightarrow\frac{1}{xyz}\geq\frac{27}{(x+y+z)^3}$$

So, if we can prove: $\frac{(\sum xy)(\sum x^2y^2)+3x^2y^2z^2}{xyz(x+y)(y+z)(z+x)} \geq \frac{3}{2}$ **(1) we are done**

$$(1) \Leftrightarrow 2\sum x^3y^3 + 2\left(\sum x^3y^2z + \sum x^2y^3z\right) + 6x^2y^2z^2 \geq$$

$$\geq 6x^2y^2z^2 + 3xyz\left(\sum x^2y + \sum xy^2\right)$$

$$\Leftrightarrow 2\sum x^3y^3 \geq xyz\left(\sum x^2y + \sum xy^2\right) \quad (2)$$

$$\left.\begin{array}{l} x^3y^3 + x^3y^3 + y^3z^3 \geq 3y^3x^2z \\ x^3y^3 + y^3z^3 + y^3z^3 \geq 3y^3z^2x \\ y^3z^3 + y^3z^3 + z^3x^3 \geq 3z^3y^2x \\ y^3z^3 + z^3x^3 + z^3x^3 \geq 3z^3x^2y \\ z^3x^3 + z^3x^3 + x^3y^3 \geq 3x^3z^2y \\ z^3x^3 + x^3y^3 + x^3y^3 \geq 3x^3y^2z \end{array}\right\} \begin{array}{c} \Rightarrow 6\sum x^3y^3 \\ \geq 3xyz\left(\sum x^3y + \sum xy^2\right) \\ \Rightarrow 2\sum x^3y^3 \\ \geq xyz\left(\sum x^2y + \sum xy^2\right) \\ \Rightarrow (2)\ \textit{is proved} \end{array}$$

Solution 2 by Kevin Soto Palacios – Peru

Probar en un triángulo ABC: $\frac{r_a^2}{a} + \frac{r_b^2}{b} + \frac{r_c^2}{c} \geq \frac{81r^2}{2p}$

Tener en cuenta lo siguiente: $r_ar_br_c = S_p,\ \ S = pr$

Por: $MA \geq MG$: $\frac{r_a^2}{a} + \frac{r_b^2}{b} + \frac{r_c^2}{c} \geq 3\sqrt[3]{\frac{r_a^2}{a}\cdot\frac{r_b^2}{b}\cdot\frac{r_c^2}{c}} = 3\sqrt[3]{\frac{p^4r^2(a+b+c)^3}{abc(a+b+c)^3}} \geq$

$$\geq 3\sqrt[3]{\frac{p^4r^2 27abc}{abc(a+b+c)^3}} \geq 3\sqrt[3]{\frac{\left(3\sqrt{3}r\right)^4 27r^2}{(a+b+c)^3}} = \frac{81r^2}{2p}$$

Se aplico lo siguiente: $(a+b+c)^3 \geq 27abc \Leftrightarrow$ **Válido por:** ($MA \geq MG$)

$p \geq 3\sqrt{3}r$ **(Lo cual demostraremos)**

$$\frac{p}{r} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geq 3\sqrt{3} \rightarrow p \geq 3\sqrt{3}r$$

22. In $\Delta\ ABC, I -$ incentre, the following relationship holds:

$$s^2\sum AI^2 > m_a m_b w_a w_b + m_b m_c w_b w_c + m_c m_a w_c w_a$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a m_b = \frac{\sqrt{2b^2+2c^2-a^2}}{2}\cdot\frac{\sqrt{2c^2+2a^2-b^2}}{2}$$

$$= \frac{\sqrt{(b+c)^2+(b-c)^2-a^2}\sqrt{(c+a)^2+(c-a)^2-b^2}}{4}$$

$$= \frac{\sqrt{(b+c)^2+(b-c+a)(b-c-a)}\sqrt{(c+a)^2+(c-a+b)(c-a-b)}}{4}$$

$$\left.\begin{matrix} a+b>c, & b-c+a>0 \\ b<c+a, & b-c-a<0\end{matrix}\right\} \Rightarrow (b-c+a)(b-c-a)<0$$

$$\left.\begin{matrix} b+c>a, & c-a+b>0 \\ c<a+b, & c-a-b<0\end{matrix}\right\} \Rightarrow (c-a+b)(c-a-b)<0$$

$$\frac{\sqrt{(b+c)^2+(b-c+a)(b-c-a)}\sqrt{(c+a)^2+(c-a+b)(c-a-b)}}{4} <$$

$$< \frac{1}{4}\sqrt{(b+c)^2}\sqrt{(c+a)^2} = \frac{(b+c)(c+a)}{4}$$

$m_a m_b < \frac{(b+c)(c+a)}{4}$. Similarly, $m_b m_c < \frac{(c+a)(a+b)}{4}$ and $m_c m_a < \frac{(a+b)(b+c)}{4}$

$$\text{Now } w_a w_b = \left(\frac{2bc\cos\frac{A}{2}}{b+c}\right)\left(\frac{2ca\cos\frac{B}{2}}{c+a}\right) = \frac{4c^2ab\cos\frac{A}{2}\cos\frac{B}{2}}{(b+c)(c+a)}$$

$m_a m_b w_a w_b < c^2ab\cos\frac{A}{2}\cos\frac{B}{2}$. Similarly, $m_b m_c w_b w_c < a^2bc\cos\frac{B}{2}\cos\frac{B}{2}$

and $m_c m_a w_c w_a < b^2 \cos\frac{C}{2}\cos\frac{A}{2}$

$$\sum m_a m_b w_a w_b < c^2 ab\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ca}} + a^2bc\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}} +$$

$$+b^2ca\sqrt{\frac{s(s-c)}{ab}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= cs(\sqrt{ab})\left(\sqrt{(s-a)(s-b)}\right) + as(\sqrt{bc})\left(\sqrt{(s-b)(s-c)}\right) +$$

$$+bs\sqrt{(s-c)(s-a)} \leq$$

$$\leq s\left(c\left(\frac{a+b}{2}\right)\left(\frac{2s-(a+b)}{2}\right) + a\left(\frac{b+c}{2}\right)\left(\frac{2s-(b+c)}{2}\right) + b\left(\frac{c+a}{2}\right)\left(\frac{2s-(c+a)}{2}\right)\right)$$

(by $AM \geq GM$)

$$\sum m_a m_b w_a w_b \leq \frac{s}{4}\left(c^2(a+b) + a^2(b+c) + b^2(c+a)\right)$$

$$= \frac{S}{4}\left(\sum a^2b + \sum ab^2\right)$$

Now, $s^2\sum AI^2 = s^2r^2\left(\frac{1}{\sin^2\left(\frac{A}{2}\right)} + \frac{1}{\sin^2\left(\frac{B}{2}\right)} + \frac{1}{\sin^2\left(\frac{C}{2}\right)}\right)$

$$\frac{r}{AI} = \sin\left(\frac{A}{2}\right) \Rightarrow AI = \frac{r}{\sin\left(\frac{A}{2}\right)} = \Delta^2\left(\frac{bc}{(s-b)(s-c)} + \frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right) =$$

$$= \frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)(s-c)}\left(bc(s-a) + ca(s-b) + ab(s-c)\right)$$

$$= s(s(ab+bc+ca) - 3abc)$$

$$= \left(\frac{S}{2}\right)\left((a+b+c)(ab+bc+ca) - 6abc\right)$$

$$= \left(\frac{S}{2}\right)\left(\sum a^2b + \sum ab^2 - 3abc\right) \quad (2)$$

Now, $\sum a^2b + \sum ab^2 = a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)$

$$AM \geq GM \leftarrow \geq 2abc + 2abc + 2abc = 6abc$$

$$\Rightarrow \sum a^2b + \sum ab^2 \geq 6abc$$

$$\Rightarrow 2\left(\sum a^2b + \sum ab^2\right) \geq 6abc + \left(\sum a^2b + \sum ab^2\right)$$

$$\Rightarrow 2\left(\sum a^2b + \sum ab^2 - 3abc\right) \geq \sum a^2b + \sum ab^2$$

$$\Rightarrow \frac{\sum a^2b + \sum ab^2 - 3abc}{2} \geq \frac{\sum a^2b + \sum ab^2}{4}$$

$$\Rightarrow \frac{S}{2}\left(\sum a^2b + \sum ab^2 - 3abc\right) \geq \frac{S}{4}\left(\sum a^2b + \sum ab^2\right)$$

$$\Rightarrow S^2 \sum AI^2 \geq \frac{S}{4}\left(\sum a^2b + \sum ab^2\right) > \sum m_a m_b w_a w_b$$

from (1) and (2)

23. In acute triangle ABC the following relationship holds:

$$2\sum \sin A > 3\pi - \prod \tan A$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Peru

Probar en un triángulo acutángulo ABC, Si es verdadero o falso:

$$2\sum \operatorname{sen} A > 3\pi - \prod \tan A$$

Tener en cuenta lo siguiente para el desarrollo:

$$2\sum \operatorname{sen} A + \tan A + \tan B + \tan C > 3\pi$$

Consideremos: $f(x) = \tan x + 2\operatorname{sen} x - 3x$

Realizamos la primera derivada: $f'(x) = \sec^2 x + 2\cos x - 3$

Realizamos la segunda derivada.

$$f''(x) = -2\,\text{sen}\,x + \frac{2\,\text{sen}\,x}{\cos^3 x} = 2\,\text{sen}\,x\frac{(1-\cos^3 x)}{\cos^3 x} > 0, x \in <0, \frac{\pi}{2}>$$

Desde que: $f(0) = f'(0) = 0$ y $f''(x) > 0$, se concluye que: $f(x) > 0$

Por lo tanto: $2\,\text{sen}\,A + \tan A > 3A \quad (x)$

$$2\,\text{sen}\,B + \tan B > 3B \quad (y)$$

$$2\,\text{sen}\,C + \tan C > 3C \quad (z)$$

Sumando $(x) + (y) + (z)$: $\rightarrow 2\sum \text{sen}\,A + \tan B + \tan C > 3\pi \rightarrow$

$$\rightarrow 2\sum \text{sen}\,A > 3\pi - \prod \tan A$$

24. In acute triangle ABC the following relationship holds:

$$\sum \frac{(\tan A)^{2n+1}}{\sqrt{\tan B \cdot \tan C}} \geq 3\left(\sum \tan A\right)^{\frac{2n}{3}}, n \in \mathbb{N}^*$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru:

Probar en un triángulo acutángulo si es (V) o (F): $\sum \frac{(\tan A)^{2n+1}}{\sqrt{\tan B \tan C}} \geq 3(\sum \tan A)^{\frac{2n}{3}}$,

$n \in \mathbb{N}$. Dado que es un triángulo acutángulo: $\tan A, \tan B, \tan C > 0$

Por: $MA \geq MG$: $\frac{\tan A^{(2n+1)}}{\sqrt{\tan B \tan C}} + \frac{\tan B^{(2n+1)}}{\sqrt{\tan A \tan C}} + \frac{\tan C^{(2n+1)}}{\sqrt{\tan B \tan C}} \geq 3\sqrt[3]{\frac{(\tan A \tan B \tan C)^{2n+1}}{\tan A \tan B \tan C}}$

$$\frac{\tan A^{(2n+1)}}{\sqrt{\tan B \tan C}} + \frac{\tan B^{(2n+1)}}{\sqrt{\tan A \tan C}} + \frac{\tan C^{(2n+1)}}{\sqrt{\tan B \tan C}} \geq 3(\tan A \tan B \tan C)^{\frac{2n}{3}}$$

Tener en cuenta que si:

$$A + B + C = \pi \rightarrow \tan A \tan B \tan C = \tan A + \tan B + \tan C$$

Por lo tanto: $\sum \frac{(\tan A)^{2n+1}}{\sqrt{\tan B \tan C}} \geq 3\left(\sum \tan A\right)^{\frac{2n}{3}}$

25. In $\Delta\ ABC$ the following relationship holds:

$$4(m_a m_b + m_a m_c + m_b m_c) \leq 2(a^2 + b^2 + c^2) + ab + ac + bc$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Peru

En un triángulo ABC. Probar que:

$$4(m_a m_b + m_b m_c + m_a m_c) \leq 2(a^2 + b^2 + c^2) + ab + ac + bc$$

$$\Rightarrow (2c^2 + ab - 4m_a m_b) + (2a^2 + bc - 4m_b m_c) + (2b^2 + ac - 4m_a m_c) \geq 0$$

Para ello demostraremos que:

$$2c^2 + ab \geq 4m_a m_b \Leftrightarrow m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad (A)$$

$m_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} \quad (B)$. Elevando al cuadrado la expresión:

$$(2c^2 + ab)^2 \geq (2b^2 + 2c^2 - a^2)(2a^2 + 2c^2 - b^2)$$

$$4c^4 + 4abc^2 + a^2b^2 \geq 4b^2a^2 + 4b^2c^2 - 2b^4 + 4a^2c^2 + 4c^4 - 2b^2c^2 - 2a^4 - 2c^2a^2 + a^2b^2$$

$$\Rightarrow 2a^4 + 2b^4 - 4a^2b^2 - 2c^2a^2 - 2b^2c^2 + 4abc^2 \geq 0$$

$$\Rightarrow 2(a^2 - b^2)^2 - 2(ac - bc)^2 \geq 0$$

$$\Rightarrow 2(a - b)^2(a + b)^2 - 2c^2(a - b)^2 \geq 0 \rightarrow 2(a - b)^2((a + b)^2 - c^2) \geq 0$$

$\Rightarrow 2(a - b)^2(a + b + c)(a + b - c) \geq 0 \Leftrightarrow a + b - c > 0$ (Por desigualdad triangular)

$\Rightarrow$ Por la tanto: $2c^2 + ab \geq 4m_a m_b \quad (C)$; de forma análoga: $2a^2 + bc \geq 4m_b m_c \quad (E)$ y

$$2b^2 + ac \geq 4m_a m_c \quad (F)$$

26. In $\Delta\ ABC$ the following relationship holds:

$$\frac{16}{(a + 3)(b + 5)(c + 7)} \leq \frac{1}{4RS} + \frac{1}{105}$$

R – circumradius, S – area

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Peru

En un triángulo ABC. Probar que: $\frac{16}{(a+3)(b+5)(c+7)} \le \frac{1}{4RS} + \frac{1}{105}$

R = **Circunradio,** S = **Área de región triangular. Tener presente lo siguiente:**

$\frac{1}{4RS} = \frac{1}{abc}$**. Desde que:** a, b, c **son lados de un triángulo** $\Leftrightarrow a, b, c > 0$

La expresión puede ser equivalente: $16 \le \frac{(a+3)(b+5)(c+7)}{4RS} + \frac{(a+3)(b+5)(c+7)}{105}$

Por: $MA \ge MG \to a + 3 \ge 2\sqrt{3a}$ **(A);** $b + 5 \ge 2\sqrt{5b}$ **(B);** $c + 7 \ge 2\sqrt{7c}$ **(C)**

Multiplicando: (A)(B)(C): $(a + 3)(b + 5)(c + 7) \ge 8\sqrt{105abc} \Rightarrow$

$$\Rightarrow \frac{(a+3)(b+5)(c+7)}{abc} \ge \frac{8\sqrt{105}}{\sqrt{abc}} \quad \textbf{(D)}$$

$$\frac{(a+3)(b+5)(c+7)}{105} \ge \frac{8\sqrt{105abc}}{105} \to \frac{(a+3)(b+5)(c+7)}{105} \ge \frac{8\sqrt{abc}}{\sqrt{105}} \quad \textbf{(E)}$$

Sumando: $(D) + (E) \to \frac{(a+3)(b+5)(c+7)}{4RS} \to \frac{(a+3)(b+5)(c+7)}{105} \ge \frac{8\sqrt{105}}{\sqrt{abc}} + \frac{8\sqrt{abc}}{\sqrt{105}} \ge 16$

(LQQD)⇔(Válido por $MA \ge MG$**)**

Solution 2 by Soumava Chakraborty-Kolkata-India

$\frac{abc}{4R} = S \Rightarrow \frac{1}{4RS} = \frac{1}{abc}$**. To prove:** $\frac{16}{(a+3)(b+5)(c+7)} \le \frac{105+abc}{105abc}$

$$\Leftrightarrow (105 + abc)(a + 3)(b + 5)(c + 7) \ge 16 \cdot 105abc$$

$$\Leftrightarrow 105abc + 105 \cdot 5ca + 105 \cdot 3bc + 105 \cdot 15c + 105 \cdot 7ab +$$

$$+105 \cdot 35a + 105 \cdot 21b + 105^2 + a^2b^2c^2 + 5abc \cdot ca + 3bc \cdot abc +$$

$$+15c \cdot abc + 7ab \cdot abc + 35a \cdot abc + 21b \cdot abc + 105abc \ge 16 \cdot 105abc$$

$$\Leftrightarrow 105(7ab + 3bc + 5ca) + 105(35a + 21b + 15c) + 105^2 + a^2b^2c^2 +$$

$$+abc(7ab + 3bc + 5ca) + abc(35a + 21b + 15c) \ge 14 \cdot 105abc$$

$$\Leftrightarrow (105 \cdot 7ab + 105 \cdot 35a + 3b^2c^2a + 15c^2ab) +$$

$$+(105\cdot 3bc+105\cdot 21b+5c^2a^2b+35a^2bc)$$
$$+(105\cdot 5ca+105\cdot 15c+7a^2b^2c+21b^2ca)$$
$$+(105^2+a^2b^2c^2)\geq 14\cdot 105abc$$ **(A). Applying** $AM\geq GM$,
$$105\cdot 7ab+105\cdot 35a+3b^2c^2a+15c^2ab\geq 4\sqrt{105^4a^4b^4c^4}\cdot 4=4(105)abc$$
$$105\cdot 3abc+105\cdot 21b+5c^2a^2b+35a^2bc\geq 4\cdot\sqrt[4]{105^4a^4b^4c^4}=4(105)abc$$
$$105\cdot 5ca+105\cdot 15c+7a^2b^2c+21b^2ca\geq 4\cdot\sqrt[4]{105^4a^4b^4c^4}=4(105)abc$$
$$105^2+a^2b^2c^2\geq 2\cdot 105\cdot abc;\ (1)+(2)+(3)+(4)\Rightarrow (A)$$

27. In $\Delta\, ABC, I-$ the incentre, $O-$ the circumcentre, $G-$ the centroid

Prove that:

$$3(OI+IG+GO)^2+52Rr\leq s^2+5r^2+18R^2$$

Proposed by Daniel Sitaru – Romania

Solution by Adil Abdullayev-Baku-Azerdbadjan

$$3(OI+IG+GO)^2+52Rr\leq s^2+5r^2+18R^2$$
$$3(OI+IG+GO)^2+52Rr\leq 3(1^2+1^2+1^2)(OI^2+IG^2+GO^2)+52Rr=$$
$$=9\left(R^2-2Rr+\frac{s^2+5r^2-16Rr}{9}+R^2-\frac{2(s^2-r^2-4Rr)}{9}\right)+52Rr=$$
$$=18R^2+26Rr+7r^2-s^2\leq s^2+5r^2+18R^2\Leftrightarrow s^2\geq 13Rr+r^2.$$
Gerretsen $\Rightarrow s^2\geq 16Rr-5r^2\geq 13Rr+r^2\Leftrightarrow R\geq 2r$ (Euler).

28. In $\Delta\, ABC$:

$$\prod_{cyc}(3a^2+2bc\cos A)\geq[(2p-a)(2p-b)(2p-c)]^2$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Peru

En un triángulo ABC. Probar que:

$$\prod(3a^2+2bc\cos A)\geq[(2p-a)(2p-b)(2p-c)]^2$$

Tener presente en un triángulo ABC, lo siguiente:

$$2bc\cos A=b^2+c^2-a^2, 2ab\cos C=a^2+b^2-c^2, 2ac=a^2+c^2-b^2, 2p=a+b+c$$

Reeemplazando en la desigualdad:

$$\left((a^2+b^2)+(a^2+c^2)\right)\left((b^2+c^2)+(b^2+a^2)\right)\left((c^2+a^2)+(c^2+b^2)\right)\geq$$

$$\geq(b+c)^2(a+c)^2(a+b)^2.$$ **Por:** $MP\geq MA$

$$a^2+b^2\geq\frac{(a+b)^2}{2}\quad(A);\ a^2+c^2\geq\frac{(a+c)^2}{2}\quad(B);\ b^2+c^2\geq\frac{(b+c)^2}{2}\quad(C)$$

Sumando: $(A)+(B)\rightarrow$ **Por:** $MA\geq MG\rightarrow 2a^2+b^2+c^2\geq\frac{(a+b)^2}{2}+\frac{(a+c)^2}{2}\geq$

$\geq(a+b)(a+c)\quad(D)$. **Análogamente para los demás términos:**

$$2b^2+a^2+c^2\geq\frac{(a+b)^2}{2}+\frac{(b+c)^2}{2}\geq(a+b)(b+c)\quad(E)$$

$$2c^2+a^2+b^2\geq\frac{(a+c)^2}{2}+\frac{(b+c)^2}{2}\geq(a+c)(b+c)\quad(F)$$

Multiplicando: $(D)(E)(F)$: $\left((a^2+b^2)+(a^2+c^2)\right)\left((b^2+c^2)+(b^2+a^2)\right)\left((c^2+a^2)+(c^2+b^2)\right)\geq$

$$\geq(b+c)^2(a+c)^2(a+b)^2$$

29. In $\Delta\ ABC$:

$$\sin A+4\sin B+4\cos\frac{C}{2}\leq 9\sin\left(\frac{\pi+B-C}{3}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios –Huarmey- Peru

En un triángulo ABC: $\operatorname{sen} A + 4\operatorname{sen} B + 4\cos\frac{C}{2} \le 9\operatorname{sen}\left(\frac{\pi+B-C}{3}\right)$

$$\Rightarrow E = \operatorname{sen} A + \operatorname{sen} B + \operatorname{sen} B + \operatorname{sen} B + \operatorname{sen} B + \operatorname{sen}\left(\frac{A+B}{2}\right) + \operatorname{sen}\left(\frac{A+B}{2}\right) + \operatorname{sen}\left(\frac{A+B}{2}\right)$$

Por desigualdad de Jensen: $\operatorname{sen} A + 4\operatorname{sen} B + 4\operatorname{sen}\left(\frac{A+B}{2}\right) \le 9\operatorname{sen}\left(\frac{A+4B+4\left(\frac{A+B}{2}\right)}{9}\right) =$

$$= 9\operatorname{sen}\left(\frac{3A+6B}{9}\right) = 9\operatorname{sen}\left(\frac{A+2B}{3}\right)$$

$$\operatorname{sen} A + 4\operatorname{sen} B + 4\operatorname{sen}\left(\frac{A+B}{2}\right) \le 9\operatorname{sen}\left(\frac{\pi+B-C}{3}\right)$$

30. In $\Delta\, ABC$:

$$45a^2 + 27b^2 + 5c^2 > 60\sqrt{7}S$$

$S -$ area

Proposed by Daniel Sitaru - Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

De la desigualdad ponderada Weizenbock (Refinamiento de Pohata):

$a^2x + b^2y + c^2z \ge 4\sqrt{xy+yz+xz}S, \to x, y, z \ge 0$. Sea: $x = 45, y = 27, z = 5$

$$45a^2 + 27b^2 + 5c^2 \ge 4\sqrt{45x27 + 27x5 + 45x5}S$$

$$45a^2 + 27b^2 + 5c^2 \ge 4\sqrt{15(9^2 + 9 + 15)}S$$

$$45a^2 + 27b^2 + 5c^2 \ge 4\sqrt{15x15x7}S;\ 45a^2 + 27b^2 + 5c^2 \ge 60\sqrt{7}S$$

31. Prove that in any triangle ABC:

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \ge \frac{1}{2}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right)$$

Proposed by Nguyen Viet Hung – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\ge\frac{1}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)$$

Tereshin's inequality $\Rightarrow m_a\ge\frac{b^2+c^2}{4R}, m_b\ge\frac{c^2+a^2}{4R}, m_c\ge\frac{a^2+b^2}{4R}$

$$h_a=\frac{2\Delta}{a}, h_b=\frac{2\Delta}{b}, h_c=\frac{2\Delta}{c}, \frac{m_a}{h_a}\ge\frac{b^2+c^2}{4R}\cdot\frac{a}{2\Delta}=\frac{a(b^2+c^2)}{2(4R\Delta)}=\frac{a(b^2+c^2)}{2abc}$$

Similarly, $\frac{m_b}{h_b}\ge\frac{b(c^2+a^2)}{2abc}, \frac{m_c}{h_c}\ge\frac{c(a^2+b^2)}{2abc}$

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\ge\frac{ab(a+b)+bc(b+c)+ca(c+a)}{2abc}$$

$$=\frac{1}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)$$

32. En un triángulo ABC: Probar que si es (V) O (F)

$$a^4c+b^4a+c^4b\ge 24\sqrt[3]{2}(RS)^{\frac{5}{3}}$$

Proposed by Daniel Sitaru-Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

Desarrollando el lado derecho tenemos:

$$24\sqrt[3]{2}(RS)^{\frac{5}{3}}=24\sqrt[3]{2}\left(\frac{abc}{4}\right)^{\frac{5}{3}}=24\sqrt[3]{2}\sqrt[3]{\frac{1}{2^{10}}(abc)^5}=24\sqrt[3]{\frac{1}{2^9}(abc)^5}=3\sqrt[3]{(abc)^5}$$

Por las desigualdades entre las medias:

Siendo: $a, b, c > 0$ (lados de un triángulo ABC)

$$a^4c+b^4a+c^4b\ge 3\sqrt[3]{(abc)^5}\quad \text{(LQQD)}$$

33. In ΔABC:

$$\sin^2 A + \sin^2 B + \sin C \le 2\sqrt{1 + \cos(A - B)\cos C}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

En un triángulo ABC: Probar que:

$$\operatorname{sen}^2 A + \operatorname{sen}^2 B + \operatorname{sen} C \le 2\sqrt{1 + \cos C\cos(A - B)}$$

Tener en cuenta lo siguiente:

$$\operatorname{sen}^2 A + \operatorname{sen}^2 B + \operatorname{sen}^2 C = 2(1 + \cos A\cos B\cos C)$$

$$2\operatorname{sen} B\operatorname{sen} A\cos C = \operatorname{sen}^2 A + \operatorname{sen}^2 B - \operatorname{sen}^2 C$$

$$\operatorname{sen}^2 A + \operatorname{sen}^2 B + \operatorname{sen} C \le 2\sqrt{1 + \cos A\cos B\cos C + \operatorname{sen} B\operatorname{sen} C\cos A}$$

$$\operatorname{sen}^2 A + \operatorname{sen}^2 B + \operatorname{sen} C \le 2\sqrt{\operatorname{sen}^2 A + \operatorname{sen}^2 B}$$

$$\operatorname{sen}^2 A + \operatorname{sen}^2 B + \operatorname{sen} A\cos B + \operatorname{sen} B\cos A \le 2\sqrt{\operatorname{sen}^2 A + \operatorname{sen}^2 B}$$

$$\operatorname{sen} A(\operatorname{sen} A + \cos B) + \operatorname{sen} B(\operatorname{sen} B + \cos A) \le 2\sqrt{\operatorname{sen}^2 A + \operatorname{sen}^2 B}$$

Por desigualdad de Cauchy:

$$\left(\operatorname{sen} A(\operatorname{sen} A + \cos B) + \operatorname{sen} B(\operatorname{sen} B + \cos A)\right)^2 \le$$

$$\le (\operatorname{sen}^2 A + \operatorname{sen}^2 B)\left((\operatorname{sen} A + \cos B)^2 + (\operatorname{sen} B + \cos A)^2\right)$$

$$\left(\operatorname{sen} A(\operatorname{sen} A + \cos B) + \operatorname{sen} B(\operatorname{sen} B + \cos A)\right)^2 \le$$

$$\le (\operatorname{sen}^2 A + \operatorname{sen}^2 B)\left(2 + 2(\operatorname{sen} A\cos B + \operatorname{sen} B\cos A)\right)$$

$$\left(\operatorname{sen} A(\operatorname{sen} A + \cos B) + \operatorname{sen} B(\operatorname{sen} B + \cos A)\right)^2 \le$$

$$\le (\operatorname{sen}^2 A + \operatorname{sen}^2 B)(2 + 2\operatorname{sen} C)$$

$$\operatorname{sen} A(\operatorname{sen} A + \cos B) + \operatorname{sen} B(\operatorname{sen} B + \cos A) \le$$

$$\leq \sqrt{2(1+\text{sen}\, C)}\sqrt{\text{sen}^2 A+\text{sen}^2 B}$$

Por lo cual nos f alta probar que:

$$\sqrt{2(1+\text{sen}\, C)}\sqrt{\text{sen}^2 A+\text{sen}^2 B}\leq 2\sqrt{\text{sen}^2 A+\text{sen}^2 B}\Leftrightarrow 1\geq \text{sen}\, C$$

34. Let ABC be a triangle with the incenter I. Prove that:

$$\frac{1}{IA\cdot IB}+\frac{1}{IB\cdot IC}+\frac{1}{IC\cdot IA}\leq\frac{R+r}{2Rr^2}$$

Proposed by Hung Nguyen Viet-HaNoi-VietNam

Solution by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo ABC con el Incentro “I”: $\frac{1}{IAIB}+\frac{1}{IBIC}+\frac{1}{IAIC}\leq\frac{R+r}{2R^2}$

Tener en cuenta lo siguinete: $IA=\frac{r}{\text{sen}\frac{A}{2}}=4R\,\text{sen}\,\frac{B}{2}\text{sen}\,\frac{C}{2}$

$$IB=\frac{r}{\text{sen}\,\frac{B}{2}}=4R\,\text{sen}\,\frac{A}{2}\text{sen}\,\frac{C}{2};\ IC=\frac{r}{\text{sen}\,\frac{C}{2}}=4\,\text{sen}\,\frac{A}{2}\text{sen}\,\frac{B}{2}$$

$$r=4R\,\text{sen}\,\frac{A}{2}\text{sen}\,\frac{B}{2}\text{sen}\,\frac{C}{2},\frac{IAIBIC}{2Rr^2}=2,ab+bc+ac=p^2+r^2+4Rr,$$

$$p^2\leq 4R^2+4Rr+3r^2\ \text{(Gerretsen)}$$

$$\Rightarrow\frac{IA+IB+IC}{IAIBIC}\leq\frac{R+r}{2Rr^2}\to IA+IB+IC\leq 2R+2r$$

Asimismo también se puede expresar de la siguiente manera:

$$IA=\frac{bc}{p}\cos\frac{A}{2}=\frac{bc}{p}\sqrt{\frac{p(p-a)}{bc}}=\frac{1}{p}\sqrt{bc}\sqrt{p(p-a)}$$

De forma análoga: $IB=\frac{1}{p}\sqrt{ac}\sqrt{p(p-b)},IC=\frac{1}{p}\sqrt{ab}\sqrt{p(p-c)}$

Por la desigualdad de Cauchy:

$$(IA+IB+IC)^2=\frac{1}{p^2}\left(\sqrt{bc}\sqrt{p(p-a)}+\sqrt{ac}\sqrt{p(p-b)}+\sqrt{ab}\sqrt{p(p-c)}\right)^2\le$$

$$\le\frac{1}{p^2}(bc+ac+ab)\big(p(p-a)+p(p-b)+p(p-c)\big)$$

$$\Rightarrow(IA+IB+IC)^2\le(ab+bc+ac)=p^2+r^2+4Rr\le$$

$$\le 4R^2+4Rr+3r^2+r^2+4Rr=4(R+r)^2\Rightarrow$$ **Por transitividad:**

$$(IA+IB+IC)^2\le 4(R+r)^2\to IA+IB+IC\le 2R+2r$$

35. Let a,b,c be the side-lengths of a triangle. Prove that:

$$\sum_{cyc}bc(bc-a^2)(b+c-a)\ge 0$$

Proposed by Nguyen Viet Hung – Hanoi - Vietnam

Solution by Kevin Soto Palacios –Huarmey- Peru

Sea: a,b,c **los lados de un triángulo. Probar que:** $\sum bc(bc-a^2)(b+c-a)\ge 0$

Denotemos: $T_1=bc(bc-a^2)(b+c-a)$

$$T_1=bc(b^2c+c^2b-abc-a^2b-a^2c+a^3)$$

$$T_1=b^3c^2+c^3b^2-a^2b^2c-a^2c^2b+a^3bc$$

De forma análoga para los siguientes términos: $T_2=ac(ac-b^2)(a+c-b)$

$$\Rightarrow T_2=a^3c^2+c^3a^2+c^3a^2-b^2a^2c-b^2c^2a+b^3ac$$

$$T_3=ab(ab-c^2)(b+c-a)\Rightarrow T_3=a^3b^2+b^3a^2-c^2a^2b-c^2b^2a+c^3ab$$

Sumando y factorizando convenientemente se llega a que:

$$\to T_1+T_2+T_3=a^3(b^2+c^2)+b^3(a^2+c^2)+c^3(a^2+b^2)-3abc(ab+bc+ac)+abc(a^2+b^2+c^2)$$

Desde que: $a,b,c>0\to$ **Por:** $MA\ge MG\to$

$$\to a^3(b^2+c^2)+b^3(a^2+c^2)+c^3(a^2+b^2)\ge 2abc(a^2+b^2+c^2)$$

$$a^3(b^2+c^2)+b^3(a^2+c^2)+c^3(a^2+b^2)-2abc(ab+bc+ac)+abc(a^2+b^2+c^2)\geq$$
$$\geq 3abc(a^2+b^2+c^2-ab-bc-ac)\geq 0$$

36. Let a, b, c be side-lenghts of a triangle with the semi-perimeter s.

Prove that

$$\frac{(s-a)(s-b)}{\sqrt{ab}}+\frac{(s-b)(s-c)}{\sqrt{bc}}+\frac{(s-c)(s-a)}{\sqrt{ca}}\leq\frac{s}{2}$$

Proposed by Nguyen Viet Hung - Vietnam

Solution by Kevin Soto Palacios –Huarmey- Peru

Sean: a, b, c lados de un triángulo con semiperímetro "s". Probar que:

$$\frac{(s-a)(s-b)}{\sqrt{ab}}+\frac{(s-b)(s-c)}{\sqrt{bc}}+\frac{(s-a)(s-c)}{\sqrt{ac}}\leq\frac{s}{2}$$

De la siguiente desigualdad ya demostrado anteriormente:

$$\frac{x^2+y^2+z^2}{2}\geq yz\sin\frac{A}{2}+zx\sin\frac{B}{2}+xy\sin\frac{C}{2}\to x,y,z\in\mathbb{R}\ y\ \wedge xyz>0$$

Es equivalente:

$$\frac{x^2+y^2+z^2}{2}\geq yz\sqrt{\frac{(s-b)(s-c)}{bc}}+zx\sqrt{\frac{(s-a)(s-c)}{ac}}+xy\sqrt{\frac{(s-a)(s-b)}{ab}}$$

Sea: $x=\sqrt{s-a}, y=\sqrt{s-b}, z=\sqrt{s-c}$

$$\Rightarrow\frac{s-a+s-b+s-c}{2}\geq\frac{(s-b)(s-c)}{bc}+\frac{(s-b)(s-c)}{ac}+\frac{(s-a)(s-b)}{ab}$$

$$\Rightarrow\frac{s}{2}\geq\frac{(s-b)(s-c)}{bc}+\frac{(s-a)(s-c)}{ac}+\frac{(s-a)(s-b)}{ab}$$

37. Prove that in any triangle ABC:

$$\frac{\cos\frac{A}{2}}{\sin^2 A}+\frac{\cos\frac{B}{2}}{\sin^2 B}+\frac{\cos\frac{C}{2}}{\sin^2 C}\ge\sqrt{3}\frac{R}{r}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{\cos\frac{A}{2}}{\text{sen}^2 A}+\frac{\cos\frac{B}{2}}{\text{sen}^2\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\text{sen}^2 C}\ge\sqrt{3}\frac{R}{r}$

$$\frac{\cos\frac{A}{2}}{\text{sen}^2 A}+\frac{\cos\frac{B}{2}}{\text{sen}^2 B}+\frac{\cos\frac{C}{2}}{\text{sen}^2 C}\ge\sqrt{3}\frac{1}{4}\csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2}$$

$$\Rightarrow\frac{\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}}{\text{sen}\frac{A}{2}\cos\frac{A}{2}}+\frac{\text{sen}\frac{A}{2}\text{sen}\frac{C}{2}}{\text{sen}\frac{B}{2}\cos\frac{B}{2}}+\frac{\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}}{\text{sen}\frac{C}{2}\cos\frac{C}{2}}\ge\sqrt{3}$$

$$\Rightarrow\frac{\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}}{\text{sen}\,A}+\frac{\text{sen}\frac{A}{2}\text{sen}\frac{C}{2}}{\text{sen}\,B}+\frac{\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}}{\text{sen}\,C}\ge\frac{\sqrt{3}}{2}$$

Tener en cuenta lo siguiente: $\text{sen}\,A=\frac{2S}{bc}$ (I); $\text{sen}\,B=\frac{2S}{ac}$ (II);

$\text{sen}\,C=\frac{2S}{ab}$ (III); $S=\sqrt{p(p-a)(p-b)(p-c)}$

$\text{sen}\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}}$ (IV); $\text{sen}\frac{B}{2}=\sqrt{\frac{(p-a)(p-c)}{ac}}$ (V); $\text{sen}\frac{C}{2}=\sqrt{\frac{(p-a)(p-b)}{ab}}$ (VI)

$$\Rightarrow\sum\frac{\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}}{\text{sen}\,A}=\frac{\frac{(p-a)\sqrt{(p-b)(p-c)}}{\sqrt{a^2bc}}}{\frac{2\sqrt{p(p-a)(p-b)(p-c)}}{bc}}=\frac{bc(p-a)}{2\sqrt{a^2bc(p)(p-a)}}=$$

$$=\frac{1}{2}\frac{bc(b+c-a)}{\sqrt{abc(a+b+c)}\sqrt{a(b+c-a)}}\ge\frac{1}{2}\frac{bc(b+c-a)}{\sqrt{abc(a+b+c)}\left(\frac{a+b+c-a}{2}\right)}$$

$$\rightarrow \frac{1}{2}\sum \frac{bc(b+c-a)}{\sqrt{abc(a+b+c)}\left(\frac{a+b+c-a}{2}\right)} = \sum \frac{\frac{bc(c+a-a)}{b+c}}{\sqrt{abc(a+b+c)}} =$$

$$= \sum \frac{bc - \frac{abc}{b+c}}{\sqrt{abc(a+b+c)}} \geq \sum \frac{bc - abc\left(\frac{1}{4b}+\frac{1}{4c}\right)}{\sqrt{abc(a+b+c)}}$$

$$\rightarrow \sum \frac{bc - abc\left(\frac{1}{4b}+\frac{1}{4c}\right)}{\sqrt{abc(a+b+c)}} = \frac{\sum bc - \sum abc\left(\frac{1}{4b}+\frac{1}{4c}\right)}{\sqrt{abc(a+b+c)}} =$$

$$= \frac{\sum bc - \sum \frac{ac}{4} - \sum \frac{ab}{4}}{\sqrt{abc(a+b+c)}} = \frac{\sum \frac{bc}{2}}{\sqrt{abc(a+b+c)}} = \frac{1}{2}\frac{\sum bc}{\sqrt{abc(a+b+c)}}$$

$$\frac{1}{2}\frac{\sum bc}{\sqrt{abc(a+b+c)}} \geq \frac{1}{2}\frac{\sqrt{3}\sqrt{abc(a+b+c)}}{\sqrt{abc(a+b+c)}} = \frac{\sqrt{3}}{2}$$

Se utilizo lo siguiente en el desarollo: $(m+n+p)^2 \geq 3(mn+np+mp)$

$\frac{1}{m}+\frac{1}{n} \geq \frac{4}{m+n}$. **Por la tanto:** $\Rightarrow \frac{\text{sen}\frac{B}{2}\sin\frac{C}{2}}{\text{sen}\,A} + \frac{\text{sen}\frac{A}{2}\text{sen}\frac{C}{2}}{\text{sen}\,B} + \frac{\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}}{\text{sen}\,C} \geq \frac{\sqrt{3}}{2}$

38. In $\Delta\ ABC$:

$$3\left(\sum \frac{a^3}{w_a}\right)\cdot\left(\sum \frac{w_a}{a}\right) \geq 4\left(\sum w_a\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Adil Abdullayev-Baku-Azerbaidjian

Lemma:

$$\sum_{cyc} w_a \leq p\sqrt{3}$$

Proof:

$$\sum_{cyc} w_a = \sum_{cyc}\left(\frac{2\sqrt{bc}}{b+c}\cdot\sqrt{p(p-a)}\right) \le \sum_{cyc}\sqrt{p(p-a)} =$$

$$= \sqrt{p}\cdot\sum_{cyc}\sqrt{p-a} \Leftrightarrow \left(\sum_{cyc} w_a\right)^2 \le p\cdot\left(\sum_{cyc}\sqrt{p-a}\right)^2 \overset{C-B-S}{\overbrace{\le}} p\cdot 3\sum_{cyc}(p-a) =$$

$$= p\cdot 3p = 3p^2 \Leftrightarrow LHS \le RHS$$

Solution:

$$LHS = 3\sum_{cyc}\left(\sqrt{\frac{a^3}{w_a}}\right)^2\cdot\sum_{cyc}\left(\sqrt{\frac{w_a}{a}}\right)^2 \overset{C-B-S}{\overbrace{\ge}} 3\cdot\left(\sum_{cyc} a\right)^2 = 12p^2 \ge RHS \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc} w_a \le p\sqrt{3}$$

39. In any triangle ABC:

$$\frac{27m_a^2m_b^2m_c^2}{S^6} \le \left(\frac{1}{(p-a)^2}+\frac{1}{(p-b)^2}+\frac{1}{(p-c)^2}\right)^3$$

m_a – median length, S – area

Proposed by Daniel Sitaru, Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{27m_a^2m_b^2m_c^2}{\Delta^6} \le \left\{\frac{1}{(p-a)^2}+\frac{1}{(p-b)^2}+\frac{1}{(p-c)^2}\right\}^3 \quad (1)$$

$$m_a^2 = \frac{2b^2+2c^2-a^2}{4}, m_b^2 = \frac{2c^2+2a^2-b^2}{4}; \ m_c^2 = \frac{2a^2+2b^2-c^2}{4}$$

(1) $\Leftrightarrow \frac{27}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2) \le$

$$\le \left\{\frac{\Delta^2}{(p-a)^2}+\frac{\Delta^2}{(p-b)^2}+\frac{\Delta^2}{(p-c)^2}\right\}^3$$

$$\Leftrightarrow \frac{3}{4}\sqrt[3]{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)} \le$$

$$\le \frac{\Delta^2}{(p-a)^2}+\frac{\Delta^2}{(p-b)^2}+\frac{\Delta^2}{(p-c)^2}$$

$$GM \le AM \Rightarrow \frac{3}{4}\sqrt[3]{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)} \le$$

$$\le \frac{3}{4}\,\frac{(2b^2+2c^2-a^2)+(2c^2+2a^2-b^2)+(2a^2+2b^2-c^2)}{3}$$

$$= \frac{3}{4}(a^2+b^2+c^2)$$

I shall show that, $\frac{3}{4}(a^2+b^2+c^2) \le \frac{\Delta^2}{(p-a)^2}+\frac{\Delta^2}{(p-b)^2}+\frac{\Delta^2}{(p-c)^2}$ **(2)**

which will prove (1)

RHS **of (2)** $= \frac{p(p-a)(p-b)(p-c)}{(p-a)^2}+\frac{p(p-a)(p-b)(p-c)}{(p-b)^2}+\frac{p(p-a)(p-b)(p-c)}{(p-c)^2}$

$$= p\left\{\frac{(p-b)(p-c)}{p-a}+\frac{(p-c)(p-a)}{p-b}+\frac{(p-a)(p-b)}{p-c}\right\}$$

$$= \frac{(a+b+c)}{4}\left\{\frac{(c+a-b)(a+b-c)}{b+c-a}+\frac{(a+b-c)(b+c-a)}{c+a-b}+\frac{(b+c-a)(c+a-b)}{a+b-c}\right\}$$

(2) $\Leftrightarrow 3(a^2+b^2+c^2) \overset{(3)}{\lesseqgtr} (a+b+c)\left\{\frac{(c+a-b)(a+b-c)}{b+c-a}+\frac{(a+b-c)(b+c-a)}{c+a-b}+\frac{(b+c-a)(c+a-b)}{a+b-c}\right\}$

Let $a+b-c=x, b+c-a=y, c+a-b=z$

$$a+b+c=x+y+z, a=\frac{z+x}{2}, b=\frac{x+y}{2}, c=\frac{y+z}{2}$$

(3) $\Leftrightarrow \frac{3}{4}\{(z+x)^2+(x+y)^2+(y+z)^2\} \le (x+y+z)\left(\frac{xy}{2}+\frac{yz}{x}+\frac{zx}{y}\right)$

$$\Leftrightarrow 6\left(\sum x^2+\sum xy\right)xyz \le 4\left(\sum x\right)\left(\sum x^2y^2\right)$$

$$\Leftrightarrow 2(x+y+z)(x^2y^2+y^2z^2+z^2x^2) \ge 3(x^2+y^2+z^2+xy+yz+zx)xyz$$

$$\Leftrightarrow 2x^3y^2+2x^2y^3+2y^3z^2+2y^2z^3+2z^3x^2+2z^2x^3+$$

$$+2xyz(xy+yz+zx) \ge 3xyz\left(\sum x^2\right)+3xyz(3xyz+zx)$$

$$\Leftrightarrow 2x^3y^2+2x^2y^3+2y^3z^2+2y^2z^3+2z^3x^2+2z^2x^3 \ge$$

$$\ge 3x^3yz+3y^3zx+3z^3xy+x^2y^2z+y^2z^2x+z^2x^2y \quad (4)$$

$$AM \ge GM \Rightarrow \frac{3}{2}(x^3y^2+x^3z^2) \overset{i}{\overbrace{\ge}} 3x^3yz; \frac{3}{2}(y^3x^2+y^3z^2) \overset{ii}{\overbrace{\ge}} 3y^3zx$$

$$\frac{3}{2}(z^3x^2+z^3y^2) \overset{iii}{\overbrace{\ge}} 3z^3xy;$$

$$i+ii+iii \Rightarrow \frac{3}{2}(x^3y^2+x^2y^3+y^3z^2+y^2z^3+z^3x^2+z^2x^3) \ge$$

$$\ge 3x^3yz+3y^3zx+3z^3xy \quad (A)$$

$$\left.\begin{cases} x^3y^2+y^2z^3+y^2z^3 \ge 3xy^2z^2 \\ y^3x^2+x^2z^3+x^2z^3 \ge 3yx^2z^2 \\ y^3z^2+z^2x^3+z^2x^3 \ge 3yz^2x^2 \\ z^3y^2+y^2x^3+y^2x^3 \ge 3zy^2x^2 \\ z^3x^2+x^2y^3+x^2y^3 \ge 3zx^2y^2 \\ x^3z^2+z^2y^3+z^2y^3 \ge 3xy^2z^2 \end{cases}\right\} AM \ge GM$$

Adding, $3(x^3y^2+x^2y^3+y^3z^2+y^2z^3+z^3x^2+z^2x^3) \ge$

$$\ge 6(x^2y^2z+y^2z^2x+z^2x^3y)$$

$$\Rightarrow \frac{1}{2}(x^3y^2+x^2y^3+y^3z^2+y^2z^3+z^3x^2+z^2x^3) \ge x^2y^2z+y^2z^2x+z^3x^3y \ (B)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{27m_a^2m_b^2m_c^2}{S^6} \le \left\{\frac{1}{(p-a)^2}+\frac{1}{(p-b)^2}+\frac{1}{(p-c)^2}\right\}^3$$

$$\Leftrightarrow 3\sqrt[3]{m_a^2m_b^2m_c^2} \le \frac{S^2}{(p-a)^2} + \frac{S^2}{(p-b)^2} + \frac{S^2}{(p-c)^2} = \sum r_a^2$$

Now, $3\sqrt[3]{m_a^2m_b^2m_c^2} \overset{AM-GM}{\le} \sum m_a^2 \le \frac{27}{4}R^2 \le \sum r_a^2$

40. Prove that in any triangle,

$$1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \ge \sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $1 + \operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2} \ge \sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Tener en cuenta lo siguiente: $\frac{r}{4R} = \operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2} \wedge \frac{p}{4R} = \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$r_a + r_b + r_c = 4R + r$. La desigualdad es equivalente:

$$1 + \frac{r}{4R} \ge \sqrt{3}\frac{p}{4R} \to 4R + r \ge \sqrt{3}p \to r_a + r_b + r_c \ge \sqrt{3}p$$

$$\to p\tan\frac{A}{2} + p\tan\frac{B}{2} + p\tan\frac{C}{2} \ge \sqrt{3}p \to \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \ge \sqrt{3}$$

En la desigualdad siguiente: $x + y + z \ge \sqrt{3(xy + yz + xz)}, \forall: x, y, z \in \mathbb{R}^+$

Sea: $x = \tan\frac{A}{2}, y = \tan\frac{B}{2}, z = \tan\frac{C}{2}, xy + yz + xz = 1$

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \ge \sqrt{3}$$

41. Prove that for any triangle ABC,

$$R \ge \frac{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{a + b + c}$$

Proposed by Nguyen Viet Hung - Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $R \geq \frac{\sqrt{(ab)^2+(bc)^2+(ac)^2}}{a+b+c}$

Tener presente lo siguiente en un triángulo ABC:

$$R = \frac{abc}{4S} \wedge \ 4S = \sqrt{(a+b+c)(b+c-a)(a+c-b)(b+a-c)}$$

$$\Rightarrow R = \frac{abc}{\sqrt{(a+b+c)(b+c-a)(a+c-b)(b+a-c)}} \Leftrightarrow b+c > a,$$

$a+c > b$; $b+a > c$. **Elevando al cuadrado la expresión tenemos:**

$$\Rightarrow \frac{(abc)^2}{(a+b+c)(b+c-a)(a+c-b)(b+a-c)} \geq \frac{(ab)^2+(bc)^2+(ac)^2}{(a+b+c)^2}$$

$$\Rightarrow \frac{(abc)^2}{(b+c-a)(a+c-b)(b+a-c)} \geq \frac{(ab)^2+(bc)^2+(ac)^2}{a+b+c} \rightarrow$$

$$\rightarrow \textit{Invirtiendo tenemos}$$

$$\Rightarrow \frac{(b+c-a)(a+c-b)(b+a-c)}{(abc)^2} \leq \frac{a+b+c}{(ab)^2+(bc)^2+(ac)^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{(ab)^2+(bc)^2+(ac)^2}{(abc)^2} \leq \frac{a+b+c}{(b+c-a)(a+c-b)(b+a-c)}$$

$$\Rightarrow \frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2} \leq \frac{(b+a-c)+(b+c-a)+(a+c-b)}{(b+c-a)(a+c-b)(b+a-c)} \rightarrow$$

$$\rightarrow \sum \frac{1}{c^2} \leq \sum \frac{1}{(a+c-b)(b+c-a)}$$

$$\Rightarrow \sum \frac{1}{(a+c-b)(b+c-a)} - \sum \frac{1}{c^2} = \sum \frac{c^2-(c+a-b)(b+c-a)}{(a+c-b)(b+c-a)c^2} =$$

$$= \sum \frac{c^2-(c^2-(a-b)^2)}{(a+c-b)(b+c-a)c^2} = \sum \frac{(a-b)^2}{(a+c-b)(b+c-a)c^2} \geq 0$$

42. In acute-angled $\Delta\ ABC$:

$$\sum|\tan^7 A-\cot^7 A|\geq 7\sum|\cot A-\tan A|$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Es bien sabido que: $|x| = |-x|$

$$\tan^7 A-\cot^7 A=(\tan A-\cot A)(\tan^6 A+\tan^5 A\cot A+\tan^4 A\cot^2 A+\tan^3 A\cot^3 A+$$
$$+\tan^2 A\cot^4 A+\tan A\cot^5 A+\cot^6 A)$$

Siendo un triángulo acutángulo: $\tan A, \tan B, \tan C>0$. Por la tanto:

$$\sum|\tan A-\cot A|\geq 0$$

$$\sum|\tan A-\cot A|\,|\tan^6 A+\tan^5 A\cot A+\tan^4 A\cot^2 A+\tan^3 A\cot^3 A+\tan^4 A\cot^2 A+\tan^5 A\cot A+\cot^6 A-7|\geq 0$$

Por: $MA\geq MG$

$$|\tan^6 A+\tan^5 A\cot A+\tan^4 A\cot^2 A+\tan^3 A\cot^3 A+\tan^4 A\cot^2 A+\tan^5 A\cot A+\cot^6 A-7|\geq 0$$

$$\rightarrow\sum|\tan A-\cot A|\,|\tan^6 A+\tan^5 A\cot A+\tan^4 A\cot^2 A+\tan^3 A\cot^3 A+\tan^4 A\cot^2 A+\tan^5 A\cot A+\cot^6 A-7|\geq 0$$

43. Prove that in any acute triangle ABC,

$$\sqrt{AH}+\sqrt{BH}+\sqrt{CH}\leq\sqrt{2(h_a+h_b+h_c)}$$

where H is the orthocenter.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Desde que es un triángulo acutángulo: $\rightarrow(\sqrt{AH},\sqrt{BH},\sqrt{CH})>0$

Por desigualdad de Cauchy:

$$(h_a+h_b+h_c)\left(\frac{AH}{h_a}+\frac{BH}{h_b}+\frac{CH}{h_c}\right)\geq\left(\sqrt{AH}+\sqrt{BH}+\sqrt{CH}\right)^2$$

Para ello desarrolaremos:

$$\frac{AH}{h_a}+\frac{BH}{h_b}+\frac{CH}{h_c}=\frac{2R\cos A}{\frac{bc}{2R}}+\frac{2R\cos B}{\frac{ac}{2R}}+\frac{2R\cos C}{\frac{ab}{2R}}=$$

$$=\frac{2R\cos A}{2R\,\text{sen}\,B\,\text{sen}\,C}+\frac{2R\cos B}{2R\,\text{sen}\,A\,\text{sen}\,C}+\frac{2R\cos C}{2R\,\text{sen}\,A\,\text{sen}\,B}$$

$$\Rightarrow -\frac{\cos(B+C)}{\text{sen}\,B\,\text{sen}\,C}-\frac{\cos(A+C)}{\text{sen}\,A\,\text{sen}\,C}-\frac{\cos(A+B)}{\text{sen}\,A\,\text{sen}\,B}=$$

$$=-(\cot B\cot C-1)-(\cot A\cot C-1)-(\cot A\cot B-1)$$

$$\Rightarrow \frac{AH}{h_b}+\frac{BH}{h_b}+\frac{CH}{h_c}=3-(\cot A\cot B+\cot B\cot C+\cot A\cot C)=2$$

$$\Rightarrow 2(h_a+h_b+h_c)\ge\left(\sqrt{AH}+\sqrt{BH}+\sqrt{CH}\right)^2\to$$

$$\to \sqrt{AH}+\sqrt{BH}+\sqrt{CH}\le\sqrt{2(h_a+h_b+h_c)}$$

44. Prove that in any acute triangle ABC,

$$\cot^2 A\cot^2 B+\cot^2 B\cot^2 C+\cot^2 C\cot^2 A\ge\frac{\cos^2 A+\cos^2 B+\cos^2 C}{\sin^2 A+\sin^2 B+\sin^2 C}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sea ABC un triángulo acutángulo. Probar que:

$$\cot^2 A\cot^2 B+\cot^2 B\cot^2 C+\cot^2 C\cot^2 A\ge\frac{\cos^2 A+\cos^2 B+\cos^2 C}{\sin^2 A+\sin^2 B+\sin^2 C}$$

$$(\cot A\cot B+\cot B\cot C+\cot A\cot C)^2-2\cot A\cot B\cot C\,(\cot A+\cot B+\cot C)\ge$$

$$\ge\frac{\cos^2 A+\cos^2 B+\cos^2 C}{\sin^2 A+\sin^2 B+\sin^2 C}\quad(1)$$

Tener en cuenta lo siguiente: → Si: $A+B+C=\pi$

$$\cot A\cot B+\cot B\cot C+\cot A\cot C=1\quad(\alpha)$$

$$\sec A\sec B\sec C+1=\tan A\tan B+\tan B\tan C+\tan A\tan C\quad(\beta)$$

$$\sin^2 A+\sin^2 B+\sin^2 C=2(1+\cos A\cos B\cos C)\quad(\gamma)\wedge\cos^2 A+\cos^2 B+\cos^2 C=$$

$$=1-2\cos A\cos B\cos C\quad(\theta)$$

Reemplazando (α) en (1):

$$\big(1-2\cot A\cot B\cot C\,(\cot A+\cot B+\cot C)\big)(\sin^2 A+\sin^2 B+\sin^2 C)\geq$$

$$\geq\cos^2 A\cos^2 B\cos^2 C.\ \textbf{De }(\gamma)\wedge(\theta)\textbf{ en }(1):$$

$$(1-2\cot A\cot B\cot C)(\cot A+\cot B+\cot C)(2+2\cos A\cos B\cos C)\geq$$

$$\geq 1-2\cos A\cos B\cos C$$

Dado que el Δ es acutángulo: $\cos A\cos B\cos C>0\Leftrightarrow$ dividamos sin alterar el sentido de la desigualdad:

$$(1-2\cot A\cot B\cot C)(\cot A+\cot B+\cot C)2(\sec A\sec B\sec C+1)\geq\sec A\sec B\sec C-2$$

$$\big(1-2\cot A\cot B\cot C\,(\cot A+\cot B+\cot C)\big)2(\tan A\tan B+\tan B\tan C+\tan A\tan C)\geq$$

$$\geq\tan A\tan B+\tan B\tan C+\tan A\tan C-3$$

$$2(\tan A\tan B+\tan B\tan C+\tan A\tan C)-$$

$$-4\cot A\cot B\cot C\left(2(\tan A+\tan B+\tan C)+\frac{\tan A\tan B}{\tan C}+\frac{\tan C\tan B}{\tan A}+\frac{\tan C\tan A}{\tan B}\right)\geq$$

$$\geq\tan A\tan B+\tan B\tan C+\tan A\tan C-3$$

$\rightarrow$ **Probaremos que:** $\frac{\tan A\tan B}{\tan C}+\frac{\tan B\tan C}{\tan A}+\frac{\tan A\tan C}{\tan B}\geq\tan A+\tan B+\tan C$

Por: $MA\geq MG$: $\frac{\tan A\tan B}{\tan C}+\frac{\tan A\tan C}{\tan B}\geq 2\tan A\quad(1)$

$$\frac{\tan A\tan B}{\tan C}+\frac{\tan B\tan C}{\tan A}\geq 2\tan B\quad(2)$$

$$\frac{\tan B\tan C}{\tan A}+\frac{\tan A\tan C}{\tan B}\geq 2\tan C\quad(3)$$

Sumando $(1)+(2)+(3)$: Se concluye que:

$$\frac{\tan A\tan B}{\tan C}+\frac{\tan B\tan C}{\tan A}+\frac{\tan A\tan C}{\tan B}\geq\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

$$2(\tan A\tan B+\tan B\tan C+\tan A\tan C)-4\cot A\cot B\cot C\,(3\tan A\tan B\tan C)\geq$$
$$\geq \tan A\tan B+\tan B\tan C+\tan A\tan C-3$$
$$2(\tan A\tan B+\tan B\tan C+\tan A\tan C)-12\geq \tan A\tan B+\tan B\tan C+\tan A\tan C-3$$
$$\Rightarrow \tan A\tan B+\tan B\tan C+\tan A\tan C\geq 9$$
$$\Rightarrow \frac{\tan A+\tan B+\tan C}{\tan C}+\frac{\tan A+\tan B+\tan C}{\tan A}+\frac{\tan A+\tan B+\tan C}{\tan B}\geq 9$$
$$\Rightarrow (\tan A+\tan B+\tan C)(\cot A+\cot B+\cot C)\geq 9\rightarrow$$ **(Válido por: $MA\geq MG$)**

45. Prove that in any triangle ABC:

$$\frac{\sin B\sin C}{\sin^2\frac{A}{2}}+\frac{\sin C\sin A}{\sin^2\frac{B}{2}}+\frac{\sin A\sin B}{\sin^2\frac{C}{2}}\geq 9$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{\operatorname{sen} B\operatorname{sen} C}{\operatorname{sen}^2\frac{A}{2}}+\frac{\operatorname{sen} C\operatorname{sen} A}{\operatorname{sen}^2\frac{B}{2}}+\frac{\operatorname{sen} A\operatorname{sen} B}{\operatorname{sen}^2\frac{C}{2}}\geq 9$

$$\frac{2\operatorname{sen} B\operatorname{sen} C}{2\operatorname{sen}^2\frac{A}{2}}+\frac{2\operatorname{sen} A\operatorname{sen} C}{2\operatorname{sen}^2\frac{B}{2}}+\frac{2\operatorname{sen} A\operatorname{sen} B}{2\operatorname{sen}^2\frac{C}{2}}\geq 9$$
$$\frac{2\operatorname{sen} B\operatorname{sen} C}{1-\cos A}+\frac{2\operatorname{sen} A\operatorname{sen} C}{1-\cos B}+\frac{2\operatorname{sen} A\operatorname{sen} B}{1-\cos C}\geq 9$$
$$\frac{2\operatorname{sen} B\operatorname{sen} C\,(1+\cos A)}{(1+\cos A)(1-\cos A)}+\frac{2\operatorname{sen} A\operatorname{sen} C\,(1+\cos B)}{(1+\cos B)(1-\cos B)}+\frac{2\operatorname{sen} A\operatorname{sen} B\,(1+\cos C)}{(1-\cos C)(1+\cos C)}\geq 9$$
$$\frac{2\operatorname{sen} B\operatorname{sen} C\,(1+\cos A)}{\operatorname{sen}^2 A}+\frac{2\operatorname{sen} A\operatorname{sen} C\,(1+\cos B)}{\operatorname{sen}^2 B}+\frac{2\operatorname{sen} A\operatorname{sen} B\,(1+\cos C)}{\operatorname{sen}^2 C}\geq 9$$
$$A=\frac{2\operatorname{sen} A\operatorname{sen} C}{\operatorname{sen}^2 A}+\frac{2\operatorname{sen} A\operatorname{sen} C}{\operatorname{sen}^2 B}+\frac{2\operatorname{sen} A\operatorname{sen} B}{\operatorname{sen}^2 C}\geq 6\rightarrow$$ **(Válido por: $MA\geq MG$)**
$$B=\frac{2\operatorname{sen} B\operatorname{sen} C\cos A}{\operatorname{sen}^2 A}+\frac{2\operatorname{sen} A\operatorname{sen} C\cos B}{\operatorname{sen}^2 B}+\frac{2\operatorname{sen} A\operatorname{sen} B\cos C}{\operatorname{sen}^2 C}$$

$$B=\frac{\text{sen}^2B+\text{sen}^2C-\text{sen}^2A}{\text{sen}^2A}+\frac{\text{sen}^2A+\text{sen}^2C-\text{sen}^2B}{\text{sen}^2B}+\frac{\text{sen}^2A+\text{sen}^2B-\text{sen}^2C}{\text{sen}^2C}$$

$$B=\frac{\text{sen}^2B}{\text{sen}^2A}+\frac{\text{sen}^2A}{\text{sen}^2B}+\frac{\text{sen}^2B}{\text{sen}^2C}+\frac{\text{sen}^2C}{\text{sen}^2B}+\frac{\text{sen}^2C}{\text{sen}^2A}+\frac{\text{sen}^2A}{\text{sen}^2C}-3\geq 3\Leftrightarrow$$

$\Leftrightarrow$ (Válido por: $MA\geq MG$) $\to A+B\geq 9$

46. In $\Delta\ ABC$ the following relationship holds:

$$\tan 2016A+\tan 2016B+\tan 2016C=\tan 2016A\tan 2016B\tan 2016C$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

En un triángulo ABC: Probar que si es (V) o (F) lo siguiente:

$$\tan 2016A+\tan 2016B+\tan 2016C=\tan 2016A\tan 2016B\tan 2016C$$

Si: $A+B+C=\pi\to$ Se cumple que: $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

Demostraré lo siguiente:

$$\tan kA+\tan kB+\tan kC=\tan kA\tan kB\tan kC\Leftrightarrow k\in\mathbb{Z}$$

$$\Rightarrow kA+kB+kC=k\pi\to kA+kB=k(\pi-C)$$

$$\Rightarrow \tan(kA+kB)=\tan\big(k(\pi-C)\big)\Rightarrow\frac{\tan kA+\tan kB}{1-\tan kA\tan kB}=-\tan kC\to$$

$\to\tan kA+\tan kB+\tan kC=\tan kA\tan kB\tan kC$. Si:

$$k=2016\to\tan 2016A+\tan 2016B+\tan 2016C=\tan 2016A\tan 2016B\tan 2016C$$

47. Prove that in any acute triangle ABC,

$$HA\sqrt{bc}+HB\sqrt{ca}+HC\sqrt{ab}\geq 2(R+r)\sqrt{2R(R+r)}$$

where H is the orthocenter.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo H ortocentro. Probar en un triángulo acutángulo ABC:

$$HA\sqrt{bc} + HB\sqrt{ac} + HC\sqrt{ab} \geq 2(R + r)\sqrt{2R(R + r)}$$

Por la desigualdad de Cauchy:

$$\left(HA\sqrt{bc} + HB\sqrt{ac} + HC\sqrt{ab}\right)\left(\frac{HA}{\sqrt{bc}} + \frac{HB}{\sqrt{ac}} + \frac{HC}{\sqrt{ab}}\right) \geq (HA + HB + HC)^2$$

$$\left(HA\sqrt{bc} + HB\sqrt{ac} + HC\sqrt{ab}\right) \geq \frac{(HA+HB+HC)^2}{\left(\frac{HA}{\sqrt{bc}}+\frac{HB}{\sqrt{ac}}+\frac{HC}{\sqrt{ab}}\right)} \quad \text{(A)}$$

Tener en cuenta lo siguiente:

1. $HA + HB + HC = 2(R + r)$ $\wedge$ 2. $\left(\frac{HA}{\sqrt{bc}} + \frac{HB}{\sqrt{ac}} + \frac{HC}{\sqrt{ab}}\right) \leq \sqrt{\frac{2(R+r)}{R}}$

Por la tanto reemplazando en (A)

$$\left(HA\sqrt{bc} + HB\sqrt{ac} + HC\sqrt{ab}\right) \geq \frac{(HA + HB + HC)^2}{\left(\frac{HA}{\sqrt{bc}} + \frac{HB}{\sqrt{ac}} + \frac{HC}{\sqrt{ab}}\right)} \geq$$

$$\geq \frac{4(R + r)^2}{\sqrt{\frac{2(R + r)}{R}}} = 2(R + r)\sqrt{2R(R + r)}$$

48. **Prove that in any triangle:**

$$\sum_{cyc}\sqrt{r_a^2 + 1} \geq \sqrt{6(4R + r)}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Nguyen Viet Hung – Hanoi – Vietnam

Using Minkowski and AM-GM inequalities respectively we get:

$$\sqrt{r_a^2+1}+\sqrt{r_b^2+1}+\sqrt{r_c^2+1}\geq\sqrt{(r_a+r_b+r_c)^2+(1+1+1)^2}$$

$$=\sqrt{(r_a+r_b+r_c)^2+9}\geq\sqrt{6(r_a+r_b+r_c)}=\sqrt{6(4R+r)}$$ as desired.

49. Prove that in any triangle:

$$\sqrt[3]{(r_a+r_b)(r_b+r_c)(r_c+r_a)}\geq\frac{2p}{\sqrt{3}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Seyran Ibrahimov-Massilli-Azerbaidian

$$\sqrt[3]{(r_a+r_b)(r_b+r_c)(r_a+r_c)}\geq\frac{2p}{\sqrt{3}}\Rightarrow p-\text{semiperimeter}$$

$$\left(\frac{S}{p-a}+\frac{S}{p-b}\right)\left(\frac{S}{p-b}+\frac{S}{p-c}\right)\left(\frac{S}{p-a}+\frac{S}{p-c}\right)\geq\frac{8p^3}{3\sqrt{3}}$$

$$\frac{S(2p-a-b)}{(p-a)(p-b)}\cdot\frac{S(2p-b-c)}{(p-b)(p-c)}\cdot\frac{S(2p-a-c)}{(p-a)(p-b)}\geq\frac{8p^3}{3\sqrt{3}}$$

$$\frac{S^3abc}{(p-a)^2(p-b)^2(p-c)^2}\geq\frac{8p^3}{3\sqrt{3}}\Rightarrow abc=4RS$$

$$\frac{p^2(p-a)^2(p-b)^2(p-c)^2\cdot 4R}{(p-a)^2(p-b)^2(p-c)^2}\geq\frac{8p^3}{3\sqrt{3}};\ 4R\geq\frac{8p}{3\sqrt{3}};\ 12\sqrt{3}R-8p\geq 0$$

$$3\sqrt{3}R-2p\geq 0\Rightarrow 27R^2\geq 4p^2\quad\text{(Gerretsen)}\Rightarrow 4p^2\leq 16R^2+16Rr+12r^2$$

$$27R^2\geq 16R^2+16Rr+12r^2;\ 11R^2-16Rr-12r^2\geq 0$$

$$(R-2r)(11R+6r)\geq 0\Rightarrow R\geq 2r\quad\text{(Euler)}$$

50. Let ABC be a triangle with incenter I. Prove that:

$$\sqrt{AI\cdot AB\cdot AC}+\sqrt{BI\cdot BC\cdot BA}+\sqrt{CI\cdot CA\cdot CB}\geq 6r\sqrt{6r}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo I incentro. Probar en un triángulo ABC:

$$\sqrt{AI\times AB\times AC}+\sqrt{BI\times BC\times BA}+\sqrt{CI\times CA\times CB}\geq 6r\sqrt{6r}$$

Tener en cuenta lo siguiente: $AI=r\csc\frac{A}{2}, BI=r\csc\frac{B}{2}, CI=r\csc\frac{C}{2}$

$$AB=2R\operatorname{sen}C, AC=2R\operatorname{sen}B, BC=2R\operatorname{sen}A, r=4R\operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}$$

La desigualdad es equivalente: $\Rightarrow \sum\sqrt{r\csc\frac{A}{2}4r^2\operatorname{sen}C\operatorname{sen}B}\geq 6r\sqrt{6r}\rightarrow$

$$\rightarrow\sum 2R\sqrt{\operatorname{sen}B\operatorname{sen}C\csc\frac{A}{2}}\geq 6r\sqrt{6}$$

$$\Rightarrow\sum\sqrt{\operatorname{sen}C\operatorname{sen}B\csc\frac{A}{2}}\geq 12\sqrt{6}\operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}$$

$$\Rightarrow\sqrt{\csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2}}\sum\sqrt{\operatorname{sen}C\operatorname{sen}B\csc\frac{A}{2}\times\csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2}}\geq 12\sqrt{6}$$

$$\Rightarrow 2\sqrt{\csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2}}\sum\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}\csc\frac{A}{2}}\geq 12\sqrt{6}$$

$\Leftrightarrow 2\sqrt{\csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2}}\geq 4\sqrt{2}\ \wedge$ **Por:** $MA\geq MG$:

$$\sum\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}\csc\frac{A}{2}}\geq 3\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}\geq 3\sqrt{3}$$

51. Let a,b,c be side – lengths of a triangle with circumradius R. Prove that:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\le R^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

De la desigualdad ponderada Weizenbock (Refinamiento de Pohoata)

$$a^2m+b^2n+c^2p\ge 4\sqrt{mn+np+mp}S\Leftrightarrow m,n,p\ge 0,$$

$(a,b,c$ lados de un triángulo$)\wedge\quad S=$ (Área). Sea: $m=\frac{x}{a},n=\frac{y}{b},p=\frac{z}{c}$

$$ax+by+cz\ge 4\sqrt{\frac{xy}{ab}+\frac{yz}{bc}+\frac{xz}{ac}}S\to ax+by+cz\ge 4\sqrt{\frac{xy}{ab}+\frac{yz}{bc}+\frac{xz}{ac}}\frac{abc}{4R}$$

$\Rightarrow\frac{R}{bc}x+\frac{R}{ac}y+\frac{R}{ab}z\ge\sqrt{\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ac}}$ (1) . En. (1). Sean: $x=\frac{bc}{a},y=\frac{ca}{b},\ z=\frac{ab}{c}$

$R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\sqrt{\frac{c^2}{ab}+\frac{a^2}{bc}+\frac{b^2}{ac}}\to$ Elevando al cuadrado, nos quenda...

$$\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}\ge R^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\quad\text{(LQQD)}$$

52. In $\Delta\ ABC$:

$$(m_a^7+m_b^7+m_c^7)\left(\frac{1}{m_a^3}+\frac{1}{m_b^3}+\frac{1}{m_c^3}\right)\ge s^4, s-\text{semiperimeter}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

$$(m_a^7+m_b^7+m_c^7)\left(\frac{1}{m_a^3}+\frac{1}{m_b^3}+\frac{1}{m_c^3}\right)$$

$$\overset{C-B-S}{\ge}(m_a^2+m_b^2+m_c^2)^2=\left\{\frac{3}{4}(a^2+b^2+c^2)\right\}^2$$

It suffices to show $\frac{3}{4}(a^2+b^2+c^2) \geq s^2 = \frac{(a+b+c)^2}{4}$

$\Leftrightarrow a^2+b^2+c^2 \geq ab+bc+ca \to$ true (Proved)

53. In $\Delta\, ABC$:

$$16\left(\sum \frac{r_a^6}{a^3}\right)\left(\sum \frac{a^3}{r_a^2}\right) \geq 9(a^2+b^2+c^2)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $16\left(\sum \frac{r_a^6}{a^3}\right)\left(\sum \frac{a^3}{r_a^2}\right) \geq 9(a^2+b^2+c^2)^2$

Tener en cuenta lo siguiente:

$\sum r_a^2 \geq \frac{27R^2}{4}$ (Demostrado anteriormente) $\wedge\ 9R^2 \geq (a^2+b^2+c^2) \to$

$\to$ (Desigualdad Leibniz). Por desigualdad de Cauchy:

$$16\left(\frac{r_a^6}{a^3}\right)\left(\sum \frac{a^3}{r_a^2}\right) \geq 16\left(\sum r_a^2\right)^2 \geq 729R^4 \geq 9(a^2+b^2+c^2)^2 \Leftrightarrow$$

$$\Leftrightarrow 9R^2 \geq (a^2+b^2+c^2)\text{(Leibniz)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$16\left(\frac{r_a^6}{a^3}+\frac{r_b^6}{b^3}+\frac{r_c^6}{c^3}\right)\left(\frac{a^3}{r_a^2}+\frac{b^3}{r_b^2}+\frac{c^3}{r_c^2}\right)$$

$$\overset{C-B-S}{\overbrace{\geq}} 16(r_a^2+r_b^2+r_c^2)^2 \overset{Thebault}{\overbrace{\geq}} 16\left(\frac{27R^2}{4}\right)^2 = (27R^2)^2$$

$$\geq \left(3(a^2+b^2+c^2)\right)^2 = 9(a^2+b^2+c^2)^2$$

$$(9R^2 \geq a^2+b^2+c^2)\textit{(Leibniz)}$$

54. In $\Delta\ ABC$:

$$(a^2+b^2+c^2)^2(AK^2+BK^2+CK^2)<4s^2(a^2b^2+b^2c^2+c^2a^2)$$

$s-$ semiperimeter, $K-$ Lemoine's point

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

In $\Delta\ ABC$, $(a^2+b^2+c^2)^2(AK^2+BK^2+CK^2)<4s^2(a^2b^2+b^2c^2+c^2a^2)$

$\frac{AK}{KK_A}=\frac{b^2+c^2}{a^2}$ (Hansberger), $AK_A\to$ symmetric from A to BC and K_A is on BC

$$\Rightarrow\frac{KK_A}{AK}=\frac{a^2}{b^2+c^2}$$

$$\Rightarrow\frac{AK_A}{AK}=\frac{a^2+b^2+c^2}{b^2+c^2}\Rightarrow AK=\left(\frac{b^2+c^2}{a^2+b^2+c^2}\right)AK_A\quad (1)$$

Again, $\frac{BK_A}{CK_A}=\frac{c^2}{b^2}\Rightarrow\frac{m}{n}=\frac{c^2}{b^2}$, where $BK_a=m, CK_A=n$

Stewart's theorem states that: $b^2m+c^2n=a(d^2+mn)$ (2), where $d=AK_A$

we have $\frac{m}{n}=\frac{c^2}{b^2}$ and $m+n=a\Rightarrow\frac{m+n}{n}=\frac{b^2+c^2}{b^2}\Rightarrow n=\frac{ab^2}{b^2+c^2}; m=\frac{ac^2}{b^2+c^2}$

$$(2)\Rightarrow\frac{b^2ac^2}{b^2+c^2}+\frac{c^2ab^2}{b^2+c^2}=a\left(d^2+\frac{a^2b^2c^2}{(b^2+c^2)^2}\right)\Rightarrow\frac{2b^2c^2}{b^2+c^2}=d^2+\frac{a^2b^2c^2}{(b^2+c^2)^2}$$

$$\Rightarrow d^2=\frac{b^2c^2}{b^2+c^2}\left(2-\frac{a^2}{b^2+c^2}\right)=\frac{b^2c^2(2b^2+2c^2-a^2)}{(b^2+c^2)^2}$$

$$\Rightarrow AK_A=d=\left(\frac{bc}{b^2+c^2}\right)\sqrt{2b^2+2c^2-a^2}\Rightarrow AK=\frac{bc\sqrt{2b^2+2c^2-a^2}}{a^2+b^2+c^2}$$

Similarly, $BK=\frac{ca\sqrt{2c^2+2a^2-b^2}}{a^2+b^2+c^2}$ and $CK=\frac{ab\sqrt{2a^2+2b^2-c^2}}{a^2+b^2+c^2}$

Given inequality $\Leftrightarrow$

$$\frac{(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)^2}(b^2c^2(2b^2+2c^2-a^2)+c^2a^2(2c^2+2a^2-b^2)+$$

$$+a^2b^2(2a^2+2b^2-c^2))$$
$$<(a+b+c)^2(a^2b^2+b^2c^2+c^2a^2)$$
$$\Leftrightarrow b^2c^2(2b^2+2c^2-a^2)+c^2a^2(2c^2+2a^2-b^2)+a^2b^2(2a^2+2b^2-c^2)<$$
$$<(a+b+c)^2(a^2b^2+b^2c^2+c^2a^2)$$

Now, $2b^2+2c^2-a^2=(b+c)^2+(b-c)^2-a^2=$

$$=(b+c)^2+(b-c+a)(b-c-a)<(b+c)^2 \quad ;$$
$$(b-c+a>0 \ \text{ and } b-c-a<0)$$

Similarly, $2c^2+2a^2-b^2<(c+a)^2$ **and** $2a^2+2b^2-c^2<(a+b)^2$

$$b^2c^2(2b^2+2c^2-a^2)+c^2a^2(2c^2+2a^2-b^2)+a^2b^2(2a^2+2b^2-c^2)$$
$$<b^2c^2(b+c)^2+c^2a^2(c+a)^2+a^2b^2(a+b)^2$$

if we can prove $b^2c^2(b+c)^2+c^2a^2(c+a)^2+a^2b^2(a+b)^2<$

$<(a+b+c)^2(a^2b^2+b^2c^2+c^2a^2)$**, we are done.**

$$\Leftrightarrow b^4c^2+b^2c^4+c^4a^2+c^2a^4+a^4b^2+a^2b^4+2\left(\sum a^3b^3\right)$$
$$<(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)+$$
$$+2(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2)$$
$$=a^4b^2+a^2b^4+b^4c^2+b^2c^4+c^4a^2+c^2a^4+3a^2b^2c^2+$$
$$+2\left(\sum a^3b^3\right)+2(ab^3c^2+ba^2c^2+cb^3a^2+bc^3a^2+ca^3b^2+ac^3b^2)$$

$\Leftrightarrow 3a^2b^2c^2+2abc(\sum a^2b+\sum ab^2)>0\rightarrow$ **true, hence proved**

55. Let a,b,c be the side – lengths of a triangle. Prove that:

$$(a+b+c)^5+9abc(a+b+c)^2+12(a+b+c)(ab+bc+ca)^2\geq$$
$$\geq 7(ab+bc+ca)(a+b+c)^3+27abc(ab+bc+ca)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c los lados de un triángulo. Probar que:

$$(a+b+c)^5+9abc(a+b+c)^2+12(a+b+c)(ab+bc+ac)^2\geq$$
$$\geq 7(ab+bc+ac)(a+b+c)^3+27abc(ab+bc+ac)$$
$$(a+b+c)[(a+b+c)^4+12(ab+bc+ac)^2-7(a+b+c)^2(ab+bc+ac)]+$$
$$+9abc[(a+b+c)^2-3(ab+bc+ac)]\geq 0$$
$$(a+b+c)[(a+b+c)^2-3(ab+bc+ac)][(a+b+c)^2-4(ab+bc+ac]+$$
$$+9abc[(a+b+c)^2-3(ab+bc+ac)]\geq 0$$
$$[(a+b+c)^2-3(ab+bc+ac)][(a+b+c)^3-4(a+b+c)(ab+bc+ac)+9abc]\geq 0$$

Desde que: a, b, c son lados de un triángulo. Por la tanto:

$$a+b+c=2p, ab+bc+ac=p^2+r^2+4Rr, abc=4pRr$$
$$[(a+b+c)^2-3(ab+bc+ac)][8p^3-8p(p^2+r^2+4Rr)+36pRr]\geq 0$$
$$[(a+b+c)^2-3(ab+bc+ac)][4pr(R-2r)]\geq 0$$
$$\Leftrightarrow (a+b+c)^2-3(ab+bc+ac)\geq 0\ \wedge 4pr(R-2r)\geq 0\Leftrightarrow$$

$\Leftrightarrow$ (Desigualdad de Euler)

56. Prove that in any triangle ABC:

$$3R\sqrt{3}\geq\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\geq 3\sqrt{6Rr}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

1. $3R\sqrt{3}\geq\sqrt{ab}+\sqrt{bc}+\sqrt{ac}$

Tener presente lo siguiente:

$$a=2R\sin A, b=2R\sin B, c=2R\sin C, r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}$$

$$3R\sqrt{3}\ge 2R\sqrt{\sin A\sin B}+2R\sqrt{\sin B\sin C}+2R\sqrt{\sin A\sin C}$$

$\frac{3\sqrt{3}}{2}\ge\sqrt{\sin A\sin B}+\sqrt{\sin B\sin C}+\sqrt{\sin A\sin C}$. **Por:** $MA\ge MG$

$$2\sqrt{\sin A\sin B}\le\sin A+\sin B \quad \text{(A)}$$

$$2\sqrt{\sin B\sin C}\le\sin B+\sin C \quad \text{(B)}$$

$$2\sqrt{\sin A\sin C}\le\sin A+\sin C \quad \text{(C)}$$

Sumando: $(A)+(B)+(C)$...

$$\Rightarrow\sqrt{\sin A\sin B}+\sqrt{\sin B\sin C}+\sqrt{\sin A\sin C}\le\sin A+\sin B+\sin C\le$$

$$\le 3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2} \quad \text{(LQQD)}$$

2. $\sqrt{ab}+\sqrt{bc}+\sqrt{ac}\ge 3\sqrt{6Rr}$. **La desigualdad es equivalente:**

$$2R\left(\sqrt{\sin A\sin B}+\sqrt{\sin B\sin C}+\sqrt{\sin A\sin B}\right)\ge 3\sqrt{6R\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}$$

$$\sqrt{\sin A\sin B}+\sqrt{\sin B\sin C}+\sqrt{\sin A\sin C}\ge 3\sqrt{6\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Rightarrow 2\sqrt{\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}}+2\sqrt{\frac{\cos\frac{A}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}}}+2\sqrt{\frac{\cos\frac{A}{2}\cos\frac{C}{2}}{\sin\frac{B}{2}}}\ge 3\sqrt{6}$$

$\Rightarrow$ **Desde que:** $\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}=\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\left(\frac{A+B}{2}\right)}=\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}}=\frac{1}{1-\tan\frac{A}{2}\tan\frac{B}{2}}$

$\Rightarrow$ **La desigualdad es equivalente:**

$$\frac{1}{\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}}+\frac{1}{\sqrt{1-\tan\frac{B}{2}\tan\frac{C}{2}}}+\frac{1}{\sqrt{1-\tan\frac{A}{2}\tan\frac{C}{2}}}\geq\frac{3\sqrt{6}}{2}$$

Por desigualdad de Cauchy:

$$\frac{1}{\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}}+\frac{1}{\sqrt{1-\tan\frac{B}{2}\tan\frac{C}{2}}}+\frac{1}{\sqrt{1-\tan\frac{A}{2}\tan\frac{C}{2}}}\geq$$

$$\geq\frac{(1+1+1)^2}{\sum\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}}\geq\frac{9}{\sum\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}}\quad \text{(A)}$$

De la siguiente desigualdad: $3(x^2+y^2+z^2)\geq(x+y+z)^2$

$\Rightarrow$ Sea: $x=\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}, y=\sqrt{1-\tan\frac{B}{2}\tan\frac{C}{2}}, z=\sqrt{1-\tan\frac{A}{2}\tan\frac{C}{2}}$

$$3\left(3-\tan\frac{A}{2}\tan\frac{B}{2}-\tan\frac{B}{2}\tan\frac{C}{2}-\tan\frac{A}{2}\tan\frac{C}{2}\right)\geq\left(\sum\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}\right)$$

$\Rightarrow\sqrt{6}\geq\sum\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}\Rightarrow$ Por tanto en (A)

$$\frac{9}{\sum\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}}\geq\frac{9}{\sqrt{6}}=\frac{3\sqrt{6}}{2}\quad \text{(LQQD)}$$

57. In acute – triangle ABC:

$$\frac{\tan^4 A}{\tan^3 B}+\frac{\tan^4 B}{\tan^3 C}+\frac{\tan^4 C}{\tan^3 A}\geq\tan A\tan B\tan C$$

Proposed by Daniel Sitaru – Romania

Solution by Seyran Ibrahimov – Maasilli – Azerbaidian

$$a=tanA, b=tanB, c=tanC$$

$$a+b+c=abc;\ \frac{a^4}{b^3}+\frac{b^4}{c^3}+\frac{c^4}{a^3}\ge abc$$

$$\frac{a^4}{b^3}+\frac{b^4}{c^3}+\frac{c^4}{a^3}\overset{Radon}{\ge}\frac{(a+b+c)^4}{(a+b+c)^3}=a+b+c=abc$$

58. In $\Delta\, ABC$:

$$2(AN^2+BN^2+CN^2)+42Rr\le 4s^2+3R\cdot ON$$

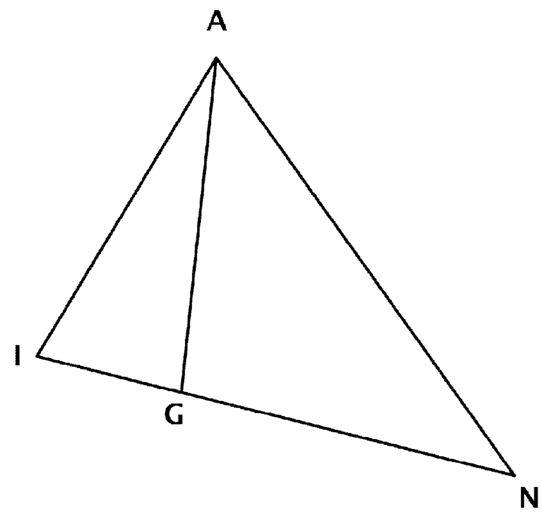
N – Nagel's point, O – circumcentre, s – semiperimeter

r – inradius, R – circumradius

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata - India

I, G, N in that order, are collinear and $GN=2IG$



By Stewart's theorem, $AI^2\cdot 2IG+AN^2\cdot IG=3IG(AG^2+2IG^2)$

$$\Rightarrow 2AI^2+AN^2=4AG^2+6IG^2\Rightarrow AN^2=3AG^2+6IG^2-2AI^2$$

Similarly, $BN^2=3BG^2+6IG^2-2BI^2;\ CN^2=3CG^2+6IG^2-2CI^2$

$$2\left(\sum AN^2\right) = 6\left(\sum AG^2\right) + 36IG^2 - 4\left(\sum AI^2\right) \quad (i)$$

$$\sum AG^2 = \frac{4}{9}\left(\sum m_a^2\right) = \frac{4}{9}\cdot\frac{3}{4}(a^2 + b^2 + c^2) = \frac{a^2 + b^2 + c^2}{3}$$

$$6\left(\sum AG^2\right) = 2\left(\sum a^2\right) = 2\left(2s^2 - 2r(4R + r)\right)$$

$$= 4s^2 - 16Rr - 4r^2 \quad \textbf{(1)}$$

$$AI^2 = r^2\csc^2\frac{A}{2}, BI^2 = r^2\csc^2\frac{B}{2}, CI^2 = r^2\csc^2\frac{C}{2}$$

$$\sum AI^2 = r^2\left\{\frac{bc}{(s-b)(s-c)} + \frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right\}$$

$$= \frac{r^2\{bc(s-a) + ca(s-b) + ab(s-c)\}}{(s-a)(s-b)(s-c)}$$

$$= \frac{r^2\{s(ab + bc + ca) - 3abc\}}{(s-a)(s-b)(s-c)} = \frac{r^2s^2(ab + bc + ca) - 3r^2sabc}{s(s-a)(s-b)(s-c)}$$

$$= \frac{r^2s^2(ab + bc + ca) - 3r^2sabc}{r^2s^2} = ab + bc + ca - \frac{3abc}{s}$$

$$= s^2 + 4Rr + r^2 - 12Rr = s^2 - 8Rr + r^2$$

$$-4(\textstyle\sum AI^2) = -4s^2 + 32Rr - 4r^2 \quad \textbf{(2)}$$

$$IG = \sqrt{\frac{-a^3 - b^3 - c^3 - 9abc + 2(\sum a^2b + \sum ab^2)}{9(a + b + c)}}$$

$$36IG^2 = 36\cdot\frac{-(a^3 + b^3 + c^3) - 9abc + 2(\sum a^2b + \sum ab^2)}{9(a + b + c)}$$

$$= \frac{2}{s}\left\{-(a^3 + b^3 + c^3) - 9abc + 2\left(\sum a^2b + \sum ab^2\right)\right\}$$

$$\sum a^2b + \sum ab^2 = a^2(b + c) + b^2(c + a) + c^2(a + b)$$

$$= a^2(2s-a)+b^2(2s-b)+c^2(2s-c)$$

$$= 2s(a^2+b^2+c^2)-(a^3+b^3+c^3)$$

$$36IG^2 = \frac{2}{s}\{-3(a^3+b^3+c^3)-9abc+4s(a^2+b^2+c^2)\}$$

$$= \frac{2}{s}\{-3(a^3+b^3+c^3-3abc+3abc)-9abc+4s(a^2+b^2+c^2)\}$$

$$= \frac{2}{s}\{-3(a^3+b^3+c^3-3abc)-18abc+4s(a^2+b^2+c^2)\}$$

$$= -\frac{6}{s}(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$-\frac{36}{s}(4Rrs)+8(a^2+b^2+c^2)$$

$$= -12(a^2+b^2+c^2-ab-bc-ca)+8(a^2+b^2+c^2)-144Rr$$

$$= 12(ab+bc+ca)-4(a^2+b^2+c^2)-144Rr$$

$$= 12\{s^2+r(4R+r)\}-4\{2s^2-2r(4R+r)\}-144Rr$$

$$= 4s^2+20r(4R+r)-144Rr = 4s^2-64Rr+20r^2 \quad \textbf{(3)}$$

$(1)+(2)+(3)$ **using** $(i) \Rightarrow 2(\sum AN^2) = 4s^2-48Rr+12r^2$

$$\Rightarrow 2\left(\sum AN^2\right)+42Rr = 4s^2-6Rr+12r^2$$

Now, $4s^2+3R\cdot ON = 4s^2+3R\cdot\left(\frac{4\Delta OI^2}{abc}\right)$

$$= 4s^2+\frac{3R\cdot 4\Delta\cdot R(R-2r)}{4R\Delta} = 4s^2+3R^2-6Rr$$

given inequality $\Leftrightarrow 4s^2-6Rr+12R^2 \le 4s^2+3R^2-6Rr$

$\Leftrightarrow 3R^2 \ge 12r^2 \Leftrightarrow R^2 \ge 4r^2 \Leftrightarrow R \ge 2r \rightarrow$ **true (Proved)**

59. Prove that in any acute triangle ABC:

$$\cos^4 A + \cos^4 B + \cos^4 C + 3\cos A\cos B\cos C \geq 4\cos^2 A\cos^2 B\cos^2 C + \frac{1}{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC:

$$\cos^4 A + \cos^4 B + \cos^4 C + 3\cos A\cos B\cos C \geq 4(\cos A\cos B\cos C)^2 + \frac{1}{2}$$

$$\Rightarrow \cos^4 A + \cos^4 B + \cos^4 C - 4(\cos A\cos B\cos C)^2 + 4\cos A\cos B\cos C \geq$$

$$\geq \frac{1}{2} + \cos A\cos B\cos C \qquad \text{(A)}$$

Desde que: $A + B + C = \pi$, se cumple lo siguiente:

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1 \rightarrow$$

$$\rightarrow \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$$

$$\Rightarrow (\cos^2 A + \cos^2 B + \cos^2 C)^2 = (1 - 2\cos A\cos B\cos C)^2$$

$$\Rightarrow \cos^4 A + \cos^4 B + \cos^4 C + 2(\cos A\cos B)^2 + 2(\cos B\cos C)^2 + 2(\cos A\cos C)^2 =$$

$$= 1 - 4\cos A\cos B\cos C + 4(\cos A\cos B\cos C)^2$$

$$\Rightarrow \cos^4 A + \cos^4 B + \cos^4 C - 4(\cos A\cos B\cos C)^2 + 4\cos A\cos B\cos C =$$

$$= 1 - 2(\cos A\cos B)^2 - 2(\cos B\cos C)^2 - 2(\cos A\cos C)^2$$

Reemplazando en (A)...

$$\Rightarrow 1 - 2(\cos A\cos B)^2 - 2(\cos B\cos C)^2 - 2(\cos A\cos C)^2 \geq \frac{1}{2} + \cos A\cos B\cos C$$

$$\Rightarrow \frac{1}{2} \geq 2(\cos A\cos B)^2 + 2(\cos B\cos C)^2 + 2(\cos A\cos C)^2 + \cos A\cos B\cos C$$

$$\Rightarrow 1 - 2\cos A\cos B\cos C \geq 4(\cos B\cos C)^2 + 4(\cos A\cos C)^2 + 4(\cos A\cos B)^2$$

$$\Rightarrow \cos^2 A + \cos^2 B + \cos^2 C \geq 4(\cos B\cos C)^2 + 4(\cos A + \cos C)^2 + 4(\cos A\cos B)^2$$

Dado que es un triángulo acutángulo: $\cos A, \cos B, \cos C > 0$

En la desigualdad, dividamos $(\div)\cos A\cos B\cos C \Leftrightarrow$ **de tal manera que el sentido no se altere:**

$$\frac{\cos A}{\cos B\cos C}+\frac{\cos B}{\cos C\cos A}+\frac{\cos C}{\cos A\cos B}\geq\frac{4\cos B\cos C}{\cos A}+\frac{4\cos A\cos C}{\cos B}+\frac{4\cos A\cos B}{\cos C}\quad \text{(B)}$$

Ahora bien: $\frac{\cos A}{\cos B\cos C}=\frac{\frac{1}{\cos B\cos C}}{\frac{1}{\cos A}}=\frac{\frac{\sin(B+C)}{\cos B\cos C}}{\frac{\sin A}{\cos A}}=\frac{\tan B+\tan C}{\tan C}$

Por lo tanto en (B)...

$$\frac{\tan B+\tan C}{4\tan A}+\frac{\tan A+\tan C}{4\tan B}+\frac{\tan A+\tan B}{4\tan C}\geq$$

$$\geq\frac{\tan A}{\tan B+\tan C}+\frac{\tan B}{\tan A+\tan C}+\frac{\tan C}{\tan A+\tan B}$$

Aplicando: $MA\geq MG$

$$(\tan B+\tan C)^2\geq 4\tan B\tan C\Rightarrow\frac{\tan B+\tan C}{4\tan B\tan C}\geq\frac{1}{\tan B+\tan C}\rightarrow$$

$$\frac{1}{4\tan B}+\frac{1}{4\tan B}\geq\frac{1}{\tan B+\tan C}$$

$$\Rightarrow\frac{1}{4}\left(\frac{\tan A}{\tan C}+\frac{\tan A}{\tan B}\right)\geq\frac{\tan A}{\tan B+\tan C}\quad \text{(M)},$$

$$\frac{1}{4}\left(\frac{\tan B}{\tan C}+\frac{\tan B}{\tan A}\right)\geq\frac{\tan B}{\tan A+\tan C}\quad \text{(N)}$$

$$\frac{1}{4}\left(\frac{\tan C}{\tan A}+\frac{\tan C}{\tan B}\right)\geq\frac{\tan C}{\tan A+\tan B}\quad \text{(P)}$$

Por último, sumando: $(M)+(N)+(P)$

$$\frac{\tan B+\tan C}{4\tan A}+\frac{\tan A+\tan C}{4\tan B}+\frac{\tan A+\tan B}{4\tan C}\geq$$

$$\geq\frac{\tan A}{\tan B+\tan C}+\frac{\tan B}{\tan A+\tan C}+\frac{\tan C}{\tan A+\tan B}$$

60. Prove that in any $\Delta\ ABC$ the following relationship holds:

$$\frac{m_a m_b m_c}{r_a r_b r_c}+\frac{w_a w_b w_c}{h_a h_b h_c}\le\frac{R}{r}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty – Kolkata – India

$$\frac{w_a w_b w_c}{h_a h_b h_c}=\frac{\frac{2bc\cos\frac{A}{2}}{b+c}\cdot\frac{2ca\cos\frac{B}{2}}{c+a}\cdot\frac{2ab\cos\frac{C}{2}}{a+b}}{\left(\frac{bc}{2R}\right)\left(\frac{ca}{2R}\right)\left(\frac{ab}{2R}\right)}$$

$$=\frac{64R^3\sqrt{\frac{s(s-a)}{bc}\cdot\frac{s(s-b)}{ca}\cdot\frac{s(s-c)}{ab}}}{(a+b)(b+c)(c+a)}=\frac{64R^3 s\cdot\frac{\Delta}{abc}}{(a+b)(b+c)(c+a)}$$

$$=\frac{64R^3 s\frac{\Delta}{4R\Delta}}{(a+b)(b+c)(c+a)}=\frac{16R^2 s}{(a+b)(b+c)(c+a)}$$

$$(a+b)(b+c)(c+a)=(2s-c)(2s-a)(2s-b)=$$

$$=8s^3-4s^2(2s)+2s\left(\sum ab\right)-abc=$$

$$=2s(s^2+4Rr+r^2)-4Rrs=2s(s^2+2Rr+r^2)$$

$\frac{w_a w_b w_c}{h_a h_b h_c}=\frac{8R^2}{s^2+12Rr+r^2}$. Now, $r_a r_b r_c=\frac{\Delta^2}{r}=\frac{r^2s^2}{r}=rs^2$

given inequality $\Leftrightarrow \frac{m_a m_b m_c}{rs^2}+\frac{8R^2}{s^2+2Rr+r^2}\le\frac{R}{r}\Leftrightarrow\frac{m_a m_b m_c}{rs^2}\le\frac{R}{r}-\frac{8R^2}{s^2+2Rr+r^2}$

$$\Leftrightarrow m_a m_b m_c\le Rs^2-\frac{8R^2rs^2}{s^2+2Rr+r^2}=$$

$$=Rs^2\left(1-\frac{8Rr}{s^2+2Rr+r^2}\right)=\frac{Rs^2(s^2-6Rr+r^2)}{s^2+2Rr+r^2}$$

$$\Leftrightarrow m_a^2m_b^2m_c^2\le\frac{R^2s^4\left(s^2-6Rr+r\right)^2}{(s^2+2Rr+r^2)^2}\qquad\textbf{(1)}$$

$$m_a^2m_b^2m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$= \frac{1}{64}\left\{-4\left(\sum a^6\right)+6\left(\sum a^4b^2+\sum a^2b^4\right)+3a^2b^2c^2\right\}$$

Now, $\sum a^6 = (\sum a^2)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$

$$(a^2+b^2)(b^2+c^2)(c^2+a^2) = \left(\sum a^2 - c^2\right)\left(\sum a^2 - a^2\right)\left(\sum a^2 - b^2\right)$$

$$= \left(\sum a^2\right)^3 - \left(\sum a^2\right)^2\left(\sum a^2\right) + \left(\sum a^2\right)\left(\sum a^2b^2\right) - a^2b^2c^2$$

$$= \left(\sum a^2\right)\left(\sum a^2b^2\right) - a^2b^2c^2$$

$$\sum a^6 = \left(\sum a^2\right)^3 - 3\left(\sum a^2\right)\left(\sum a^2b^2\right) + 3a^2b^2c^2$$

Also, $\sum a^4b^2 + \sum a^2b^4 =$

$$= a^2b^2\left(\sum a^2 - c^2\right) + b^2c^2\left(\sum a^2 - a^2\right) + c^2a^2\left(\sum a^2 - b^2\right)$$

$$= \left(\sum a^2\right)\left(\sum a^2b^2\right) - 3a^2b^2c^2$$

$$m_a^2m_b^2m_c^2 = \frac{1}{64}\left\{-4\left(\sum a^2\right)^3 + 12\left(\sum a^2\right)\left(\sum a^2b^2\right) - 12a^2b^2c^2 + 6\left(\sum a^2\right)\left(\sum a^2b^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2\right\}$$

$$= \frac{1}{32}\left\{-2\left(\sum a^2\right)^3 + 9\left(\sum a^2\right)\left(\sum a^2b^2\right) - \frac{27}{2}a^2b^2c^2\right\}$$

$$\sum a^2b^2 = \left(\sum ab\right)^2 - 2abc(2s) = (s^2+4Rr+r^2)^2 - 16Rrs^2 =$$

$$= s^4 + 16R^2r^2 + r^4 - 8Rrs^2 + 8Rr^3 + 2s^2r^2$$

$$m_a^2m_b^2m_c^2 = \frac{1}{32}\left\{\begin{array}{c} -16(s^2-4Rr-r^2)^3 + 18(s^2-4Rr-r^2)\cdot \\ \cdot\left(s^4+16R^2r^2+r^4-8Rrs^2+8Rr^3+2s^2r^2 - \frac{27}{2}(4Rrs)^2\right)\end{array}\right\}$$

$$= \frac{1}{16}\left(s^6 - 12s^4Rr + 33s^4r^2 - 60s^2R^2r^2 - 120s^2Rr^3 - 33s^2r^4 - 64R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6\right) \quad \textbf{(2)}$$

Given inequality:

$$\Leftrightarrow \frac{1}{16}\left(s^6 - 12s^4Rr + 33s^4r^2 - 60s^2R^2r^2 - 120s^2Rr^3 - 33s^2r^4 - 64R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6\right) \le$$

$$\le \frac{R^2s^4\left(s^2-6Rr+r^2\right)^2}{(s^2+2Rr+r^2)^2}. \text{ (from (1), (2))}$$

$$\Leftrightarrow s^{10} + s^8(35r^2 - 8Rr - 16R^2) + s^6(34r^4 - 136R^2r^2 - 8Rr^3 + 192R^3r) -$$

$$-s^4\left(160R^3r^3 + 576R^4r^2 + 580R^2r^4 + 264Rr^5 + 34r^6\right) -$$

$$-s^2\left(496R^4r^4 + 1040R^3r^5 + 816R^2r^6 + 280Rr^7 + 35r^8\right) - 256R^5r^5 -$$

$$-448R^4r^6 - 304R^3r^7 - 100R^2r^8 - 16Rr^9 - r^{10} \le 0 \quad \text{(A)}$$

$$LHS \text{ of (A)} \le (4R^2 + 4Rr + 3r^2)^5 +$$

$$+(4R^2 + 4Rr + 3r^2)^4(35r^2 - 8Rr - 16R^2) +$$

$$+(4R^2 + 4Rr + 3r^2)^3(34r^4 - 136R^2r^2 - 8Rr^3 + 192R^3r) -$$

$$-(16Rr - 5r^2)^2\left(160R^3r^3 + 576R^4r^2 + 580R^2r^4 + 264Rr^5 + 34r^6\right) -$$

$$-(16Rr - 5r^2)\left(496R^4r^4 + 1040R^3r^5 + 816R^2r^6 + 280Rr^7 + 35r^8\right) -$$

$$-256R^5r^5 - 448R^4r^6 - 304R^3r^7 - 100R^2r^8 - 16Rr^9 - r^{10} =$$

$$= -3072R^{10} - 1024R^9r + 6144R^8r^2 + 27648R^7r^3 - 91776R^6r^4 +$$

$$+130176R^5r^5 - 52240R^4r^6 + 97984R^3r^7 + 69116R^2r^8 +$$

$$+19212Rr^9 + 3320r^{10}.$$ **it suffices to prove:**

$$-3072R^{10} - 1024R^9r + 6144R^8r^2 + 27648R^7r^3 -$$

$$-91776R^6r^4 + 130176R^5r^5 - 52240R^4r^6 + 97984R^3r^7 + 69116R^2r^8 +$$

$$+19212Rr^9 + 3320r^{10} \le 0$$

$$\Leftrightarrow 3072t^{10} + 1024t^9 - 6144t^8 - 27648t^7 + 91776t^6 - 130176t^5 +$$

$$+52240t^4 - 97984t^3 - 69116t^2 - 19212t - 3320 \ge 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow 768t^{10} + 256t^9 - 153t^8 - 6912t^7 + 22944t^6 - 32544t^5 + 13060t^4 -$$

$$-24496t^3 - 17279t^2 - 4803t - 830 \geq 0$$

$$\Leftrightarrow (t-2)(768t^9 + 1792t^8 + 2048t^7 - 2816t^6 + 17312t^5 + 2080t^4 + 17220t^3 + 9944t^2 + 2609t + 415) \geq 0 \text{ (B)}$$

Now, $768t^9 - 2816t^6 + 17220t^3 = t^3(768t^6 - 2816t^3 + 17220)$

$$= t^3 \underbrace{\underset{e}{\left(768z^2 - 2816z + 17220\right)}}_{\Delta=-44969984<0 \quad e>0} \ (z = t^3)$$

$768t^9 - 2816t^6 + 17220t^3 > 0 \Rightarrow$ **(B) is true** $\left(t = \frac{R}{r} \geq 2 \to \text{Euler}\right)$

$\Rightarrow$ **(A) is true (Hence proved)**

61. In ΔABC the following relationship holds:

$$\sum \frac{r_a}{r_b}\left(1 - \frac{r}{r_c}\right) \geq 2$$

Proposed by Nicolae Nica - Romania

Solution 1 by Adil Abdullayev – Baku – Azerbaidian

$$LHS = \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} - \frac{(4R+r)^2}{p^2} + 2 =$$

$$= \frac{r_a^2}{r_a r_b} + \frac{r_b^2}{r_c r_b} + \frac{r_c^2}{r_a r_c} - \frac{(4R+r)^2}{p^2} + 2 \overset{C-B-S}{\geq}$$

$$\overset{C-B-S}{\geq} \frac{(4R+r)^2}{p^2} - \frac{(4R+r)^2}{p^2} + 2 = 2.$$

Solution 2 by Daniel Sitaru – Romania

$$\sum \frac{r_a}{r_b}\left(1 - \frac{r}{r_c}\right) = \sum \frac{s-b}{s-a}\left(1 - \frac{s-c}{s}\right) = \frac{1}{s}\sum \frac{(s-b)c}{s-a} =$$

$$= \frac{1}{s}\sum \frac{y(x+y)}{x} \geq 2 \leftrightarrow \sum \frac{y(x+y)}{x} \geq 2(x+y+z) \leftrightarrow$$

$$(a = y+z, b = z+x, c = x+y)$$

$$\leftrightarrow \frac{yz}{x}+\frac{xy}{z}+\frac{xz}{y}\geq x+y+z \leftrightarrow x^2y^2+y^2z^2+z^2x^2\geq xyz(x+y+z)$$

$(2,2,0)\succcurlyeq(2,1,1)$**(Muirhead)**

62. Prove that in any triangle ABC:

$$\frac{r_a}{(s-b)(s-c)}+\frac{r_b}{(s-c)(s-a)}+\frac{r_c}{(s-a)(s-b)}\geq\frac{2}{r}\sqrt{\frac{4R+r}{2R}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en en triángulo ABC, la siguiente desigualdad:

$$\frac{r_a}{(s-b)(s-c)}+\frac{r_b}{(s-a)(s-c)}+\frac{r_c}{(s-b)(s-a)}\geq\frac{2}{r}\sqrt{\frac{4R+r}{2R}}$$

Recordar lo siguiente en un triángulo ABC: $r_a=\frac{S}{s-a},r_b=\frac{S}{s-b},r_c=\frac{S}{s-c},S=\frac{abc}{4R}$

$$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=4R\frac{(s-a)(s-b)(s-c)}{abc}$$

Reemplazando en la desigualdad...

$$\frac{S}{(s-a)(s-b)(s-c)}+\frac{S}{(s-a)(s-b)(s-c)}+\frac{S}{(s-a)(s-b)(s-c)}\geq$$

$$\geq\frac{2abc}{4R(s-a)(s-b)(s-c)}\sqrt{\frac{4R+r}{2R}}$$

$$\Rightarrow\frac{6SR}{abc}\geq\sqrt{\frac{4R+r}{2R}}\rightarrow\frac{3}{2}\geq\sqrt{\frac{4R+r}{2R}}\rightarrow\frac{9}{4}\geq 2+\frac{r}{2R}\rightarrow\frac{1}{4}\geq\frac{r}{2R}\Leftrightarrow$$

$\Leftrightarrow R\geq 2r$**.... (Desigualdad de Euler)**

63. In ΔABC the following relationship holds:

$$3\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\geq\sum\frac{b^2+bc+c^2}{bcm_a}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

Using Tereshin's inequality, $m_a\geq\frac{b^2+c^2}{4R}, m_b\geq\frac{c^2+a^2}{4R}, m_c\geq\frac{a^2+b^2}{4R}$

$$\sum\frac{b^2+bc+c^2}{bcm_a}\leq 4R\left\{\frac{b^2+c^2+bc}{bc(b^2+c^2)}+\frac{c^2+a^2+ca}{ca(c^2+a^2)}+\frac{a^2+b^2+ab}{ab(a^2+b^2)}\right\}=$$

$$=4R\left\{\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)+\left(\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2}\right)\right\}=$$

$$\leq 4R\left\{\frac{a+b+c}{abc}+\frac{1}{2}\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)\right\}=$$

$$(b^2+c^2\geq 2bc, c^2+a^2\geq 2ca, a^2+b^2\geq 2ab)$$

$$=4R\left(\frac{a+b+c}{abc}+\frac{a+b+c}{2abc}\right)=4R\cdot\frac{3}{2}\left(\frac{2s}{4Rrs}\right)=\frac{3}{r}\quad (1)$$

Now, $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=2R\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)=2R\frac{2s}{abc}=\frac{4Rs}{4Rrs}=\frac{1}{r}$

$$3\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{3}{r}\quad (2)$$

(1) and (2) $\Rightarrow$ the desired inequality

64. Prove that in any triangle ABC,

$$\frac{h_ah_b}{r_ar_b}+\frac{h_bh_c}{r_br_c}+\frac{h_ch_a}{r_cr_a}\geq 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{h_ah_b}{r_ar_b}+\frac{h_bh_c}{r_br_c}+\frac{h_ah_c}{r_ar_c}\geq 3$. Recordar lo siguiente:

$$h_a=\frac{2S}{a}, h_b=\frac{2S}{b}, h_c=\frac{2S}{c};\ r_a=\frac{S}{p-a}, r_b=\frac{S}{p-b}, r_c=\frac{S}{p-c}$$

La desigualdad es equivalente:

$$\frac{2(p-a)}{a}\cdot\frac{2(p-b)}{b}+\frac{2(p-b)}{b}\cdot\frac{2(p-c)}{c}+\frac{2(p-a)}{a}\cdot\frac{2(p-c)}{c}\geq 3$$

$$\frac{(b+c-a)(a+c-b)}{ab}+\frac{(a+c-b)(a+b-c)}{bc}+\frac{(b+c-a)(a+b-c)}{ac}\geq 3$$

Sean: $b=z+x, c=x+y, a=y+z$

Por la tanto: $b+c-a=2x\geq 0, a+c-b=2y\geq 0, a+b-c=2z\geq 0$

$$\frac{xy}{(z+y)(z+x)}+\frac{yz}{(x+z)(x+y)}+\frac{xz}{(y+z)(y+z)}\geq\frac{3}{4}$$

Por desigualdad de Cauchy: $\frac{(xy)^2}{x(y+z)y(z+x)}+\frac{(yz)^2}{y(z+x)z(x+y)}+\frac{(xz)^2}{x(y+z)z(x+y)}\geq$

$$\geq\frac{(xy+yz+xz)^2}{\sum xy(z+x)(z+y)}\geq\frac{\sum(xy)^2+2xyz(x+y+z)}{\sum z^2xy+\sum x^2yz+\sum y^2xz+\sum(xy)^2}$$

$$\Rightarrow\frac{\sum(xy)^2+2xyz(x+y+z)}{3xyz(x+y+z)+\sum(xy)^2}\geq\frac{3}{4}\Leftrightarrow\sum(xy)^2\geq xyz(x+y+z)$$

65. Prove that in any triangle ABC,

$$\frac{AI}{b+c}+\frac{BI}{c+a}+\frac{CI}{a+b}\leq\frac{\sqrt{3}}{2}$$

where I is the incenter.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: "I" incentro. Probar en un triángulo ABC: $\frac{AI}{b+c}+\frac{BI}{a+c}+\frac{CI}{a+b}\le\frac{\sqrt{3}}{2}$

Tener presente lo siguiente: $IA=\frac{bc}{p}\cos\frac{A}{2}=\frac{bc}{p}\sqrt{\frac{p(p-a)}{bc}}=\frac{1}{p}\sqrt{bc}\sqrt{p(p-a)}$,

$$IB=\frac{1}{p}\sqrt{ac}\sqrt{p(p-b)},\ IC=\frac{1}{p}\sqrt{ab}\sqrt{p(p-c)}$$

Tenemos en la desigualdad: $\frac{\sqrt{bc}\sqrt{p(p-a)}}{p(b+c)}+\frac{\sqrt{ac}\sqrt{p(p-b)}}{p(a+c)}+\frac{\sqrt{ab}\sqrt{p(p-c)}}{p(a+b)}\le\frac{\sqrt{3}}{2}$

Aplicando: $MA\ge MG$ $\frac{\sqrt{bc}\sqrt{p(p-a)}}{p(b+c)}+\frac{\sqrt{ac}\sqrt{p(p-b)}}{p(a+c)}+\frac{\sqrt{ab}\sqrt{p(p-c)}}{p(a+b)}\le$

$\le\frac{\sqrt{p-a}}{2\sqrt{p}}+\frac{\sqrt{p-b}}{2\sqrt{p}}+\frac{\sqrt{p-c}}{2\sqrt{p}}$. De la desigualdad de Cauchy:

$3(x^2+y^2+z^2)\ge(x+y+z)^2$. Sean: $x=\sqrt{p-a}, y=\sqrt{p-b}, z=\sqrt{p-c}$,

por la tanto se tendrá: $\Rightarrow 3(p-a+p-b+p-c)\ge$

$\ge\left(\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\right)^2\Leftrightarrow\sqrt{3p}\ge\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}$. Para

culminar: $\Rightarrow\frac{\sqrt{bc}\sqrt{p(p-a)}}{p(b+c)}+\frac{\sqrt{ac}\sqrt{p(p-b)}}{p(a+c)}+\frac{\sqrt{ab}\sqrt{p(p-c)}}{p(a+b)}\le$

$$\le\frac{\sqrt{p-a}}{2\sqrt{p}}+\frac{\sqrt{p-b}}{2\sqrt{p}}+\frac{\sqrt{p-c}}{2\sqrt{p}}\le\frac{\sqrt{3p}}{2\sqrt{p}}=\frac{\sqrt{3}}{2}$$

66. Let ABC be a triangle with $a=BC, b=CA, c=AB$ and the incenter I.

Prove that:

$$\frac{IB\cdot IC}{a}+\frac{IC\cdot IA}{b}+\frac{IA\cdot IB}{c}\le\frac{a+b+c}{3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sea “I” incentro. Probar en un triángulo ABC la siguiente desigualdad:

$$\frac{IB\cdot IC}{a}+\frac{IA\cdot IC}{b}+\frac{IA\cdot IB}{c}\leq\frac{a+b+c}{3}=\frac{2p}{3}$$

$$IA=\frac{bc}{p}\cos\frac{A}{2}=\frac{bc}{p}\sqrt{\frac{p(p-a)}{bc}}=\frac{1}{p}\sqrt{bc}\sqrt{p(p-a)}$$

$IB=\frac{1}{p}\sqrt{ac}\sqrt{p(p-b)}$;$IC=\frac{1}{p}\sqrt{ab}\sqrt{p(p-c)}$. **Reemplazando en la desigualdad:**

$$\frac{1}{p}\sqrt{bc}\sqrt{(p-b)(p-c)}+\frac{1}{p}\sqrt{ac}\sqrt{(p-a)(p-c)}+\frac{1}{p}\sqrt{ab}\sqrt{(p-a)(p-b)}\leq$$

$\leq\frac{a+b+c}{3}$. **Por desigualdad de Cauchy:**

$$\frac{1}{p}\left(\sqrt{bc}\sqrt{(p-b)(p-c)}+\sqrt{ac}\sqrt{(p-a)(p-c)}+\sqrt{ab}\sqrt{(p-a)(p-b)}\right)^2\leq$$

$$\leq\left(\sum bc\right)\left(\sum(p-b)(p-c)\right)$$

$$\rightarrow\sum bc=ab+bc+ac\leq\frac{(a+b+c)^2}{3}=\frac{4p^2}{3}$$

$$\sum(p-b)(p-c)=3p^2-2p(a+b+c)+ab+bc+ac$$

$$\rightarrow\sum(p-b)(p-c)=3p^2-2p(2p)+ab+bc+ac\leq-p^2+\frac{4p^2}{3}=\frac{p^2}{3}$$

Por la tanto: $(\sum bc)(\sum(p-b)(p-c))\leq\left(\frac{4p^2}{3}\right)\left(\frac{p^2}{3}\right)=\frac{4p^4}{9}$. **Por transitividad:**

$$\frac{1}{p}\left(\sqrt{bc}\sqrt{(p-b)(p-c)}+\sqrt{ac}\sqrt{(p-a)(p-c)}+\sqrt{ab}\sqrt{(p-a)(p-b)}\right)\leq$$

$$\leq\frac{1}{p}\sqrt{\frac{4p^4}{9}}=\frac{2p}{3}$$

67. In $\Delta\ ABC$ the following relationship holds:

$$\sin^2 A\cos^4 A+\sin^2 B\cos^4 B+\sin^2 C\cos^4 C<\frac{4}{9}$$

Proposed by Daniel Sitaru - Romania

Solution by Soumava Chakraborty – Kolkata – India

Let $\sin^2 A=x',\cos^2 A=x,x+x'=1;\sin^2 A\cos^4 A=(1-x)x^2=x^2-x^3$

$$\sin^2 A\cos^4 A-\frac{4}{27}=x^2-x^3-\frac{4}{27}=\frac{-(3x-2)^2(3x+1)}{27}$$

Similarly, letting $\cos^2 B=y$ and $\cos^2 C=z$, $\sin^2 B\cos^4 B-\frac{4}{27}=\frac{-(3y-2)^2(3y+1)}{27}$

and $\sin^2 C\cos^4 C-\frac{4}{27}=\frac{-(3z-2)^2(3z+1)}{27}$

$$\sum\sin^2 A\cos^4 A-\frac{4}{9}=\frac{-\{(3x-2)^2(3x+1)+(3y-2)^2(3y+1)+(3z-2)^2(3z+1)\}}{27}\quad(*)$$

Now, $0\le x<1,0\le y<1,0\le z<1$

(equality holding in case of only 1 variable, only angle $=90^\circ$)

$3x+1,3y+1,3z+1\ all>0$ (1)

If $3x-2=0,3y-2=0,3z-2=0,\cos^2 A=\cos^2 B=\cos^2 C=\frac{2}{3}$

Case 1: ΔABC is acute, $\cos A=\cos B=\cos C=\sqrt{\frac{2}{3}}$

But $\cos A=\cos B=\cos C\Rightarrow A=B=C=\frac{\pi}{3}\Rightarrow\cos A=\cos B=\cos C=\frac{1}{2}$

$\Rightarrow\cos A=\cos B=\cos C=\sqrt{\frac{2}{3}}$ is impossible

$\Rightarrow$ all of $(3x-2),(3y-2),(3z-2)$ can't be $=0$ at the same time (2)

(1), (2) $\Rightarrow-\sum(3x-2)^2(3x+1)<0\overset{*}{\Rightarrow}\sum\sin^2 A\cos^4 A-\frac{4}{9}<0$

$$\Rightarrow \sum \sin^2 A \cos^4 A < \frac{4}{9}$$

Case 2: ΔABC **is obtuse** $\Rightarrow$ **2 of** $\cos A, \cos B, \cos C = \sqrt{\frac{2}{3}}$

and 1 of them $= -\sqrt{\frac{2}{3}} \Rightarrow \sum \cos A = \sqrt{\frac{2}{3}} < 1$. **But** $\sum \cos A = 1 + \frac{r}{R} > 1 \Rightarrow$

$\Rightarrow$ **all of** $(3x-2), (3y-2), (3z-2)$ **can't be** $= 0$ **at the same time in case of obtuse** Δ **too (3)**

$$\text{(1), (3)} \Rightarrow -\sum (3x-2)^2(3x+1) < 0 \overset{*}{\Leftrightarrow} \sum \sin^2 A \cos^4 A - \frac{4}{9} < 0 \Rightarrow$$

$$\Rightarrow \sum \sin^2 A \cos^4 A < \frac{4}{9}$$

68. Let ABC be a triangle and let P be any point in its plane. Prove that:

$$(b+c)PA + (c+a)PB + (a+b)PC \geq 2\sqrt{abc(a+b+c)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

$$\frac{PA}{a}(ab+ac) + (cb+ca)\frac{PB}{b} + \frac{PC}{C}(ca+cb) \geq 2\sqrt{abc(a+b+c)}$$

1. Para todos los R^+: m, n, p, x, y, z se cumple la siguiente desigualdad:

$$(n+p)x + (p+m)y + (m+n)z \geq 2\sqrt{(mn+np+mp)(xy+yz+zx)}$$

(Demostrado anteriormente) ... (A)

Sea: $x = \frac{PA}{a}, y = \frac{PB}{b}, z = \frac{PC}{c}, n = ab, p = ac, m = cb$

2. De la desigualdad: "HAYASHI" se llego a demonstrar lo siguiente:

$$\frac{PA}{a}\cdot\frac{PB}{b}+\frac{PB}{b}\cdot\frac{PC}{c}+\frac{PA}{a}\cdot\frac{PC}{c}=xy+yz+zx\geq 1$$

Reemplazando en ... (A)

$$\frac{PA}{a}(ab+ac)+(cb+ca)\frac{PB}{b}+\frac{PC}{c}(ca+cb)\geq$$

$$\geq 2\sqrt{abc(a+b+c)\left(\frac{PA}{a}\cdot\frac{PB}{b}+\frac{PB}{b}\cdot\frac{PC}{c}+\frac{PA}{a}\cdot\frac{PC}{c}\right)}=2\sqrt{abc(a+b+c)}$$

69. Prove that in two any triangles ABC and $A'B'C'$ the following inequality holds:

$$(b+c)a'+(c+a)b'+(a+b)c'\geq 8\sqrt{3SS'}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

1. Para todos los $\mathbb{R}^+$: m, n, p, x, y, z, se cumple la siguiente desigualdad:

Demostración:

$$(n+p)x+(m+p)y+(m+n)z\geq 2\sqrt{(mn+np+pm)(xy+yz+zx)}$$

Aplicando: Cauchy – Schwarz:

$$P=(n+p)x+(m+p)y+(m+n)z=(m+n+p)(x+y+z)-(mx+ny+pz)$$

$$P=\sqrt{\left((m^2+n^2+p^2)+2(mn+np+mp)\right)\left((x^2+y^2+z^2)+2(xy+yz+zx)\right)}-(mx+ny+pz)$$

$$\geq\sqrt{(m^2+n^2+p^2)(x^2+y^2+z^2)}+2\sqrt{(mn+np+mp)(xy+yz+zx)}-(mx+ny+pz)\geq$$

$\geq 2\sqrt{(mn+np+pm)(xy+yz+zx)}$**. La igualdad se alcanza cuando:** $\frac{m}{x}=\frac{n}{y}=\frac{p}{z}$

2. Siendo $a, b, c\ \wedge a', b', c'$ los lados de los triángulos ABC y$A'B'C$, se cumple la siguiente desigualdad:

$$ab + bc + ac \geq 4\sqrt{3}S \ \wedge a'b' + b'c' + a'c' \geq 4\sqrt{3}S'$$

Demostración:

$$ab + bc + ac = (2\text{R}\sin \text{A})(2\text{R}\sin \text{B}) + (2\text{R}\sin \text{B})(2\text{R}\sin \text{C}) + (2\text{R}\sin \text{A})(2\text{R}\sin \text{C}) \geq$$

$$\geq 4\sqrt{3}(2\text{R}^2 \sin \text{A} \sin \text{B} \sin \text{C})$$

$\Rightarrow \csc \text{C} + \csc \text{A} + \csc \text{B} \geq 2\sqrt{3}$ **... (Válido en un triángulo** ABC**)**

Realizamos los siguientes cambios: $m = a, n = b, p = c, x = a', y = b', z = c'$

$$mn + np + mp = ab + bc + ac \geq 4\sqrt{3}S$$

$$xy + yz + zx = a'b' + b'c' + a'c' \geq 4\sqrt{3}S'$$

Aplicando las desigualdad (1) $\wedge$ **(2) se tiene lo siguiente:**

$$(b + c)a' + (c + a)b' + (a + b)c' \geq 2\sqrt{(mn + np + pm)(xy + yz + zx)} \geq$$

$$\geq 2\sqrt{(4\sqrt{3}S)(4\sqrt{3}S')} = 8\sqrt{3SS'}$$

70. Prove that for any triangle ABC and all positive real numbers x, y, z the following inequality holds:

$$\frac{x}{y + z} \cdot \frac{\sin A}{\sin B \sin C} + \frac{y}{z + x} \cdot \frac{\sin B}{\sin C \sin A} + \frac{z}{x + y} \cdot \frac{\sin C}{\sin A \sin B} \geq \sqrt{3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Prove that for any triangle ABC and all positive real numbers x, y, z the following inequality holds: $\frac{x}{y+z} \cdot \frac{\sin A}{\sin B \sin C} + \frac{y}{z+x} \cdot \frac{\sin B}{\sin C \sin A} + \frac{z}{x+y} \cdot \frac{\sin C}{\sin A \sin B} \geq \sqrt{3}$ **... (A)**

1) Para todos los $\mathbb{R}^+$: m, n, p, u, v, w, se cumple la siguiente desigualdad:

Demostración:

$$(n+p)u+(m+p)v+(m+n)w \geq 2\sqrt{(mn+np+pm)(uv+vw+wu)}$$

Aplicando: Cauchy – Schwarz:

$$P=(n+p)u+(m+p)v+(m+n)w=(m+n+p)(u+v+w)-(mu+nv+pw)$$

$$P=\sqrt{\left((m^2+n^2+p^2)+2(mn+np+mp)\right)\left((u^2+v^2+w^2)+2(uv+vw+wu)\right)}-$$

$$-(mu+nv+pw)$$

$$\geq \sqrt{(m^2+n^2+p^2)(u^2+v^2+w^2)}+2\sqrt{(mn+np+mp)(uv+vw+wu)}-$$

$$-(mu+nv+pw) \geq 2\sqrt{(mn+np+pm)(uv+vw+wu)}$$

La igualdad se alcanza cuando: $\frac{m}{u}=\frac{n}{v}=\frac{p}{w}$

2) Siendo: $x,y,z>0 \rightarrow$ **Se cumple la siguiente la desigualdad:**

$$\frac{xy}{(z+x)(z+y)}+\frac{yz}{(x+z)(x+y)}+\frac{zx}{(y+z)(y+x)} \geq \frac{3}{4}$$

La desigualdad puede ser equivalente en ... (A) como:

$$\frac{x}{y+z}\cdot\frac{\sin(B+C)}{\sin B\sin C}+\frac{y}{z+x}\cdot\frac{\sin(A+C)}{\sin C\sin B}+\frac{z}{x+y}\cdot\frac{\sin(A+B)}{\sin A\sin B} \geq \sqrt{3}$$

$$\Rightarrow \frac{x}{y+z}(\cot B+\cot C)+\frac{y}{z+x}(\cot A+\cot C)+\frac{z}{x+y}(\cot A+\cot B) \geq \sqrt{3}$$

Realizamos los siguientes cambios:

$$u=\cot A,\ v=\cot B,\ w=\cot C,\ m=\frac{x}{y+z},\ n=\frac{y}{z+x},\ p=\frac{z}{x+y}$$

$$mn+np+pm=\frac{xy}{(z+x)(z+y)}+\frac{yz}{(x+z)(x+y)}+\frac{zx}{(y+z)(y+x)} \geq \frac{3}{4}$$

Si: $A+B+C=\pi$

$$uv+vw+wu=\cot A\cot B+\cot B\cot C+\cot C\cot A=1$$

Aplicando las desigualdad (1) ∧ (2) se tiene lo siguiente:

$$\frac{x}{y+z}(\cot B+\cot C)+\frac{y}{z+x}(\cot A+\cot C)+\frac{z}{x+y}(\cot A+\cot B)\geq$$

$$\geq 2\sqrt{(uv+vw+wu)(mn+np+pm)}\geq 2\sqrt{1\left(\frac{3}{4}\right)}=\sqrt{3}$$

Solution 2 by Myagmarsuren Yadamsuren – Mongolia

$$I=\sum_{\substack{x;y;z\\A;B;C}}\frac{x}{y+z}\cdot\frac{\sin A}{\sin B\cdot\sin C}\geq 3$$

$$I\underbrace{\geq}_{Chebyshev}\frac{1}{3}\cdot\Bigg(\underbrace{\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}}_{I_1}\Bigg)\cdot\Bigg(\underbrace{\frac{\sin A}{\sin B\cdot\sin C}+\frac{\sin B}{\sin A\cdot\sin C}+\frac{\sin C}{\sin A\cdot\sin B}}_{I_2}\Bigg)$$

$$I_1\overbrace{\geq}^{Chebyshev}\frac{1}{3}\cdot(x+y+z)\cdot\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)\overbrace{\geq}^{Cauchy-Schwarz}$$

$$\geq\frac{1}{3}\cdot(x+y+z)\cdot\left(\frac{(1+1+1)^2}{2(x+y+z)}\right)=\frac{3}{2};\ I_1\geq\frac{3}{2}$$

$$I_2\overbrace{\geq}^{Chebyshev}\frac{1}{3}(\sin A+\sin B+\sin C)\cdot\left(\frac{1}{\sin A\sin B}+\frac{1}{\sin B\cdot\sin C}+\frac{1}{\sin C\cdot\sin A}\right)\overbrace{\geq}^{Cauchy}$$

$$\geq 3\cdot\frac{1}{3}\cdot 3\cdot\sqrt[3]{\sin A\cdot\sin B\cdot\sin C}\cdot\frac{1}{\sqrt[3]{(\sin A\cdot\sin B\cdot\sin C)^2}}=$$

$$=3\cdot\frac{1}{\sqrt[3]{\sin A\cdot\sin B\cdot\sin C}}\overbrace{\geq}^{Cauchy}\frac{9}{\sin A+\sin B+\sin C}$$

$$I_2\geq\frac{9}{\sin A+\sin B+\sin C}\Rightarrow y=\sin x;\ y''=-\sin x<0$$

$$I_2\geq\frac{9}{\sin A+\sin B+\sin C}\overbrace{\geq}^{Jensen}\frac{9}{3\cdot\sin\left(\frac{A+B+C}{3}\right)}=\frac{3}{\frac{\sqrt{3}}{2}}=2\sqrt{3}$$

$$I \geq \frac{1}{3} \cdot I_1 \cdot I_2 \geq \frac{1}{3} \cdot \frac{3}{2} \cdot 2\sqrt{3} = \sqrt{3};\ I \geq \sqrt{3}$$

Solution 3 by Marian Dincă – Romania

$$\sum \frac{x}{y+z} \cdot \frac{\sin A}{\sin B \sin C} = \frac{1}{\sin A \sin B \sin C} \cdot \sum \frac{x}{y+z} \cdot \sin^2 A =$$

$$\frac{1}{\sin A \sin B \sin C} \cdot \sum \frac{x}{y+z} \cdot \frac{a^2}{4R^2} = \frac{1}{2F} \cdot \sum \frac{x}{y+z} \cdot a^2 \geq$$

Use the well – known inequality: $ua^2 + vb^2 + wc^2 \geq \sqrt{uv + vw + wu} \cdot 4F$

where: a, b, c, F **the sides and the area triangle and:** $\sum \frac{x}{y+z} \cdot \frac{y}{z+x} \geq \frac{3}{4}$ **–well –known**

$$\geq \frac{1}{2F} \sqrt{\sum \frac{x}{y+z} \cdot \frac{y}{z+x}} \cdot 4F \geq \frac{1}{2F} \sqrt{\frac{3}{4}} \cdot 4F = \sqrt{3}$$

71. Prove that in any triangle ABC and for all positive real numbers x, y, z the following inequality holds

$$\left(\frac{x}{r_a} + \frac{y}{r_b} + \frac{z}{r_c}\right)\left(\frac{x}{r_b} + \frac{y}{r_c} + \frac{z}{r_a}\right)\left(\frac{x}{r_c} + \frac{y}{r_a} + \frac{z}{r_b}\right) \geq \frac{xyz}{r^3}.$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Marian Dincă – Romania

Use the identity:

$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$ **and weighted AM-GM for ponders:** $\frac{1}{r_a}; \frac{1}{r_b}; \frac{1}{r_c}$ **and variables** x, y, z

we obtain:

$$\frac{x}{r_a} + \frac{y}{r_b} + \frac{z}{r_c} \geq \left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) \cdot x^{\frac{1}{r_a}:\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)} \cdot y^{\frac{1}{r_b}:\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)} \cdot z^{\frac{1}{r_c}:\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)} \quad (1)$$

and similarly:

$$\frac{x}{r_b}+\frac{y}{r_c}+\frac{z}{r_a}\ge\left(\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_a}\right)\cdot x^{\frac{1}{r_b}:\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)}\cdot y^{\frac{1}{r_c}:\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)}\cdot z^{\frac{1}{r_a}:\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)}\quad (2)$$

$$\frac{x}{r_c}+\frac{y}{r_a}+\frac{z}{r_b}\ge\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\cdot x^{\frac{1}{r_c}:\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)}\cdot y^{\frac{1}{r_a}:\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)}\cdot z^{\frac{1}{r_b}:\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)}\quad (3)$$

and multiplying (1); (2); (3) proved the inequality

72. In ΔABC the following relationship holds:

$$\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\le\frac{3s}{\sqrt[3]{abc(\sin A\sin B\sin C)^2}}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

1. $\frac{R}{r}\ge\frac{b}{c}+\frac{c}{b}$**. Tener presente lo siguiente:**

$$\frac{r}{R}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{4(s-a)(s-b)(s-c)}{abc}$$ **. Reemplazando en la desigualdad:**

$$\frac{abc}{4(s-a)(s-b)(s-c)}\ge\frac{b^2+c^2}{bc}\Leftrightarrow ab^2c^2\ge(b^2+c^2)(s-a)(s-b)(s-c)$$

Realizamos los siguientes cambios de variables:

$$x=s-a\ge 0, y=s-b\ge 0, z=s-c\ge 0$$

$$y+z=2s-b-c=a, z+x=2s-a-c=b, x+y=2s-a-b=c$$

La desigualdad es equivalente:

$$(y+z)(z+x)^2(x+y)^2\ge 4((z+x)^2+(x+y)^2)(xyz)$$

$$(y+z)(z+x)^2(x+y)^2-4((z+x)^2+(x+y)^2)(xyz)\ge 0$$

$$y(z+x)^2(x+y)^2+z(z+x)^2(x+y)^2-4xyz(z+x)^2-4xyz(x+y)^2\ge 0$$

$$y(x+z)^2((x+z)^2-4xz)+z(x+z)^2((x+y)^2-4xy)\ge 0\Leftrightarrow$$

$$\Leftrightarrow y(x+y)^2(x-z)^2+z(x+z)^2(x-y)^2\ge 0$$

De forma análoga: $\frac{R}{r} \geq \frac{a}{c} + \frac{c}{a}, \frac{R}{r} \geq \frac{a}{b} + \frac{b}{a}$**. Probar en un triángulo** ABC**:**

$$\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{a}{c} \leq \frac{3s}{\sqrt[3]{abc(\sin A \sin B \sin C)^2}}, s - \text{semiperimeter}$$

Recordar lo siguiente:

$$\frac{a}{2R} = \sin A, \frac{b}{2R} = \sin B, \frac{c}{2R} = \sin C, s = \frac{a+b+c}{2}, abc = 4sRr$$

$$\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{a}{c} \leq \frac{3}{2} \cdot \frac{a+b+c}{\sqrt[3]{abc\left(\frac{abc}{8R^3}\right)^2}} \Leftrightarrow \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{a}{c} \leq \frac{3}{2}\left(\frac{a+b+c}{abc}\right) 4R^2$$

$$\Rightarrow \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{a}{c} \leq \frac{3}{2}\left(\frac{2s}{4sRr}\right) 4R^2 = \frac{3R}{r}$$ **. Desde que:**

$\frac{R}{r} \geq \frac{b}{c} + \frac{c}{b}$ **... (I);** $\frac{R}{r} \geq \frac{a}{c} + \frac{c}{a}$ **...(II);** $\frac{R}{r} \geq \frac{a}{b} + \frac{b}{a}$ **...(III); Sumando:** $(I) + (II) + (III)$

$$\frac{3R}{r} \geq \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{a}{c} \text{ ... (LQQD)}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$LHS = \frac{a+c}{b} + \frac{b+c}{a} + \frac{a+b}{c} = \frac{2s-b}{b} + \frac{2s-a}{a} + \frac{2s-c}{c} =$$

$$= 2s\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3 = \frac{2s(s^2 + 4Rr + r^2)}{4Rrs} - 3 =$$

$$= \frac{s^2 + 4Rr + r^2}{2Rr} - 3 = \frac{s^2 - 2Rr + r^2}{2Rr}$$

$$RHS = \frac{3s}{\sqrt[3]{abc \cdot \frac{a^2b^2c^2}{64R^6}}} = \frac{3s}{\left(\frac{abc}{4R^2}\right)} = \frac{3s}{\left(\frac{4Rrs}{4R^2}\right)} = \frac{3R}{r}$$

Given inequality $\Leftrightarrow \frac{s^2-2Rr+r^2}{2Rr} \leq \frac{3R}{r}$

$$\Leftrightarrow s^2 - 2Rr + r^2 \leq 6R^2 \Leftrightarrow s^2 \leq 6R^2 + 2Rr - r^2$$

Gerretsen $\Rightarrow s^2 \leq 4R^2 + 4Rr + 3r^2$

It suffices to prove: $4R^2+4Rr+3r^2\le 6R^2+2Rr-r^2$

$\Leftrightarrow 2R^2-2Rr-4r^2\ge 0\Leftrightarrow R^2-Rr-2r^2\Leftrightarrow (R-2r)(R+r)\ge 0\to$ true

$R\ge 2r$ (Euler) (Hence proved)

73. *In $\Delta\ ABC$ the following relationship holds:*

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^4\left(\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{c}\right)^6\ge 2^4\cdot 3^9\cdot S^2, S=[ABC]-\textit{area}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^4\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^6\ge 2^4\cdot 3^9\cdot S^2$$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\ge 3(abc)^{\frac{1}{6}}\text{ (AM}\ge\text{GM); }\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\ge 3(abc)^{\frac{1}{9}}\text{ (AM}\ge\text{GM)}$$

$$LHS\ge 3^4\cdot 3^6(abc)^{\frac{4}{6}}(abc)^{\frac{6}{9}}=3^{10}(abc)^{\frac{4}{3}}$$

It suffices to prove: $3^{10}(abc)^{\frac{4}{3}}\ge 2^4\cdot 3^9r^2s^2\ (S=rs)\Leftrightarrow 3^3(abc)^4\ge 2^{12}r^6s^6$

$$\Leftrightarrow 27(4Rrs)^4\ge 2^{12}r^6s^6\Leftrightarrow 27\cdot 2^8\cdot R^4\cdot r^4\cdot s^4\ge 2^{12}r^6s^6$$

$\Leftrightarrow 27R^4\ge 2^4r^2s^2\Leftrightarrow s^2\le\frac{27R^4}{16r^2}$. Gerretsen $\Rightarrow s^2\le 4R^2+4Rr+3r^2$

It suffices to prove: $4R^2+4Rr+3r^2\le\frac{27R^4}{16r^2}$

$$\Leftrightarrow 27R^4-64R^2r^2-64Rr^3-48r^4\ge 0$$

$$\Leftrightarrow 27t^4-64t^2-64t-48\ge 0\left(t=\frac{R}{r}\right)$$

$\Leftrightarrow (t-2)(27t^3+54t^2+44t+24)\ge 0\to$ true, $t=\frac{R}{r}\ge 2$ (Euler)

(Hence proved)

Solution 2 by Seyran Ibrahimov – Maasilli – Azerbaidian

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\ge 3\sqrt[6]{abc};\ \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\ge 3\sqrt[9]{abc}$$

$$3^4\cdot\sqrt[3]{a^2b^2c^2}\cdot 3^6\cdot\sqrt[3]{a^2b^2c^2}\ge 2^4 3^9 S^2;\ 3abc\sqrt[3]{abc}\ge 16S^2$$

$$12R\cdot S\sqrt[3]{abc}\ge 16S^2;\ 3R\sqrt[3]{abc}\ge 4S;\ 27R^3\cdot 4RS\ge 64S^3$$

$$27R^4\ge 16S^2;\ 3\sqrt{3}R^2\ge 4S;\ S\le\frac{3\sqrt{3}R^2}{4}$$

$$2R^2 sinAsinBsinC\le\frac{3\sqrt{3}R^2}{4};\ sinAsinBsinC\le\frac{3\sqrt{3}}{8}$$

74. Prove that for two any triangles ABC and $A'B'C'$

$$\frac{\sqrt{r_b}+\sqrt{r_c}}{s'-a'}+\frac{\sqrt{r_c}+\sqrt{r_a}}{s'-b'}+\frac{\sqrt{r_a}+\sqrt{r_b}}{s'-c'}\ge\frac{2}{r'}\sqrt{h_a+h_b+h_c}\ge\frac{6\sqrt{r}}{r'}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

1. Para todos los $\mathbb{R}^+$: m, n, p, u, v, w, se cumple la siguiente desigualdad:

Demostración:

$$(n+p)u+(m+p)v+(m+n)w\ge 2\sqrt{(mn+np+pm)(uv+vw+wu)}$$

Aplicando: Cauchy – Schwarz: $P = (n+p)u+(m+p)v+(m+n)w =$

$$=(m+n+p)(u+v+w)-(mu+nv+pw)$$

$$P=\sqrt{\left((m^2+n^2+p^2)+2(mn+np+mp)\right)\left((u^2+v^2+w^2)+2(uv+vw+wu)\right)}-$$

$$-(mu+nv+pw)$$

$$\ge\sqrt{(m^2+n^2+p^2)(u^2+v^2+w^2)}+2\sqrt{(mn+np+mp)(uv+vw+wu)}-$$

$$-(mu+nv+pw)\ge 2\sqrt{(mn+np+pm)(uv+vw+wu)}$$

La igualdad se alcanza cuando: $\frac{m}{u}=\frac{n}{v}=\frac{p}{w}$

2. Tener en cuenta las siguientes desigualdades:

$$\sqrt{r_b r_c} \ge h_a, \sqrt{r_c r_a} \ge h_b, \sqrt{r_a r_b} \ge h_c, h_a + h_b + h_c \ge 9r$$

Sean: $m = \sqrt{r_a}, n = \sqrt{r_b}, p = \sqrt{r_c}, u = \frac{1}{s'-a'}, v = \frac{1}{s'-b'}, w = \frac{1}{s'-c'}$

$$mn + np + pm = \sqrt{r_a r_b} + \sqrt{r_b r_c} + \sqrt{r_a r_c} \ge h_a + h_b + h_c$$

$$uv + vw + wu = \frac{1}{(s'-a')(a'-b')} + \frac{1}{(s'-b')(s'-c')} + \frac{1}{(s'-c')(s'-a')}$$

$$uv + vw + wu = \frac{(s'-a') + (s'-b') + (s'-c')}{(s'-a')(s'-b')(s'-c')} = \frac{(s')^2}{s'(s'-a)(s'-b')(s'-c')} =$$

$= \frac{(s')^2}{(S')^2} = \frac{1}{(r')^2}$. **Aplicando las desigualdades (1) ∧ (2), se tiene lo siguiente:**

$$\frac{\sqrt{r_b} + \sqrt{r_c}}{s'-a'} + \frac{\sqrt{r_c} + \sqrt{r_a}}{s'-b'} + \frac{\sqrt{r_a} + \sqrt{r_b}}{s'-c'} \ge 2\sqrt{(mn + np + pm)(uv + vw + wu)} \ge$$

$$\ge \frac{2}{r'}\sqrt{h_a + h_b + h_c} \ge \frac{2}{r'}\sqrt{9r} \ge \frac{6\sqrt{r}}{r'} \dots \text{(LQQD)}$$

75. Let ABC be a triangle and let P be any point in the plane. Prove that:

$$\left(\frac{1}{b} + \frac{1}{c}\right)\frac{PA}{a} + \left(\frac{1}{c} + \frac{1}{a}\right)\frac{PB}{b} + \left(\frac{1}{a} + \frac{1}{b}\right)\frac{PC}{c} \ge \sqrt{\frac{2}{Rr}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Let ABC be a triangle and let P be any point in the plane. Prove that:

$$\left(\frac{1}{b} + \frac{1}{c}\right)\frac{PA}{a} + \left(\frac{1}{c} + \frac{1}{a}\right)\frac{PB}{b} + \left(\frac{1}{a} + \frac{1}{b}\right)\frac{PC}{c} \ge \sqrt{\frac{2}{Rr}} \dots \text{(A)}$$

1. Para todos los $\mathbb{R}^+$: m, n, p, u, v, w, se cumple la siguiente desigualdad:

$$(n + p)u + (m + p)v + (m + n)w \ge 2\sqrt{(mn + np + pm)(uv + vw + wu)} \Leftrightarrow$$

$\Leftrightarrow$ **(Demostrado anteriormente)**

Sean: $n=\frac{1}{b}, p=\frac{1}{c}, m=\frac{1}{a}, u=\frac{PA}{a}, v=\frac{PB}{b}, w=\frac{PC}{c}$

2. Hayashi Inequality:

Siendo P un punto arbitrario en el plano de un triángulo ABC, se cumple lo siguiente: $anp+bmp+cmn\geq abc$

Donde: $m=PA, n=PB, p=PC, a=BC, b=AC, c=AB$

$$(BC)\cdot(PB)\cdot(PC)+(AC)\cdot(PA)\cdot(PC)+(AB)\cdot(PA)\cdot(PB)\geq(AB)\cdot(AC)\cdot(BC)$$

$\Rightarrow\frac{PB\cdot PC}{AB\cdot AC}+\frac{PA\cdot PC}{BA\cdot BC}+\frac{PA\cdot PB}{CA\cdot CB}\geq 1$ **... (A continuación lo demostraremos)**

La manera clásica para demostrar esta desigualdad es trabajando en el plano complejo.

Siendo P el origen, la asignación de los puntos A, B, C, los afijos x, y, z. Entonces:

$$\frac{PB\cdot PC}{AB\cdot AC}=\frac{|y-0|\cdot|z-0|}{|x-y|\cdot|x-z|}=\frac{|y|\cdot|z|}{|x-y|\cdot|x-z|}=\left|\frac{yz}{(x-y)(x-z)}\right|$$

$$\frac{PA\cdot PC}{BA\cdot BC}=\left|\frac{xz}{(y-x)(y-z)}\right|\wedge\frac{PA\cdot PB}{CA\cdot CB}=\left|\frac{xy}{(z-x)(z-y)}\right|$$

$\Rightarrow\frac{PB\cdot PC}{AB\cdot AC}+\frac{PA\cdot PC}{BA\cdot BC}+\frac{PA\cdot PB}{CA\cdot CB}=\left|\frac{yz}{(x-y)(x-z)}\right|+\left|\frac{xz}{(y-x)(y-z)}\right|+\left|\frac{xy}{(z-x)(z-y)}\right|$ **... (B)**

Por desigualdad triangular:

$\left|\frac{yz}{(x-y)(x-z)}\right|+\left|\frac{xz}{(y-x)(y-z)}\right|+\left|\frac{xy}{(z-x)(z-y)}\right|\geq\left|\frac{yz}{(x-y)(x-z)}+\frac{xz}{(y-x)(y-z)}+\frac{xy}{(z-x)(z-y)}\right|$ **... (B)**

Tener presente lo siguiente:

$$\frac{yz}{(x-y)(x-z)}+\frac{xz}{(y-x)(y-z)}+\frac{xy}{(z-x)(z-y)}=\frac{yz}{(x-y)(x-z)}+\frac{xz}{(x-y)(z-y)}-\frac{xy}{(x-z)(z-y)}$$

$$\frac{yz}{(x-y)(x-z)}+\frac{xz}{(x-y)(z-y)}+\frac{xy}{(x-z)(z-y)}=\frac{yz(z-y)+zx(x-z)-xy(x-y)}{(x-y)(x-z)(z-y)}=1$$

$\Rightarrow(x-y)(x-z)(z-y)=yz(z-y)+zx(x-z)-xy(x-y)$ **... (Lo cual es cierto)**

Por la tanto tenemos en ... (B)

$$\left|\frac{yz}{(x-y)(x-z)}\right|+\left|\frac{xz}{(y-x)(y-z)}\right|+\left|\frac{xy}{(z-x)(z-y)}\right|\geq\left|\frac{yz}{(x-y)(x-z)}+\frac{xz}{(y-x)(y-z)}+\frac{xy}{(z-x)(z-y)}\right|=1$$

Finalmente tenemos en ... (A)

$$\left(\frac{1}{b}+\frac{1}{c}\right)\frac{PA}{a}+\left(\frac{1}{a}+\frac{1}{c}\right)\frac{PB}{b}+\left(\frac{1}{a}+\frac{1}{b}\right)\frac{PC}{c}\geq$$

$$\geq 2\sqrt{\left(\frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab}\right)\left(\frac{PB\cdot PC}{AB\cdot AC}+\frac{PA\cdot PC}{BA\cdot BC}+\frac{PA\cdot PB}{CA\cdot CB}\right)}\geq$$

$$\geq 2\sqrt{\left(\frac{1}{2Rr}\right)1}=\sqrt{\frac{2}{Rr}}\text{ ... (LQQD)}$$

76. In $\Delta\ ABC, I-$ incentre

$$\frac{R}{r}\sum\left(AI\cdot\cos\frac{B-C}{2}\right)^2\geq\frac{6abc}{a+b+c}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Apostolopoulos – Messolonghi – Greece

We have $AI=4R\sin\frac{B}{2}\sin\frac{C}{2}, BI=4R\sin\frac{C}{2}\sin\frac{A}{2}, CI=4R\sin\frac{A}{2}\sin\frac{B}{2}$.

So $AI\cdot BI\cdot CI=(4R)^3\left(\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\right)^2$

It is well – known that $\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=\frac{r}{4R}$, $AI\cdot BI\cdot CI=4Rr^2$.

Also, we have by the Law of Sines that $\frac{b-c}{a}=\frac{\sin B-\sin C}{\sin A}=\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}$, namely

$$\left(\frac{b-c}{a}\right)^2=\frac{\sin^2\frac{B-C}{2}}{\cos^2\frac{A}{2}}\geq\sin^2\frac{B-C}{2}\text{ so }\left(\frac{b-c}{a}\right)^2\geq 1-\cos^2\frac{B-C}{2}\Leftrightarrow$$

$$\Leftrightarrow\cos^2\frac{B-C}{2}\geq 1-\left(\frac{b-c}{a}\right)^2=\frac{4(s-b)(4-c)}{a^2}\text{ but}$$

$$\sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}\text{ so }\cos^2\frac{B-C}{2}\geq\frac{4bc}{a^2}\sin^2\frac{A}{2}$$

Similarly $\cos^2\frac{C-A}{2}\ge\frac{4ca}{b^2}\sin^2\frac{B}{2}$, and $\cos^2\frac{A-B}{2}\ge\frac{4ab}{c^2}\sin^2\frac{C}{2}$

So $\cos^2\frac{A-B}{2}\cdot\cos^2\frac{B-C}{2}\cdot\cos^2\frac{C-A}{2}\ge 64\left(\frac{r}{4R}\right)^2=4\frac{r^2}{R^2}$.

Now from the Cauchy – Schwarz Inequality, we get

$$\frac{R}{r}\sum\left(AI\cdot\cos\frac{B-C}{2}\right)^2\ge\frac{R}{r}\cdot 3\sqrt[3]{(AI\cdot BI\cdot CI)^2\cdot\cos^2\frac{A-B}{2}\cdot\cos^2\frac{B-C}{2}\cdot\cos^2\frac{C-A}{2}}\ge$$

$$\ge\frac{R}{r}\cdot 3\sqrt[3]{(4Rr^2)^2\cdot 4\frac{r^2}{R^2}}=\frac{3R}{r}\cdot\sqrt[3]{4^3\cdot r^6}=\frac{3R}{r}\cdot r^2=12Rr=$$

$$=6\cdot(2Rr)=\frac{6abc}{a+b+c}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo: Probar que: I – incentre

$$\frac{R}{r}\sum\left(AI\times\cos\left(\frac{B-C}{2}\right)\right)^2\ge\frac{6abc}{a+b+c}\ \ldots\ (A)$$

Recordar lo siguiente en un triángulo ABC:

1) $\frac{R}{r}=\frac{1}{4\operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}}=\frac{abc}{4(s-a)(s-b)(s-c)}$

Ahora bien:

2) $\cos\left(\frac{B-C}{2}\right)=\cos\frac{B}{2}\cos\frac{C}{2}+\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}$

$$\Rightarrow\cos\left(\frac{B-C}{2}\right)=\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}+\sqrt{\frac{(s-c)(s-a)}{ac}}\sqrt{\frac{(s-b)(s-a)}{ab}}$$

$$\Rightarrow\cos\left(\frac{B-C}{2}\right)=\left(\frac{s}{a}+\frac{s-a}{a}\right)\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\Rightarrow \cos\left(\frac{B-C}{2}\right)=\left(\frac{b+c}{a}\right)\sqrt{\frac{(s-b)(s-c)}{bc}}$$

De forma análoga para los demás:

$$\cos\left(\frac{A-B}{2}\right)=\left(\frac{a+b}{c}\right)\sqrt{\frac{(s-a)(s-b)}{ab}},\cos\left(\frac{C-A}{2}\right)=\left(\frac{a+c}{b}\right)\sqrt{\frac{(s-a)(s-c)}{ac}}$$

3) Los incentros de un triángulo ABC, se puede expresar de la siguiente manera:

$$IA=\frac{bc}{s}\cos\frac{A}{2}=\frac{bc}{s}\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{bc(s-a)}{s}};\ IB=\sqrt{\frac{ac(s-b)}{s}},IC=\sqrt{\frac{ab(s-c)}{s}}$$

Reemplazando en la desigualdad ... (A):

$$\frac{abc}{4(s-a)(s-b)(s-c)}\left(\sum\frac{bc(s-a)}{s}\left(\frac{b+c}{a}\right)^2\frac{(s-b)(s-c)}{bc}\right)\geq\frac{6abc}{2s}$$

La cual es suficiente probar:

$$\frac{1}{(p-a)(p-b)(p-c)}\left(\sum(s-a)\left(\frac{b+c}{a}\right)^2(s-b)(s-c)\right)\geq 12$$

$$\Rightarrow\left(\frac{b+c}{a}\right)^2\geq 3\sqrt[3]{\left(\frac{(b+c)(a+c)(a+b)}{abc}\right)^2}\geq 3\sqrt[3]{8^2}=12\ \text{... (LQQD)}$$

Solution 3 by Soumava Chakraborty – Kolkata – India

$$AI=\frac{r}{\sin\frac{A}{2}};\ AI^2\cos^2\frac{B-C}{2}=\frac{r^2\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}}$$

given inequality $\Leftrightarrow \frac{R}{r}\cdot r^2\sum\frac{\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}}\geq\frac{24Rrs}{2s}\Leftrightarrow\sum\frac{\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}}\geq 12$

$$\frac{\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}}=\frac{1}{\sin^2\frac{A}{2}\sec^2\left(\frac{B-C}{2}\right)}=\frac{1}{\sin^2\frac{A}{2}\left(1+\tan^2\frac{B-C}{2}\right)}=$$

using Napier's analogy: $\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$

$$=\frac{1}{\sin^2\frac{A}{2}\left\{1+\frac{(b-c)^2}{(b+c)^2}\cot^2\frac{A}{2}\right\}}=\frac{1}{\sin^2\frac{A}{2}+\frac{(b-c)^2}{(b+c)^2}\cos^2\frac{A}{2}}=$$

$$=\frac{1}{1-\cos^2\frac{A}{2}\left\{1-\frac{(b-c)^2}{(b+c)^2}\right\}}=\frac{1}{1-\frac{s(s-a)}{bc}\cdot\frac{4bc}{(b+c)^2}}=$$

$$=\frac{(b+c)^2}{(2s-a)^2-4s(s-a)}=\frac{(b+c)^2}{a^2}$$

Similarly, $\frac{\cos^2\left(\frac{C-A}{2}\right)}{\sin^2\frac{B}{2}}=\frac{(c+a)^2}{b^2}$ and $\frac{\cos^2\left(\frac{A-B}{2}\right)}{\sin^2\frac{C}{2}}=\frac{(a+b)^2}{c^2}$

$$\sum\frac{\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}}=\frac{(b+c)^2}{a^2}+\frac{(c+a)^2}{b^2}+\frac{(a+b)^2}{c^2}=$$

$$=\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)+\left(\frac{c^2}{b^2}+\frac{b^2}{c^2}\right)+2\left(\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}\right)\ge$$

$$\underbrace{\ge}_{AM-GM}2+2+2+2\cdot3\sqrt[3]{\frac{a^2b^2c^2}{a^2b^2c^2}}=12$$

(Proved)

Solution 4 by Soumitra Mandal- Chandar Nagore – India

$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}\Rightarrow\csc\frac{A}{2}\cos\frac{B-C}{2}=\frac{b+c}{a}$$

so,

$$\sum_{cyc}\left(AI\cos\frac{B-C}{2}\right)^2 \ge \frac{1}{3}\left(\sum_{cyc} AI\cos\frac{B-C}{2}\right)^2 = \frac{1}{3}\left(\sum_{cyc} r\csc\frac{A}{2}\cos\frac{B-C}{2}\right)^2$$

since, $AI = r\csc\frac{A}{2}, BI = r\csc\frac{B}{2}$ and $CI = r\csc\frac{C}{2}$

$$= \frac{r^2}{3}\left(\sum_{cyc}\frac{b+c}{a}\right)^2 \ge \frac{r^3}{3}\left(3\sqrt[3]{\prod_{cyc}\left(\frac{b+c}{a}\right)}\right)^2 \ge 12r^2$$

so, $\frac{R}{r}\sum_{cyc}\left(AI\cos\frac{B-C}{2}\right)^2 \ge 12Rr = 12\cdot\frac{abc}{4S}\cdot\frac{S}{s} = \frac{3abc}{s} = \frac{6abc}{a+b+c}$

77. In acute-angled $\Delta\, ABC$:

$$\left(\sum\frac{1}{\sqrt{s-a}}\right)^2 \le (m_a + w_b + h_c)\cdot\frac{4R+r}{3sr^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

$m_a \ge w_a \ge h_a$ is any triangle

$$m_a + w_b + h_c \ge h_a + h_b + h_c = \frac{\sum ab}{2R} = \frac{s^2+4Rr+r^2}{2R} \quad (1)$$

Now, $\left(\sum\frac{1}{\sqrt{s-a}}\right)^2 \le 3\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)$ (CBS)

$$= \frac{3s\sum(s-a)(s-b)}{s(s-a)(s-b)(s-c)} = \frac{3\sum(s-a)(s-b)}{r^2s} =$$

$$= \frac{3\{3s^2 - s\cdot 2(a+b+c) + \sum ab\}}{r^2S} = \frac{3(3s^2-4s^2+s^2+4Rr+r^2)}{r^2s} =$$

$= \frac{3r(4R+r)}{r^2S}$ (2). It suffices to show: $\frac{3r(4R+r)}{r^2s} \le \frac{s^2+4Rr+r^2}{2R}\cdot\frac{(4R+r)}{2sr^2}$ from (1), (2)

$$\Leftrightarrow s^2 + 4Rr + r^2 \ge 18Rr \Leftrightarrow s^2 \ge 14Rr - r^2$$

Gerretsen $\Rightarrow s^2 \geq 16Rr - 5r^2$. It suffices to prove: $16Rr - 5r^2 \geq 14Rr - r^2$

$\Leftrightarrow 2Rr \geq 4r^2 \Leftrightarrow Rr \geq 2r^2 \Leftrightarrow R \geq 2r$ (true → Euler) (proved)

78. **Prove that for two any triangles ABC and $A'B'C'$ we have:**

$$\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\tan\frac{A'}{2}+\left(\frac{1}{h_c}+\frac{1}{h_a}\right)\tan\frac{B'}{2}+\left(\frac{1}{h_a}+\frac{1}{h_b}\right)\tan\frac{C'}{2}\geq\frac{12}{a+b+c}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

1) Para todos los $\mathbb{R}^+$: m, n, p, u, v, w, se cumple la siguiente desigualdad:

$$(n+p)u+(m+p)v+(m+n)w\geq 2\sqrt{(mn+np+pm)(uv+vw+wu)}\ \ldots\ (A)$$

Realizamos los siguientes cambios de variables:

$$n=\frac{1}{h_b}, p=\frac{1}{h_c}, m=\frac{1}{h_a}, u=\tan\frac{A'}{2}, v=\tan\frac{B'}{2}, w=\tan\frac{C'}{2}$$

2) Recordar los siguiente en los triángulos ABC y $A'B'C'$:

M) Si: $\frac{A'}{2}+\frac{B'}{2}+\frac{C'}{2}=\frac{\pi'}{2}\rightarrow uv+vw+wu=\tan\frac{A'}{2}\tan\frac{B'}{2}+\tan\frac{B'}{2}\tan\frac{C'}{2}+\tan\frac{C'}{2}\tan\frac{A'}{2}=1$

N) $mn+np+pm=\frac{1}{h_ah_b}+\frac{1}{h_bh_c}+\frac{1}{h_ch_a}=\frac{2R}{bc}\cdot\frac{2R}{ac}+\frac{2R}{ac}\cdot\frac{2R}{ab}+\frac{2R}{ab}\cdot\frac{2R}{bc}=\frac{4R^2}{abc}\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)$

Por la desigualdad de Cauchy y Euler $(R\geq 2r)$:

$$\Rightarrow mn+np+pm=\frac{4R^2}{abc}\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)\geq\frac{4R^2}{4sRr}\left(\frac{9}{a+b+c}\right)\geq$$

$$\geq\frac{4}{(a+b+c)}\cdot\frac{9}{(a+b+c)}=\frac{36}{(a+b+c)^2}.$$ Finalmente en ... (A)

$$\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\tan\frac{A'}{2}+\left(\frac{1}{h_c}+\frac{1}{h_a}\right)\tan\frac{B'}{2}+\left(\frac{1}{h_a}+\frac{1}{h_b}\right)\tan\frac{C'}{2}\geq$$

$$\geq 2\sqrt{(1)\frac{36}{(a+b+c)^2}}=\frac{12}{a+b+c}$$ ... (LQQD)

Solution 2 by Soumitra Mandal- Chandar Nagore – India

We have,

$$(m+n)x+(n+p)y+(p+m)z\geq 2\sqrt{(mn+np+pm)(xy+yz+zx)}$$

equality at $\frac{m}{x}=\frac{n}{y}=\frac{p}{z}$**. Again we have** $\sum_{cyc}\tan\frac{A'}{2}\tan\frac{B'}{2}=1$

putting $x=\tan\frac{A'}{2},y=\tan\frac{B'}{2},z=\tan\frac{C'}{2},m=\frac{1}{h_a},n=\frac{1}{h_b}$ **and** $p=\frac{1}{h_c}$**.**

$$\sum_{cyc}\left(\frac{1}{h_c}+\frac{1}{h_b}\right)\tan\frac{A'}{2}\geq 2\sqrt{\left(\sum_{cyc}\frac{1}{h_ah_b}\right)\left(\sum_{cyc}\tan\frac{A'}{2}\tan\frac{B'}{2}\right)}=2\sqrt{\frac{ab+bc+ca}{4\Delta^2}}$$

$$=\sqrt{\frac{ab+bc+ca}{s(s-a)(s-b)(s-c)}}\geq\sqrt{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\frac{8}{s}}\geq\sqrt{\frac{144}{(a+b+c)^2}}=\frac{12}{a+b+c}$$

Solution 3 by Soumava Chakraborty – Kolkata – India

$$\forall x,y,z,m,n,p\in\mathbb{R}^+,$$

$$(n+p)x+(p+m)y+(m+n)z\geq 2\sqrt{(mn+np+mp)(xy+yz+zx)}$$

Let $m=\frac{1}{h_a},n=\frac{1}{h_b},p=\frac{1}{h_c};\ x=\tan\frac{A'}{2},y=\tan\frac{B'}{2},z=\tan\frac{C'}{2}$

$$LHS\geq 2\sqrt{\frac{1}{h_ah_b}+\frac{1}{h_bh_c}+\frac{1}{h_ch_a}}\sqrt{\sum\tan\frac{A'}{2}\tan\frac{B'}{2}}=2\sqrt{\frac{\sum h_a}{h_ah_bh_c}}\cdot 1=2\sqrt{\frac{\frac{\sum ab}{2R}}{\frac{a^2b^2c^2}{8R^3}}}=$$

$$\left(h_a=\frac{bc}{2R},h_b=\frac{ca}{2R},h_c=\frac{ab}{2R}\right)$$

$$=2\sqrt{\frac{s^2+4Rr+r^2}{a^2b^2c^2}\cdot 4R^2}=\frac{4R}{4Rrs}\sqrt{s^2+4Rr+r^2}=\frac{\sqrt{s^2+4Rr+r^2}}{rs}$$

Given inequality $\Leftrightarrow\frac{\sqrt{s^2+4Rr+r^2}}{rs}\geq\frac{6}{s}$

$\Leftrightarrow s^2+4Rr+r^2\ge 36r^2 \Leftrightarrow s^2\ge 35r^2-4Rr$. **Gerretsen** $\Rightarrow s^2\ge 16Rr-5r^2$

It suffices to prove: $16Rr-5r^2\ge 35r^2-4Rr \Leftrightarrow 20Rr\ge 40r^2 \Leftrightarrow R\ge 2r$

(true → Euler)

79. In $\Delta\ ABC$:

$$\frac{8r^3}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}\le\frac{1}{27}$$

Proposed by Daniel Sitaru, Romania

Solution 1 by George Apostolopoulos – Messolonghi – Greece

We have, by (AM – GM)

$$\frac{8r^3}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}\le\frac{8r^3}{2\sqrt{m_am_b}\cdot 2\sqrt{m_bm_c}\cdot 2\sqrt{m_cm_a}}=\frac{r^3}{m_a\cdot m_b\cdot m_c}\quad (*)$$

Now we use the notation Δ for the area of the triangle and h_a for the altitude from vertex A, etc. Since h_a is the shortest segment from A to $\overline{BC}$, we have $h_a\le m_a$, and similarly for the medians and altitudes from B and C. Therefore,

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\le\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{a+b+c}{2\Delta}=\frac{1}{r}\quad (1)$$

From the geometric mean – harmonic mean inequality, we have

$(m_am_bm_c)^{\frac{1}{3}}\ge\frac{3}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}\ge 3r$**, where we used (1) in the last step.**

So $m_am_bm_c\ge(3r)^3=27r^3$. So (*) gives

$$\frac{8r^3}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}\le\frac{r^3}{m_am_bm_c}\le\frac{r^3}{27r^3}=\frac{1}{27}$$

Equality holds when the triangle ABC is equilateral.

Solution 2 by Rovsen Pirguliev-Sumgait-Azerbaidian

We prove that: $(m_a + m_b)(m_b + m_c)(m_c + m_a) \geq 27 \cdot 8r^3$

$$\left.\begin{array}{r} m_a + m_b \geq 2\sqrt{m_a m_b} \\ m_b + m_c \geq 2\sqrt{m_b m_c} \\ m_c + m_a \geq 2\sqrt{m_a m_c} \end{array}\right\} \otimes$$

$$(m_a + m_b)(m_b + m_c)(m_c + m_a) \geq 8m_a m_b m_c$$

Now we prove that $m_a m_b m_c \geq p^2 r$; $\prod\left(\frac{m_a}{\cos\frac{A}{2}}\right)^2 \geq \prod bc = a^2b^2c^2$

or $m_a m_b m_c \geq p^2 r$ it is known that $p \geq 3\sqrt{3}r$; $m_a m_b m_c \geq p^2 r \overset{p\geq 3\sqrt{3}r}{\overbrace{\geq}} 27r^3$

Solution 3 by Soumava Chakraborty – Kolkata – India

$m_a \geq h_a, m_b \geq h_b, m_c \geq h_c$ and $h_a = \frac{bc}{2R}, h_b = \frac{ca}{2R}, h_c = \frac{ab}{2R}$

$$\prod(m_a + m_b) \geq \frac{abc(a+b)(b+c)(c+a)}{8R^3} \overset{A-G}{\overbrace{\geq}} \frac{abc(2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca})}{8R^3}$$

$$= \frac{(abc)^2}{R^3} = \frac{(4Rrs)^2}{R^3} = \frac{16r^2s^2}{R} \Rightarrow \frac{1}{\prod(m_a + m_b)} \leq \frac{R}{16r^2s^2}$$

$$\Rightarrow \frac{8r^3}{\prod(m_a+m_b)} \leq \frac{Rr}{2s^2} \quad \text{it suffices to prove: } \frac{Rr}{2s^2} \leq \frac{1}{27}$$

$$\frac{Rr}{2s^2} \leq \frac{1}{27} \Leftrightarrow 2s^2 \geq 27Rr \quad \text{Gerretsen} \Rightarrow 2s^2 \geq 32Rr - 10r^2$$

it suffices to prove: $32Rr - 10r^2 \geq 27Rr$

$\Leftrightarrow 5Rr \geq 10r^2 \Leftrightarrow R \geq 2r$ (true → Euler)

Solution 4 by Adil Abdullayev – Baku – Azerbaidian

$$LHS \leq RHS \Leftrightarrow (m_a + m_b)(m_b + m_c)(m_c + m_a) \geq 27 \cdot 8r^3$$

Lemma 1. $(m_a + m_b)(m_b + m_c)(m_c + m_a) \geq 8r_a r_b r_c$

Lemma 2. $p^2 \geq 27r^2$.

$$(m_a + m_b)(m_b + m_c)(m_c + m_a) \overset{\text{Lemma 1}}{\overbrace{\geq}} 8rp^2 \overset{\text{Lemma 2}}{\overbrace{\geq}} 27 \cdot 8r^3.$$

80. Let a, b, c be three side – lengths of a triangle with the area S and let x, y, z be positive real numbers. Prove that

$$\frac{x(b+c)}{y+z} + \frac{y(c+a)}{z+c} + \frac{z(a+b)}{x+y} \geq \sqrt{6\sqrt{3}S + \frac{1}{2}(a^2 + b^2 + c^2)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Let a, b, c be three side – lengths of a triangle with the area S and let x, y, z be positive real numbers. Prove that:

$$\frac{x(b+c)}{y+z} + \frac{y(c+a)}{z+c} + \frac{z(a+b)}{x+y} \geq \sqrt{6\sqrt{3}S + \frac{1}{2}(a^2 + b^2 + c^2)} \text{ ... (A)}$$

1. Para todos los $\mathbb{R}^+$: m, n, p, u, v, w, se cumple la siguiente desigualdad:

$$(n+p)u + (m+p)v + (m+n)w \geq 2\sqrt{(mn + np + pm)(uv + vw + wu)} \Leftrightarrow$$

$\Leftrightarrow$ (Demostrado anteriormente). Realizando los siguientes cambios de variables:

$$u = \frac{x}{y+z}, v = \frac{y}{x+z}, w = \frac{z}{x+y}, n = b, p = c, m = a$$

2. Siendo: $x, y, z > 0$, se cumple la siguiente desigualdad:

$$\frac{xy}{(z+x)(z+y)} + \frac{yz}{(x+z)(x+y)} + \frac{zx}{(y+x)(y+z)} \geq \frac{3}{4} \text{ ... (A continuación lo demostraremos)}$$

De la siguiente identidad:

$$(x+y)(y+z)(z+x) = xy(x+y) + yz(y+z) + zx(z+x) + 2xyz$$

$$\Rightarrow \frac{xy}{(z+x)(z+y)} + \frac{yz}{(x+z)(x+y)} + \frac{zx}{(y+x)(y+z)} = 1 - \frac{2xyz}{(x+y)(y+z)(z+x)} \geq$$

$\geq 1 - \frac{1}{4} = \frac{3}{4}$ ... (LQQD). Por consiguiente:

$$mn + np + pm = ab + bc + ac \wedge uv + vw + wu = \frac{xy}{(z+x)(z+y)} + \frac{yz}{(x+z)(x+y)} + \frac{zx}{(y+x)(y+z)} \geq \frac{3}{4}$$

Luego, en (A) ...

$$\frac{x}{y+z}(b+c) + \frac{y}{x+z}(a+c) + \frac{z}{x+y}(a+b) \geq \sqrt{3(ab+bc+ac)} \geq$$

$\geq \sqrt{6\sqrt{3}S + \frac{3}{2}(a^2+b^2+c^2)}$. Lo cual nos falta probar que:

$\sqrt{3(ab+bc+ac)} \geq \sqrt{6\sqrt{3}S + \frac{3}{2}(a^2+b^2+c^2)}$ ... (B). Recordar lo siguiente:

$$a^2+b^2+c^2 = 4S(\cot A + \cot B + \cot C),\ ab = 2S\csc C,\ bc = 2S\csc A,\ ac = 2S\csc B$$

Elevando al cuadrado ... (B), nos queda:

$$3(ab+bc+ac) \geq 6\sqrt{3}S + \frac{3}{2}(a^2+b^2+c^2) \Leftrightarrow 2(ab+bc+ac) \geq$$

$$\geq 4\sqrt{3}S + a^2+b^2+c^2$$

$$4S(\csc C + \csc A + \csc B) \geq 4\sqrt{3}S + 4S(\cot A + \cot B + \cot C)$$

$(\csc A - \cot A) + (\csc B - \cot B) + (\csc C - \cot C) = \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{3}$ ...(LQQD)

Por transitividad:

$\frac{x}{y+z}(b+c) + \frac{y}{x+z}(a+c) + \frac{z}{x+y}(a+b) \geq \sqrt{6\sqrt{3}S + \frac{3}{2}(a^2+b^2+c^2)}$ ... (LQQD)

81. Prove that in any triangle ABC the following relationship holds:

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{2R}{R+2r}$$

Proposed by Nica Nicolae - Romania

Solution by Adil Abdullayev - Baku Azerbaidjian

$$\sum a^2 = 2(s^2 - r^2 - 4Rr), \sum ab = s^2 + r^2 + 4Rr$$

$$\frac{\sum a^2}{\sum ab} = \frac{2(s^2 - r^2 - 4Rr)}{s^2 + r^2 + 4Rr} \geq \frac{2R}{R + 2r} \leftrightarrow$$

$$\leftrightarrow (s^2 - r^2 - 4Rr)(R + 2r) \geq (s^2 + r^2 + 4Rr)R \leftrightarrow$$

$$\leftrightarrow 5Rr + 4R^2 + r^2 \geq s^2$$

$$5Rr + 4R^2 + r^2 \overbrace{\geq}^{to\ prove} 4R^2 + 4Rr + 3r^2 \overbrace{\geq}^{GERRETSEN} s^2$$

$$5Rr + 4R^2 + r^2 \geq 4R^2 + 4Rr + 3r^2 \leftrightarrow Rr \geq 2r^2 \leftrightarrow R \geq 2r \textit{ (EULER)}$$

82. In ΔABC:

$$\sqrt[4]{\frac{ab^2}{R}} + \sqrt[4]{\frac{bc^2}{R}} + \sqrt[4]{\frac{ca^2}{R}} \leq \sqrt{a + b + c} \cdot 3^{\frac{5}{8}}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Pro la desigualdad de Cauchy:

$$\left(\sqrt[4]{ab^2} + \sqrt[4]{bc^2} + \sqrt[4]{ca}\right)^2 \leq \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)\left(\sqrt{b^2} + \sqrt{c^2} + \sqrt{a^2}\right) \dots \text{(A)}$$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \sqrt{3(a + b + c)} \leq \sqrt{3\left(3\sqrt{3}R\right)} = 3\sqrt[4]{3}\sqrt{R} = 3^{\frac{5}{4}}\sqrt{R}$$

Por lo cual reemplanzando en ... (A)

$$\Rightarrow \sqrt[4]{ab^2} + \sqrt[4]{bc^2} + \sqrt[4]{ca^2} \leq \left(\sqrt{3^{\frac{5}{4}}\sqrt{R}}\right)\left(\sqrt{b + c + a}\right)$$

$$\Rightarrow \sqrt[4]{\frac{ab^2}{R}} + \sqrt[4]{\frac{bc^2}{R}} + \sqrt[4]{\frac{ca^2}{R}} \le \sqrt{a+b+c} \cdot 3^{\frac{5}{8}} \ldots \text{(LQQD)}$$

83. In $\Delta\, ABC$ the following relationship holds:

$$m_a h_a w_a + m_b h_b w_b + m_c h_c w_c \ge \sin A \sin B \sin C \left(\frac{b^2c^2}{a} + \frac{c^2a^2}{b} + \frac{a^2b^2}{c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$m_a h_a w_a + m_b h_b w_b + m_c h_c w_c \ge \sin A \sin B \sin C \left(\sum \frac{b^2c^2}{a}\right)$$

$$LHS \ge h_a^3 + h_b^3 + h_c^3 \qquad (m_a \ge w_a \ge h_a) \text{ etc}$$

$$= \frac{\sum a^3b^3}{8R^3} \qquad \left(h_a = \frac{bc}{2R}\right) \quad \text{etc}$$

$$RHS = \frac{abc}{8R^3}\left(\frac{b^3c^3 + c^3a^3 + a^3b^3}{abc}\right) = \frac{\sum a^3b^3}{8R^3};\ LHS \ge \frac{\sum a^3b^3}{8R^3} = RHS$$

Solution 2 by Soumava Pal – Kolkata – India

In $\Delta\, ABC$, we have $m_a \ge h_a$ and $w_a \ge h_a$,

$$\text{so } m_a h_a w_a \ge h_a^3 = \left(\frac{bc}{2R}\right)^3 = \frac{b^2c^2 \cdot 2R\sin(B) \cdot 2R\sin(C)}{2R \cdot 2R \cdot \frac{a}{\sin(A)}} \overset{\text{by sin rule}}{=}$$

$$= \frac{\sin(A)\sin(B)\sin(C)\, b^2c^2}{a}$$

$$\Rightarrow \sum_{cycl} m_a w_a h_a \ge \sum_{cycl} \frac{\sin(A)\sin(B)\sin(C)\, b^2c^2}{a}$$

84. **In $\Delta\ ABC$ the following relationship holds:**

$$\frac{a^3(2s-a)^3}{b(2s-b)}+\frac{b^3(2s-b)^3}{c(2s-c)}+\frac{c^3(2s-c)^3}{a(2s-a)}\geq\frac{27a^2b^2c^2}{s^2}$$

s **– semiperimeter**

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$\frac{a^3(2s-a)^3}{b(2s-b)}+\frac{b^3(2s-b)^3}{c(2s-c)}+\frac{c^3(2s-c)^3}{a(2s-a)}\geq\frac{27a^2b^2c^2}{s^2}$$

$$\Rightarrow s^2\left(\frac{a^3(2s-a)^3}{b(2s-b)}+\frac{b^3(2s-b)^3}{c(2s-c)}+\frac{c^3(2s-c)^3}{a(2s-a)}\right)\geq 27a^2b^2c^2$$

Por: $MA\geq MG\Rightarrow\frac{(a+b+c)^2}{4}\left(\frac{a^3(2s-a)^3}{b(2s-b)}+\frac{b^3(2s-b)^3}{c(2s-c)}+\frac{c^3(2s-c)}{a(2s-a)}\right)\geq$

$$\geq\frac{9\sqrt[3]{(abc)^2}}{4}\cdot 3\sqrt[3]{(abc)^2\big((2s-a)(2s-b)(2s-c)\big)^2}$$

Lo cual es suficiente probar:

$$\frac{9\sqrt[3]{(abc)^2}}{4}\cdot 3\sqrt[3]{(abc)^2\big((2s-a)(2s-b)(2s-c)\big)^2}\geq 27a^2b^2c^2\ \textbf{... (A)}$$

Desde que: $(2s-a)(2s-b)(2s-c)=(b+c)(c+a)(a+b)\geq 8abc$

Utilizando en ... (A): $\frac{9\sqrt[3]{(abc)^2}}{4}\cdot 3\sqrt[3]{(abc)^2\big((2s-a)(2s-b)(2s-c)\big)^2}\geq$

$$\geq\frac{9\sqrt[3]{(abc)^2}}{4}\cdot 3\sqrt[3]{(abc)^2 64(abc)^2}=27a^2b^2c^2\ \textbf{... (LQQD)}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$LHS=\frac{(ab+ca)^3}{(ab+bc)}+\frac{(bc+ab)^3}{(bc+ca)}+\frac{(bc+ca)^3}{(ca+ab)}\geq$$

$$\overset{AM-GM}{\geq} 3\sqrt[3]{a^2b^2c^2(a+b)^2(b+c)^2(c+a)^2}$$

it suffices to show: $a^2b^2c^2\prod(a+b)^2 \geq \frac{729a^6b^6c^6}{s^6}$

$$\Leftrightarrow \prod(a+b) \geq \frac{27(abc)^2}{s^3} = \frac{432R^2r^2}{s}$$

$$\Leftrightarrow 2abc + ab(2s-c) + bc(2s-a) + ca(2s-b) \geq \frac{432R^2r^2}{s}$$

$$\Leftrightarrow 2s^2\left(\sum ab\right) - 4Rrs^2 \geq 432R^2r^2$$

$$\Leftrightarrow s^2(s^2+4Rr+r^2) - 2Rrs^2 - 216R^2r^2 \geq 0$$

$$\Leftrightarrow s^4 + s^2(2Rr+r^2) - 216R^2r^2 \geq 0 \quad \textbf{(1)}$$

Gerretsen $\Rightarrow s^2 \geq 16Rr - 5r^2$

LHS of (1) $\geq (16Rr-5r^2)^2 + (16Rr-5r^2)(2Rr+r^2) - 216R^2r^2$

It suffices to show: $(16R-5r)^2 + (16R-5r)(2R+r) - 216R^2 \geq 0$

$$\Leftrightarrow 36R^2 - 77Rr + 10r^2 \geq 0$$

$\Leftrightarrow (R-2r)(36R-5r) \geq 0$**, which is true** $R \geq 2r$ **(Euler)**

Solution 3 by Soumitra Mandal- Chandar Nagore – India

$$\sum_{cyc}\frac{a^3(2s-a)^3}{b(2s-b)} \geq \frac{1}{3}\frac{\left(\sum_{cyc}(2sa-a^2)\right)^3}{\sum_{cyc}b\,(2s-b)}$$

Applying Holder's Inequality $= \frac{1}{3}\frac{\{2s(a+b+c)-\sum_{cyc}a^2\}^3}{2s(a+b+c)-\sum_{cyc}a^2} = \frac{4(ab+bc+ca)^3}{3(ab+bc+ca)} \geq$

$$\geq \frac{4(ab+bc+ca)^3}{(a+b+c)^2} \geq \frac{27a^2b^2c^2}{s^2}$$

85. **Prove that for two any triangles ABC and $A'B'C'$ we have:**

$$\text{a. } \left(\frac{m_b}{b}+\frac{m_c}{c}\right)\tan\frac{A'}{2}+\left(\frac{m_c}{c}+\frac{m_a}{a}\right)\tan\frac{B'}{2}+\left(\frac{m_a}{a}+\frac{m_b}{b}\right)\tan\frac{C'}{2}\geq 3$$

$$\text{b. } \left(\frac{b}{m_b}+\frac{c}{m_c}\right)\tan\frac{A'}{2}+\left(\frac{c}{m_c}+\frac{a}{m_a}\right)\tan\frac{B'}{2}+\left(\frac{a}{m_a}+\frac{b}{m_b}\right)\tan\frac{C'}{2}\geq 4.$$

Proposed by Hung Nguyen Viet-Ha Noi-Viet Nam

Solution by Kevin Soto Palacios – Huarmey – Peru

$$\text{a. } \left(\frac{m_b}{b}+\frac{m_c}{c}\right)\tan\frac{A'}{2}+\left(\frac{m_c}{c}+\frac{m_a}{a}\right)\tan B'_2+\left(\frac{m_a}{a}+\frac{m_b}{b}\right)\tan\frac{C'}{2}\geq 3 \text{ ... (A)}$$

1) Para todos los $\mathbb{R}^+$: m, n, p, u, v, w, se cumple la siguiente desigualdad:

$$(n+p)u+(m+p)v+(m+n)w\geq 2\sqrt{(mn+np+pm)(uv+vw+wu)} \text{ ...(B)}$$

Sean: $n=\frac{m_b}{b}, p=\frac{m_c}{c}, m=\frac{m_a}{a}, u=\tan\frac{A'}{2}, v=\tan\frac{B'}{2}, w=\tan\frac{C'}{2}$

2. Hayashi's Inequality:

Siendo P un punto arbitrario en el plano de un triángulo ABC, se cumple lo siguiente: $\frac{PB\cdot PC}{AB\cdot AC}+\frac{PA\cdot PC}{BA\cdot BC}+\frac{PA\cdot PB}{CA\cdot CB}\geq 1 \Leftrightarrow$ Sea $P=G$ (Centroid)

$$\frac{GB\cdot GC}{bc}+\frac{GA\cdot GC}{ac}+\frac{GA\cdot GB}{ab}=\frac{\left(\frac{2}{3}m_b\right)\left(\frac{2}{3}m_c\right)}{bc}+\frac{\left(\frac{2}{3}m_a\right)\left(\frac{2}{3}m_c\right)}{ac}+\frac{\left(\frac{2}{3}m_a\right)\left(\frac{2}{3}m_b\right)}{ab}\geq 1$$

$\Rightarrow mn+np+pm=\frac{m_a m_b}{ab}+\frac{m_b m_c}{bc}+\frac{m_a m_c}{ca}\geq\frac{9}{4}$. Además: si:

$$\frac{A'}{2}+\frac{B'}{2}+\frac{C'}{2}=\frac{\pi'}{2}\to uv+vw+wu=\tan\frac{A'}{2}\tan\frac{B'}{2}+\tan\frac{B'}{2}\tan\frac{C'}{2}+\tan\frac{C'}{2}\tan\frac{A'}{2}=1$$

Finalmente reemplazando en ... (A)

$$\left(\frac{m_b}{b}+\frac{m_c}{c}\right)\tan\frac{A'}{2}+\left(\frac{m_c}{c}+\frac{m_a}{a}\right)\tan\frac{B'}{2}+\left(\frac{m_a}{a}+\frac{m_b}{b}\right)\tan\frac{C'}{2}\geq$$

$\geq 2\sqrt{\left(\frac{9}{4}\right)(1)}=3$ ... (LQQD) . En un triángulo ABC. Demonstrar que:

$$2c^2 + ab \geq 4m_am_b \Leftrightarrow m_a = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} \text{ ... (B)}$$

Elevando al cuadrado la expression:

$$(2c^2 + ab)^2 \geq (2b^2 + 2c^2 - a^2)(2a^2 + 2c^2 - b^2)$$

$$4c^4 + 4abc^2 + a^2b^2 \geq$$

$$\geq 4b^2a^2 + 4b^2c^2 - 2b^4 + 4a^2c^2 + 4c^4 - 2b^2c^2 - 2a^4 - 2c^2a^2 + a^2b^2$$

$$\Rightarrow 2a^4 + 2b^4 - 4a^2b^2 - 2c^2a^2 - 2b^2c^2 + 4abc^2 \geq 0$$

$$\Rightarrow 2(a^2 - b^2)^2 - 2(ac - bc)^2 \geq 0$$

$$\Rightarrow 2(a-b)^2(a+b)^2 - 2c^2(a-b)^2 \geq 0 \to 2(a-b)^2((a+b)^2 - c^2) \geq 0$$

$$\Rightarrow 2(a-b)^2(a+b+c)(a+b-c) \geq 0 \Leftrightarrow$$

$\Leftrightarrow a + b - c > 0$ (Por desigualdad triangular) ... (LQQD)

Análogamente se cumple lo siguiente: $a^2 + bc \geq 4m_bm_c \wedge 2b^2 + ac \geq 4m_am_c$

De las siguientes desigualdades:

$$4m_am_b \leq 2c^2 + ab, 4m_bm_c \leq 2a^2 + bc, 4m_cm_a = 2c^2 + ab$$

Demostraremos la desigualdad pedida:

$$\frac{ab}{m_am_b} + \frac{bc}{m_bm_c} + \frac{ca}{m_am_a} \geq 4 \Leftrightarrow \frac{ab}{4m_am_b} + \frac{bc}{4m_bm_c} + \frac{ca}{4m_cm_a} \geq 1$$

Aplicando las desigualdades iniciales:

$$\frac{ab}{4m_am_b} + \frac{bc}{4m_bm_c} + \frac{ca}{4m_cm_a} \geq \frac{(ab)^2}{2c^2ab+(ab)^2} + \frac{(bc)^2}{2a^2bc+(bc)^2} + \frac{(ca)^2}{2b^2ac+(ca)^2}$$

Por la desigualdad de Cauchy: $\frac{(ab)^2}{2c^2ab+(ab)^2} + \frac{(bc)^2}{2a^2bc+(bc)^2} + \frac{(ca)^2}{2b^2ac+(ca)^2} \geq$

$$\geq \frac{(ab+bc+ca)^2}{(ab)^2+(bc)^2+(ac)^2+2abc(a+b+c)} \geq 1 \text{ ... (LQQD)}$$

b. $\left(\frac{b}{m_b} + \frac{c}{m_c}\right)\tan\frac{A'}{2} + \left(\frac{c}{m_c} + \frac{a}{m_a}\right)\tan\frac{B'}{2} + \left(\frac{a}{m_a} + \frac{b}{m_b}\right)\tan\frac{C'}{2} \geq 4.$

1) Para todos los $\mathbb{R}^+$: m, n, p, u, v, w, se cumple la siguiente desigualdad:

$$(n+p)u+(m+p)v+(m+n)w \geq 2\sqrt{(mn+np+pm)(uv+vw+wu)} \text{ ... (B)}$$

Sean: $n=\frac{b}{m_b}, p=\frac{c}{m_c}, m=\frac{a}{m_a}, u=\tan\frac{A'}{2}, v=\tan\frac{B'}{2}, w=\tan\frac{C'}{2}$

Además ya se demostro anteriormente lo siguiente:

$$\rightarrow mp+np+pm=\frac{ab}{m_a m_b}+\frac{bc}{m_b m_c}+\frac{ca}{m_c m_a} \geq 4$$

$$\rightarrow uv+vw+wu=\tan\frac{A'}{2}\tan\frac{B'}{2}+\tan\frac{B'}{2}\tan\frac{C'}{2}+\tan\frac{C'}{2}\tan\frac{A'}{2}=1$$

Finalmente reemplazando en ... (B):

$$\left(\frac{b}{m_b}+\frac{c}{m_c}\right)\tan\frac{A'}{2}+\left(\frac{c}{m_c}+\frac{a}{m_a}\right)\tan\frac{B'}{2}+\left(\frac{a}{m_a}+\frac{b}{m_b}\right)\tan\frac{C'}{2} \geq 2\sqrt{(4)(1)}=4$$

86. In $\Delta\, ABC$ the following relationship holds:

$$\left(\sqrt{\frac{m_a}{w_a}}+\sqrt{\frac{m_b}{w_b}}+\sqrt{\frac{m_c}{w_c}}\right)\left(\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}\right) \leq \frac{9R}{2r}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC, se cumple la siguiente desigualdad: $\frac{3R}{2r} \geq \frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}$

Lo cual es suficiente demostrar lo siguiente:

$$\frac{R}{2r} \geq \frac{m_a}{h_a} \rightarrow \frac{R}{2r} \geq \frac{m_a}{\frac{2rs}{a}} \rightarrow Rs \geq am_a \rightarrow \frac{abc}{4r} \geq am_a \rightarrow \frac{bc}{4r} \geq m_a \rightarrow$$

$$\rightarrow b^2c^2 \geq 4r^2(2b^2+2c^2-a^2)$$

$$\Rightarrow b^2c^2 \geq 4\frac{(s-a)(s-b)(s-c)}{s}(2b^2+2c^2-a^2) \rightarrow$$

$$\rightarrow b^2c^2s - 4(s-a)(s-b)(s-c)(2b^2+2c^2-a^2) \geq 0$$

Sea: $x = s-a, y = s-b, z = s-c \Leftrightarrow a = y+z, b = x+z \wedge c = y+z$:

La desigualdad es equivalente:

$$(x+z)^2(x+y)^2(x+y+z) - 4xyz(2(x+z)^2+2(x+y)^2-(y+z)^2) \geq 0$$

$$x(x+z)^2(x+y)^2 + y(x+z)^2(x+y)^2 + z(x+z)^2(x+y)^2 - 4xyz(x+z)^2 -$$

$$-4xyz(x+z)^2 - 4xyz(x+y)^2 - 4xyz(x+y)^2 + 4xyz(x+y)^2 \geq 0$$

$$\Rightarrow y(x+y)^2\left[(x+z)^2-4xz\right] + z(x+z)^2\left[(x+y)^2-4xy\right] +$$

$$+x\left[(x+z)^2(x+y)^2 - 4yz(x+z)^2 - 4yz(x+y)^2 + 4yz(y+z)^2\right] \geq 0$$

$$\Rightarrow y(x+y)^2(x-z)^2 + z(x+z)^2(x-y)^2 +$$

$$+x(x^4+2x^3y+2x^3z+x^2y^2+x^2z^2-4x^2yz-6xy^2z-6xyz^2+9y^2z^2) \geq 0$$

$$\Rightarrow y(x+y)^2(x-z)^2 + z(x+z)^2(x-y)^2 + x(x^2+xy+xz-3yz)^2 \geq 0$$

(LQQD). De forma análoga: $\frac{R}{2r} \geq \frac{m_b}{h_b}, \frac{R}{2r} \geq \frac{m_c}{h_c}$**. En un triángulo** ABC **probar que:**

$$\left(\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}\right)\left(\sqrt{\frac{m_a}{w_a}}+\sqrt{\frac{m_b}{w_b}}+\sqrt{\frac{m_c}{w_c}}\right) \leq \frac{9R}{2r}$$

Tener en cuenta lo siguiente, lo cual ya se ha demostrado:

1) $w_a \geq h_a, w_b \geq h_b, w_c \geq h_c$

2) $m_a + m_b + m_c \leq 4R + r \leq \frac{9R}{2}, \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} = \frac{1}{r}$

Por la desigualdad de Cauchy:

3) $(m_a+m_b+m_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right) \leq \left(\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}\right)^2$

$$\Rightarrow 3)\ \sqrt{\left(\frac{9R}{2}\right)\left(\frac{1}{r}\right)} \geq \sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}}$$

4) $(m_a + m_b + m_c)\left(\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c}\right) \leq \left(\sqrt{\frac{m_a}{w_a}} + \sqrt{\frac{m_b}{w_b}} + \sqrt{\frac{m_c}{w_c}}\right)^2$

$$\Rightarrow 4)\ (m_a + m_b + m_c)\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) \geq \left(\sqrt{\frac{m_a}{w_a}} + \sqrt{\frac{m_b}{w_b}} + \sqrt{\frac{m_c}{w_c}}\right)^2 \Rightarrow$$

$$\Rightarrow \sqrt{\left(\frac{9R}{2}\right)\left(\frac{1}{r}\right)} \geq \sqrt{\frac{m_a}{w_a}} + \sqrt{\frac{m_b}{w_b}} + \sqrt{\frac{m_c}{w_c}}$$

Multiplicando (3) × (4): $\left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}}\right)\left(\sqrt{\frac{m_a}{w_a}} + \sqrt{\frac{m_b}{w_b}} + \sqrt{\frac{m_c}{w_c}}\right) \leq \frac{9R}{2r}$ **... (LQQD)**

En un triángulo ABC **probar que:** $\left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}}\right)\left(\sqrt{\frac{m_a}{w_a}} + \sqrt{\frac{m_b}{w_b}} + \sqrt{\frac{m_c}{w_c}}\right) \leq \frac{9R}{2r}$

Desde que: $w_a \geq h_a, w_b \geq h_b, w_c \geq h_c$ **2)** $\frac{R}{2r} \geq \frac{m_a}{h_a}, \frac{R}{2r} \geq \frac{m_b}{h_b}, \frac{R}{2r} \geq \frac{m_c}{h_c}$

Por lo cual de desigualdad es equivalente: $\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} \leq \sqrt{\frac{3R}{2r}}$ **... (A),**

$$\sqrt{\frac{m_a}{w_a}} + \sqrt{\frac{m_b}{w_b}} + \sqrt{\frac{m_c}{w_c}} \leq \sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} \leq \sqrt{\frac{3R}{2r}} \ \text{... (B)}$$

Multiplicando: (A) × (B):

$$\left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}}\right)\left(\sqrt{\frac{m_a}{w_a}} + \sqrt{\frac{m_b}{w_b}} + \sqrt{\frac{m_c}{w_c}}\right) \leq \sqrt{\frac{3R}{2r}}\sqrt{\frac{3R}{2r}} = \frac{9R}{2r} \ \text{... (LQQD)}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$\text{LHS} \leq \left(\sqrt{\sum m_a}\sqrt{\sum \frac{1}{w_a}}\right)\left(\sqrt{\sum m_a}\sqrt{\sum \frac{1}{h_a}}\right) \leq \left(\sum m_a\right)\sqrt{\sum \frac{1}{h_a}}\sqrt{\sum \frac{1}{h_a}}$$

$$\left(w_a \ge h_a \text{ etc}, \frac{1}{w_a} \le \frac{1}{h_a} \text{ etc} \Rightarrow \sum \frac{1}{w_a} \le \sum \frac{1}{h_a}\right)$$

$$LHS \le \left(\sum m_a\right)\left(\sum \frac{1}{h_a}\right) = \left(\sum m_a\right)\left(\frac{a+b+c}{2\Delta}\right) =$$

$$\overset{\text{Bottema-Bager}}{\le} (4R+r)\left(\frac{2s}{2rs}\right) = \frac{4R+r}{r}$$

it suffices to prove that $\frac{4R+r}{r} \le \frac{9R}{2r} \Leftrightarrow 8R + 2r \le 9R \Leftrightarrow R \ge 2r \rightarrow$ **true (Euler)**

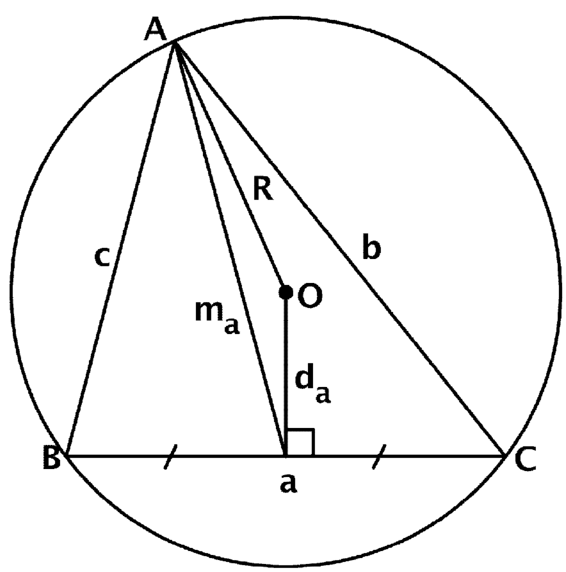
Solution 3 by George Apostolopoulous – Messolonghi – Greece

It is well – known that $m_a \ge l_a \ge h_a$**, etc.**

We have $\left(\sqrt{\frac{m_a}{l_a}} + \sqrt{\frac{m_b}{l_b}} + \sqrt{\frac{m_c}{l_c}}\right) \le \sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}}$**, so**

$$\left(\sum_{cyc}\sqrt{\frac{m_a}{l_a}}\right)\cdot\left(\sum_{cyc}\sqrt{\frac{m_a}{h_a}}\right) \le \left(\sum_{cyc}\sqrt{\frac{m_a}{h_a}}\right)^2 \le 3\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)$$

$(x+y+z)^2 \le 3(x^2+y^2+z^2)$**. Now we have**



$m_a \leq R + d_a$, etc. Then $\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \leq R\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) + \frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} =$

$$= R\left(\frac{a+b+c}{2[ABC]}\right) + \frac{\frac{ad_a}{2}}{\frac{ah_a}{2}} + \frac{\frac{bd_b}{2}}{\frac{bh_b}{2}} + \frac{\frac{cd_c}{2}}{\frac{ch_c}{2}} =$$

$= \frac{R\cdot 2s}{2[ABC]} + \frac{[ABC]}{\underbrace{[ABC]}_{\text{area of } \Delta ABC}} = \frac{R\cdot s}{r\cdot s} + 1 = \frac{R}{r} + 1$. So $\left(\sum_{cyc}\sqrt{\frac{m_a}{l_a}}\right)\cdot\left(\sum_{cyc}\sqrt{\frac{m_a}{h_a}}\right) \leq 3\left(\frac{R}{r} + 1\right)$.

Now, we have (EULER) $R \geq 2r \Leftrightarrow 6r \leq 3R \Leftrightarrow$

$6R + 6r \leq 9R \Leftrightarrow \frac{3R}{r} + 3 \leq \frac{9R}{2r} \Leftrightarrow 3\left(\frac{R}{r} + 1\right) \leq \frac{9R}{2r}$. So

$$\left(\sum_{cyc}\sqrt{\frac{m_a}{l_a}}\right)\cdot\left(\sum_{cyc}\sqrt{\frac{m_a}{h_a}}\right) \leq \frac{9R}{2r}.$$

Equality holds when the triangle ABC is equilateral.

87. In $\Delta\ ABC, I -$ incentre

$$\frac{R}{r}\sum\left(AI\cdot\cos\frac{B-C}{2}\right)^2 \geq \frac{6abc}{a+b+c}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo: Probar que: $\mathrm{I} -$ incentre

$$\frac{\mathrm{R}}{\mathrm{r}}\sum\left(\mathrm{AI}\times\cos\left(\frac{\mathrm{B}-\mathrm{C}}{2}\right)\right)^2 \geq \frac{6\mathrm{abc}}{\mathrm{a+b+c}} \ldots \text{(A)}$$

Recordar lo siguiente en un triángulo ABC:

1) $\frac{\mathrm{R}}{\mathrm{r}} = \frac{1}{4\,\mathrm{sen}\frac{A}{2}\mathrm{sen}\frac{B}{2}\mathrm{sen}\frac{C}{2}} = \frac{\mathrm{abc}}{4(s-a)(s-b)(s-c)}$. Ahora bien:

2) $\cos\left(\frac{B-C}{2}\right)=\cos\frac{B}{2}\cos\frac{C}{2}+\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}$

$$\Rightarrow \cos\left(\frac{B-C}{2}\right)=\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}+\sqrt{\frac{(s-c)(s-a)}{ac}}\sqrt{\frac{(s-b)(s-a)}{ab}}$$

$$\Rightarrow \cos\left(\frac{B-C}{2}\right)=\left(\frac{s}{a}+\frac{s-a}{a}\right)\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\Rightarrow \cos\left(\frac{B-C}{2}\right)=\left(\frac{b+c}{a}\right)\sqrt{\frac{(s-b)(s-c)}{bc}}$$

De forma análoga para los demás:

$$\cos\left(\frac{A-B}{2}\right)=\left(\frac{a+b}{c}\right)\sqrt{\frac{(s-a)(s-b)}{ab}},\ \cos\left(\frac{C-A}{2}\right)=\left(\frac{a+c}{b}\right)\sqrt{\frac{(s-a)(s-c)}{ac}}$$

3) Los incentros de un triángulo ABC, se puede expresar de la siguiente manera:

$$IA=\frac{bc}{s}\cos\frac{A}{2}=\frac{bc}{s}\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{bc(s-a)}{s}};\ IB=\sqrt{\frac{ac(s-b)}{s}},\ IC=\sqrt{\frac{ab(s-c)}{s}}$$

Reemplazando en la desigualdad ... (A):

$$\frac{abc}{4(s-a)(s-b)(s-c)}\left(\sum\frac{bc(s-a)}{s}\left(\frac{b+c}{a}\right)^2\frac{(s-b)(s-c)}{bc}\right)\geq\frac{6abc}{2s}$$

La cual es suficiente probar:

$$\frac{1}{(p-a)(p-b)(p-c)}\left(\sum(s-a)\left(\frac{b+c}{a}\right)^2(s-b)(s-c)\right)\geq 12$$

$$\Rightarrow\left(\frac{b+c}{a}\right)^2\geq 3\sqrt[3]{\left(\frac{(b+c)(a+c)(a+b)}{abc}\right)^2}\geq 3\sqrt[3]{8^2}=12$$ **... (LQQD)**

Solution 2 by Soumava Chakraborty – Kolkata – India

$AI = \frac{r}{\sin\frac{A}{2}}$; $AI^2\cos^2\frac{B-C}{2} = \frac{r^2\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}}$ **given inequality** $\Leftrightarrow \frac{R}{r}\cdot r^2\sum\frac{\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}} \geq \frac{24Rrs}{2s}$

$$\Leftrightarrow \sum\frac{\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}} \geq 12$$

$$\frac{\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}} = \frac{1}{\sin^2\frac{A}{2}\sec^2\left(\frac{B-C}{2}\right)} = \frac{1}{\sin^2\frac{A}{2}\left(1+\tan^2\frac{B-C}{2}\right)}$$

using Napier's analogy: $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2} =$

$$= \frac{1}{\sin^2\frac{A}{2}\left\{1+\frac{(b-c)^2}{(b+c)^2}\cot^2\frac{A}{2}\right\}} = \frac{1}{\sin^2\frac{A}{2}+\frac{(b-c)^2}{(b+c)^2}\cos^2\frac{A}{2}} =$$

$$= \frac{1}{1-\cos^2\frac{A}{2}\left\{1-\frac{(b-c)^2}{(b+c)^2}\right\}} = \frac{1}{1-\frac{s(s-a)}{bc}\cdot\frac{4bc}{(b+c)^2}} =$$

$= \frac{(b+c)^2}{(2s-a)^2-4s(s-a)} = \frac{(b+c)^2}{a^2}$. **Similarly,** $\frac{\cos^2\left(\frac{C-A}{2}\right)}{\sin^2\frac{B}{2}} = \frac{(c+a)^2}{b^2}$ **and** $\frac{\cos^2\left(\frac{A-B}{2}\right)}{\sin^2\frac{C}{2}} = \frac{(a+b)^2}{c^2}$

$$\sum\frac{\cos^2\left(\frac{B-C}{2}\right)}{\sin^2\frac{A}{2}} = \frac{(b+c)^2}{a^2}+\frac{(c+a)^2}{b^2}+\frac{(a+b)^2}{c^2} =$$

$$= \left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)+\left(\frac{c^2}{b^2}+\frac{b^2}{c^2}\right)+2\left(\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}\right) \geq$$

$$\overset{AM-GM}{\geq} 2+2+2+2\cdot 3\sqrt[3]{\frac{a^2b^2c^2}{a^2b^2c^2}} = 12$$

Solution 3 by Soumitra Mandal- Chandar Nagore – India

$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}\Rightarrow \csc\frac{A}{2}\cos\frac{B-C}{2}=\frac{b+c}{a}\text{ so,}$$

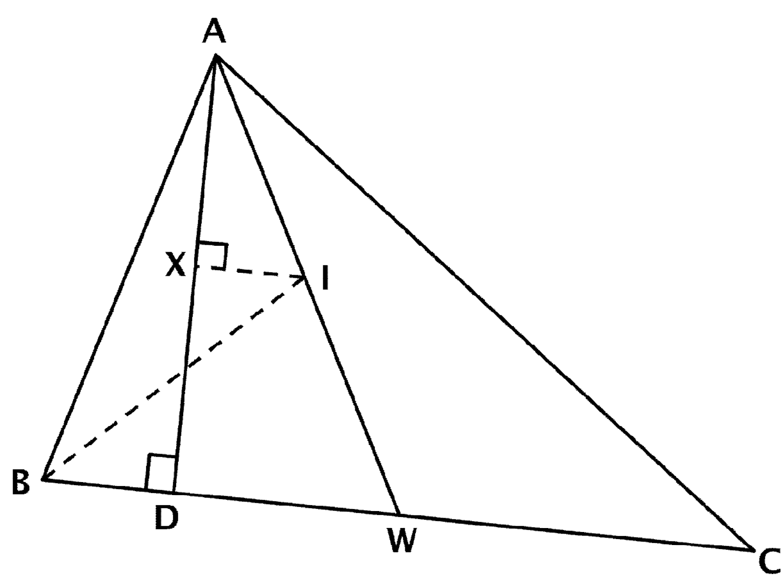
$$\sum_{cyc}\left(AI\cos\frac{B-C}{2}\right)^2\ge\frac{1}{3}\left(\sum_{cyc}AI\cos\frac{B-C}{2}\right)^2=\frac{1}{3}\left(\sum_{cyc}r\csc\frac{A}{2}\cos\frac{B-C}{2}\right)^2$$

$$\text{since, } AI=r\csc\frac{A}{2}, BI=r\csc\frac{B}{2}\text{ and } CI=r\csc\frac{C}{2}$$

$$=\frac{r^2}{3}\left(\sum_{cyc}\frac{b+c}{a}\right)^2\ge\frac{r^2}{3}\left(3\sqrt[3]{\prod_{cyc}\left(\frac{b+c}{a}\right)}\right)^2\ge 12r^2\text{ so,}$$

$$\frac{R}{r}\sum_{cyc}\left(AI\cos\frac{B-C}{2}\right)^2\ge 12Rr=12\cdot\frac{abc}{4S}\cdot\frac{S}{s}=\frac{3abc}{s}=\frac{6abc}{a+b+c}$$

Solution 4 by Soumava Pal – Kolkata – India



$AD\rightarrow$ **altitude,** $AW\rightarrow$ **angle bisector,** $I\rightarrow$ incentre, $IX\perp AD$

From angle bisector theorem, in $\Delta\ ABC$

$$\frac{BW}{WC}=\frac{AB}{AC}=\frac{c}{b}\Rightarrow BW=\frac{ac}{b+c}$$

From angle bisector theorem in $\Delta\ ABW,\frac{AI}{ID}=\frac{AB}{BW}$

$$\Rightarrow\frac{AI}{WI}=\frac{c}{\frac{ac}{b+c}}=\frac{b+c}{a}\Rightarrow\frac{AI}{AW}=\frac{b+c}{a+b+c}$$

$$\angle DAW=\angle BAW-\angle BAD=\frac{A}{2}-(90-B)=\frac{A}{2}-\left(\frac{A+B+C}{2}-B\right)=\frac{B-C}{2}$$

$$AX=AI\cos\angle IAX=AI\cos\angle DAW=AI\cos\left(\frac{B-C}{2}\right)\quad (2)$$

$\frac{AX}{AD}=\frac{AI}{AW}$ ($\Delta\ AXI$ and $\Delta\ ADW$ are similar)

$$\Rightarrow\frac{AX}{\left(\frac{2\Delta}{BC}\right)}=\frac{b+c}{a+b+c}\Rightarrow AX=\left(\frac{2\Delta}{a}\right)\frac{(b+c)}{2s}=\frac{b+c}{a}\cdot\frac{\Delta}{s}=r\left(\frac{b+c}{a}\right)\quad (3)$$

From (2) and (3), $AI\cos\left(\frac{B-C}{2}\right)=AX=r\left(\frac{b+c}{a}\right)$

$$X=\frac{R}{r}\sum\left(AI\cos\left(\frac{B-C}{2}\right)\right)^2\geq\frac{R}{r}\sum\left(\frac{r(b+c)}{a}\right)^2=$$

$$=Rr\left(\sum\left(\frac{b+c}{a}\right)^2\right)\quad (1)$$

$$\sum\left(\frac{b+c}{a}\right)^2\geq 3\sqrt[3]{\prod\left(\frac{b+c}{a}\right)^2}=3\sqrt[3]{\frac{(b+c)^2(c+a)^2(a+b)^2}{a^2b^2c^2}}=$$

$$=3\sqrt[3]{\frac{(b+c)^2}{bc}\cdot\frac{(c+a)^2}{ca}\cdot\frac{(a+b)^2}{ab}}=3\sqrt[3]{\left(\frac{b+c}{\sqrt{bc}}\right)^2\left(\frac{c+a}{\sqrt{ca}}\right)^2\left(\frac{a+b}{\sqrt{ab}}\right)^2}\geq$$

$\geq 3\sqrt[3]{2^2\cdot 2^2\cdot 2^2}$. (by AM – GM, $\frac{b+c}{\sqrt{bc}}\geq 2$)

$$=12\quad (2)$$

From (1) and (2), $X \geq 12Rr = 12 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = \frac{3abc}{(a+b+c)12} = \frac{6abc}{a+b+c}$

$$\Rightarrow \frac{R}{r}\sum\left(AI\cos\left(\frac{B-C}{2}\right)\right)^2 \geq \frac{6abc}{a+b+c}$$

88. **Let ABC be a triangle. Prove that:**

$$\frac{\sin\frac{A}{2}}{\cot\frac{A}{2}} + \frac{\sin\frac{B}{2}}{\cot\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cot\frac{C}{2}} \geq \frac{\sqrt{3}}{2}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Marian Dincă – Romania

$$\sum\frac{\sin\frac{A}{2}}{\cot\frac{A}{2}} = \sum\frac{\left(\sin\frac{A}{2}\right)^2}{\cos\frac{A}{2}} = \sum\frac{1-\left(\cos\frac{A}{2}\right)^2}{\cos\frac{A}{2}} = \sum\frac{1}{\cos\frac{A}{2}} - \sum\cos\frac{A}{2} \geq$$

$$\geq \frac{9}{\sum\cos\frac{A}{2}} - \sum\cos\frac{A}{2} \geq \frac{9}{3\cos\left(\frac{A+B+C}{6}\right)} - 3\cos\left(\frac{A+B+C}{6}\right) =$$

$$= \frac{3}{\frac{\sqrt{3}}{2}} - 3\frac{\sqrt{3}}{2} = 2\sqrt{3} - 3\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

because: $\sum\cos\frac{A}{2} \leq 3\cos\left(\frac{A+B+C}{6}\right)$ – Jensen inequality

89. **In $\Delta\ ABC$:**

$$\sum\frac{1}{\sqrt{\sin A}} \leq \frac{a^2+b^2+c^2+3}{2\sqrt{2S}}$$

Proposed by Daniel Sitaru – Romania

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Solution 1 by Kevin Soto Palacios – Huarmey – Peru

$$\sum\sqrt{\frac{1}{\sin A}}\le\frac{a^2+b^2+c^2+3}{2\sqrt{2S}}$$. **Recordar lo siguiente, en un triángulo ABC:**

$S=\frac{abc}{4R},\frac{1}{\sin A}=\frac{2R}{a},\frac{1}{\sin B}=\frac{2R}{b},\frac{1}{\sin C}=\frac{2R}{c}$. **La desigualdad es equivalente:**

$$2\sqrt{2\cdot\frac{abc}{4R}}\sum\sqrt{\frac{2R}{a}}\le a^2+b^2+c^2+3$$

$\Rightarrow 2\left(\sqrt{bc}+\sqrt{ac}+\sqrt{ab}\right)\le a^2+b^2+c^2+3$. **De la siguiente desigualdad:**

$$a^2+b^2+c^2+3\ge(ab+1)+(bc+1)+(ac+1)\ge 2\left(\sqrt{bc}+\sqrt{ac}+\sqrt{ab}\right)$$

Solo basta probar: $(ab+1)+(bc+1)+(ac+1)\ge 2\left(\sqrt{bc}+\sqrt{ac}+\sqrt{ab}\right)$

$$\Rightarrow\left(\sqrt{ab}-1\right)^2+\left(\sqrt{bc}-1\right)^2+\left(\sqrt{ac}-1\right)^2\ge 0$$

Solution 2 by Soumitra Mandal - Chandar Nagore – India

$$\frac{3+\sum_{cyc}a^2}{2\sqrt{2S}}=\frac{\sum_{cyc}(a^2+1)}{2}\cdot\sqrt{\frac{2R}{abc}}\text{ since, }R=\frac{abc}{4S}$$

$$\ge\left(\sum_{cyc}a\right)\sqrt{\frac{2R}{abc}}=\sqrt{\frac{2R}{abc}\cdot\left(\sum_{cyc}a\right)^2}\ge\sqrt{2R}\cdot\sqrt{3\frac{\sum_{cyc}ab}{abc}}=$$

$$=\sqrt{2R}\cdot\sqrt{3\left(\sum_{cyc}\frac{1}{a}\right)}\ge\sqrt{2R}\left(\sum_{cyc}\frac{1}{\sqrt{a}}\right)\left[\text{since, }\frac{1}{3}\left(\sum_{cyc}\frac{1}{\sqrt{a}}\right)\le\sqrt{\frac{1}{3}\left(\sum_{cyc}\frac{1}{a}\right)}\right]=$$

$$=\sum_{cyc}\sqrt{\frac{2R}{a}}=\sum_{cyc}\frac{1}{\sqrt{\sin A}}$$

Solution 3 by Soumava Pal – Kolkata – India

$$a^2+b^2+c^2+3\geq ab+bc+ca+3;\ (a^2+b^2+c^2\geq ab+bc+ca)$$

$$ab+1\geq 2\sqrt{ab}=2\sqrt{\frac{2\Delta}{\sin C}}\quad\left(\frac{1}{2}ba\sin C=\Delta\right)$$

$$\text{Similarly }bc+1\geq 2\sqrt{\frac{2\Delta}{\sin A}},ca+1\geq 2\sqrt{\frac{2\Delta}{\sin B}}$$

$$a^2+b^2+c^2+3\geq ab+bc+ca+3\geq 2\sqrt{2\Delta}\left(\frac{1}{\sqrt{\sin A}}+\frac{1}{\sqrt{\sin B}}+\frac{1}{\sqrt{\sin C}}\right)$$

$$\Rightarrow\frac{a^2+b^2+c^2}{2\sqrt{2\Delta}}\geq\sum_{cycl}\frac{1}{\sqrt{\sin A}}$$

90. In $\Delta ABC,\Delta A'B'C'$:

$$s'\left(\sum\frac{a}{\sqrt{a^2+b^2}}\right)+R'\left(\sum\frac{b}{\sqrt{a^2+b^2}}\right)\left(\sum\cos A'\right)\leq 9R'$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$\text{LHS}\leq\frac{3\sqrt{3}R'}{2}\left(\sum\frac{a}{\sqrt{a^2+b^2}}\right)+R'\left(\sum\frac{b}{\sqrt{a^2+b^2}}\right)\left(\frac{3}{2}\right)$$

($s'\leq\frac{3\sqrt{3}R'}{2}$ (Mitrinovic) and $\sum\cos A'=1+\frac{r}{R}\leq\frac{3}{2}$) it suffices to prove:

$$\frac{\sqrt{3}}{2}\left(\sum\frac{a}{\sqrt{a^2+b^2}}\right)+\frac{1}{2}\left(\sum\frac{b}{\sqrt{a^2+b^2}}\right)\leq 3$$

$$\Leftrightarrow\frac{\sqrt{3}}{2}\left(\sum\frac{a}{\sqrt{a^2+b^2}}\right)+\frac{1}{2}\left(\sum\frac{b}{\sqrt{a^2+b^2}}\right)\leq 3$$

$$\Leftrightarrow\left(\frac{\sqrt{3}}{2}\cdot\frac{a}{\sqrt{a^2+b^2}}+\frac{1}{2}\cdot\frac{b}{\sqrt{a^2+b^2}}\right)+\left(\frac{\sqrt{3}}{2}\cdot\frac{b}{\sqrt{b^2+c^2}}+\frac{1}{2}\cdot\frac{c}{\sqrt{b^2+c^2}}\right)+$$

$$+\frac{\sqrt{3}}{2}\left(\frac{c}{\sqrt{c^2+a^2}}+\frac{1}{2}\cdot\frac{a}{\sqrt{c^2+a^2}}\right)\le 3 \quad \textbf{(1)}$$

$$\frac{\sqrt{3}}{2}\cdot\frac{a}{\sqrt{a^2+b^2}}+\frac{1}{2}\cdot\frac{b}{\sqrt{a^2+b^2}}\overset{C-B-S}{\le}\sqrt{\frac{3+1}{4}}\cdot\sqrt{\frac{a^2+b^2}{a^2+b^2}}=1 \quad \textbf{(i)}$$

$$\frac{\sqrt{3}}{2}\cdot\frac{b}{\sqrt{b^2+c^2}}+\frac{1}{2}\cdot\frac{c}{\sqrt{b^2+c^2}}\overset{C-B-S}{\le}\sqrt{\frac{3+1}{4}}\cdot\sqrt{\frac{b^2+c^2}{b^2+c^2}}=1 \quad \textbf{(ii)}$$

$$\frac{\sqrt{3}}{2}\cdot\frac{c}{\sqrt{c^2+a^2}}+\frac{1}{2}\cdot\frac{a}{\sqrt{c^2+a^2}}\overset{C-B-S}{\le}\sqrt{\frac{3+1}{4}}\cdot\sqrt{\frac{c^2+a^2}{c^2+a^2}} \quad \textbf{(iii)}$$

(i) + (ii) + (iii) ⇒ (1) ; (1) is true (Proved)

Solution 2 by Soumitra Mandal- Chandar Nagore – India

$$s'\left(\sum_{cyc}\frac{a}{\sqrt{a^2+b^2}}\right)+R'\left(\sum_{cyc}\frac{b}{\sqrt{a^2+b^2}}\right)\left(\sum_{cyc}\cos A'\right)\le$$

$$\le s'\left(\sum_{cyc}\frac{a}{\sqrt{a^2+b^2}}\right)+R'\left(\sum_{cyc}\frac{b}{\sqrt{a^2+b^2}}\right)\left(\sum_{cyc}\sin A'\right)=$$

[since $\sin x\ge\cos x$ ***for all*** $x>0$***]***

$$=s'\left(\sum_{cyc}\frac{a}{\sqrt{a^2+b^2}}\right)+\left(\sum_{cyc}\frac{b}{\sqrt{a^2+b^2}}\right)\left(\frac{\sum_{cyc}a'}{2}\right)=$$

[since, $\sin A'=\frac{a'}{2R'}$**]**

$$=s'\left(\sum_{cyc}\frac{a+b}{\sqrt{a^2+b^2}}\right)\le\frac{a'+b'+c'}{\sqrt{2}}\le 9R'$$

(proved)

91. In $\Delta\ ABC$ the following relationship holds:

$$\left(5+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)\sum\frac{w_bw_c}{m_ah_a}\le\frac{8(2R+r)}{r}$$

Proposed by Bogdan Fustei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{w_bw_c}{m_ah_a}\overset{(a)}{=}\left(\prod w_a\right)\left(\sum\frac{1}{m_aw_ah_a}\right)$$

Now, $\prod w_a=\prod\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)=\frac{8(abc)^2}{\prod(a+b)}\left(\frac{s}{4R}\right)=\frac{32Rr^2s^3}{\prod(a+b)}$

$$=\frac{32Rr^2s^3}{2abc+\sum ab(2s-c)}=\frac{32Rr^2s^3}{2s(s^2+4Rr+r^2)}\overset{(1)}{=}\frac{16Rr^2s^2}{s^2+2Rr+r^2}$$

Again $\because m_a\ge\frac{b+c}{2}\cos\frac{A}{2}\therefore m_aw_ah_a\ge\left(\frac{b+c}{2}\cos\frac{A}{2}\right)\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)\left(\frac{bc}{2R}\right)=\frac{b^2c^2}{2R}\left(\frac{s(s-a)}{bc}\right)$

$$=\frac{sbc(s-a)}{2R}\Rightarrow\frac{1}{m_aw_ah_a}\overset{(i)}{\le}\frac{2R}{sbc(s-a)}$$

Similarly, $\frac{1}{m_aw_bh_b}\overset{(ii)}{\le}\frac{2R}{sca(s-b)}$ ***and,*** $\frac{1}{m_cw_ch_c}\overset{(iii)}{\le}\frac{2R}{sab(s-c)}$

(i)+(ii)+(iii) $\Rightarrow\sum\frac{1}{m_aw_ah_a}\le\frac{2R}{s}\left(\sum\frac{1}{bc(s-a)}\right)=\frac{2R}{s}\left(\frac{\sum a(s-b)(s-c)}{abc\prod(s-a)}\right)=$

$$=\left(\frac{2R}{s\cdot 4Rrs\cdot sr^2}\right)\left(\sum a\left(s^2-s(b+c)+bc\right)\right)$$

$$=\frac{1}{2r^3s^3}\{s^2(2s)-2s(s^2+4Rr+r^2)+12Rrs\}\overset{(2)}{=}\frac{2R-r}{r^2s^2}$$

Also, $5+\sum\frac{h_a}{r_a}=5+2\left(\sum\frac{s-a}{a}\right)=5+2s\sum\frac{1}{a}-6=\frac{s^2+4Rr+r^2}{2Rr}-1$

$\overset{(3)}{=}\frac{s^2+2Rr+r^2}{2Rr}$ ***(a), (1), (2), (3)*** $\Rightarrow LHS\le$

$$\left(\frac{16Rr^2s^2}{s^2+2Rr+r^2}\right)\left(\frac{2R-r}{r^2s^2}\right)\left(\frac{s^2+2Rr+r^2}{2Rr}\right)=\frac{8(2R-r)}{r}$$

(proved)

92. Prove that in any $\Delta\ ABC$,

$$\sqrt{abc}\left(\frac{a^2}{\sqrt{b}}+\frac{b^2}{\sqrt{c}}+\frac{c^2}{\sqrt{a}}\right)^2 \geq 16\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)S^2,$$

where $S = [\Delta\ ABC]$ is the area of $\Delta\ ABC$.

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Pal-Kolkata-India

***WLOG*, assume $a \geq b \geq c$. Then $a^2 \geq b^2 \geq c^2$ but $\frac{1}{\sqrt{a}} \leq \frac{1}{\sqrt{b}} \leq \frac{1}{\sqrt{c}}$. First employing the *Rearrangement inequality*, and then the *AM – GM inequality*,**

$$\sum_{cycl}\frac{a^2}{\sqrt{b}} \geq \sum_{cycl}\frac{a^2}{\sqrt{a}} = a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}} \geq 3\sqrt{abc}$$

Thus, (1) $\sum_{cycl}\frac{a^2}{\sqrt{b}} \geq 3\sqrt{abc}$.

Using the Rearrangement inequality the second time and then the *Chebyshev's inequality*,

$$\sum_{cycl}\frac{a^2}{\sqrt{b}} \geq \sum_{cycl}\frac{a^2}{\sqrt{a}} = a\sqrt{a}+b\sqrt{b}+c\sqrt{c} > \frac{1}{3}(a+b+c)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

So that, (2) $\sum_{cycl}\frac{a^2}{\sqrt{b}} \geq \frac{1}{3}\left(\sum_{cycl}a\right)\left(\sum_{cycl}\sqrt{a}\right)$

Multiplying (1) and (2), we get:

(3) $\sqrt{abc}\left(\sum_{cycl}\frac{a^2}{\sqrt{b}}\right) \geq (abc)(a+b+c)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).$

Define $x = a+b-c, y = b+c-a, z = c+a-b$. Then $x+y \geq 2\sqrt{xy}$ and $b \geq \sqrt{xy}$. Similarly, $a \geq \sqrt{xz}$ and $c \geq \sqrt{yz}$. It follows that $abc \geq xyz$. Thus we may continue (3):

$$\sqrt{abc}\left(\sum_{cycl}\frac{a^2}{\sqrt{b}}\right) \geq (abc)(a+b+c)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \geq$$

$$\geq xyz(a+b+c)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) = 16S^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right),$$ **as required.**

Solution 2 by Soumitra Moukherjee-Chandar Nagore-India

Since

$$(x+y)(y+z)(z+x) \geq 8xyz, abc \geq (a+b-c)(b+c-a)(c+a-b).$$

Further,

$$\sqrt{abc}\left(\sum_{cycl}\frac{a^2}{\sqrt{b}}\right)^2 \geq 3\sqrt[3]{\prod_{cycl}\frac{a^2}{\sqrt{b}}}\left(\sum_{cycl}\frac{a^2}{\sqrt{b}}\right)\sqrt{abc} = 3abc\left(\sum_{cycl}\frac{a^2}{\sqrt{b}}\right) \geq$$

$$\geq \left(\prod_{cycl}(a+b-c)\right)\frac{3(a+b+c)^2}{\sum_{cycl}\sqrt{a}} \geq$$

$$\geq \left(\sum_{cycl}\sqrt{a}\right)(a+b+c)\left(\prod_{cycl}(a+b-c)\right) = \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)16S^2.$$

Solution 3 by George Apostolopoulos – Messolonghi – Greece

We have $2s = a+b+c$, **so** $\frac{a^3(2s-a)^3}{b(2s-b)} + \frac{b^3(2s-b)^3}{c(2s-c)} + \frac{c^3(2s-c)^3}{a(2s-a)} =$

$$\frac{a^3(b+c)^3}{b(c+a)} + \frac{b^3(c+a)^3}{c(a+b)} + \frac{c^3(a+b)^3}{a(b+c)} \overset{\text{by AM-GM Inequality}}{\geq}$$

$$\geq 3\sqrt[3]{\frac{[a(b+c)b(c+a)c(a+b)]^3}{abc(a+b)(b+c)(c+a)}} =$$

$$=\frac{3abc\cdot(a+b)(b+c)(c+a)}{\sqrt[3]{abc}\cdot\sqrt[3]{(a+b)(b+c)(c+a)}}\ge\frac{3abc\cdot(2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca})}{\frac{a+b+c}{3}\cdot\frac{a+b+b+c+c+d}{3}}=$$

$$=\frac{24a^2b^2c^2}{\frac{2}{9}(a+b+c)^2}=\frac{24a^2b^2c^2}{\frac{2}{9}\cdot 4\cdot S^2}=\frac{27a^2b^2c^2}{S^2}$$

Equality holds when the triangle ABC is equilateral.

Solution 4 by Tuk Zaya – Ulanbaataar – Mongolia

$$\sqrt{abc}\cdot\left(\frac{a^2}{\sqrt{b}}+\frac{b^2}{\sqrt{c}}+\frac{c^2}{\sqrt{a}}\right)^2\ge 16(\sqrt{a}+\sqrt{b}+\sqrt{c})\cdot S^2 \qquad /3\sqrt{abc}$$

$$3abc\cdot\left(\frac{a^2}{\sqrt{b}}+\frac{b^2}{\sqrt{c}}+\frac{c^2}{\sqrt{a}}\right)^2\ge 48\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})\cdot S^2$$

$$3\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{c}(\sqrt{a}+\sqrt{b}+\sqrt{c})\le(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})^2$$

$$abc\ge 8(p-a)(p-b)(p-c)=\frac{8S^2}{P}$$

$$3abc\cdot\left(\frac{a^2}{\sqrt{b}}+\frac{b^2}{\sqrt{c}}+\frac{c^2}{\sqrt{a}}\right)^2\ge\frac{24S^2}{P}\left(\frac{a^2}{\sqrt{b}}+\frac{b^2}{\sqrt{c}}+\frac{c^2}{\sqrt{a}}\right)^2$$

$$48\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})S^2\le 16(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})^2\cdot S^2$$

$$\frac{24S^2}{P}\cdot\left(\frac{a^2}{\sqrt{b}}+\frac{b^2}{\sqrt{c}}+\frac{c^2}{\sqrt{a}}\right)^2\ge 16(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})^2\cdot S^2$$

$$\sqrt{3}\cdot\left(\frac{a^2}{\sqrt{b}}+\frac{b^2}{\sqrt{c}}+\frac{c^2}{\sqrt{a}}\right)\ge\sqrt{a+b+c}(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})$$

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ac}\le\sqrt{3(ab+bc+ac)}$$

$$\sqrt{3}\left(\frac{a^2}{\sqrt{b}}+\frac{b^2}{\sqrt{c}}+\frac{c^2}{\sqrt{a}}\right)\ge\sqrt{3}\sqrt{(a+b+c)(ab+bc+ac)}$$

$$\frac{a^2}{\sqrt{b}}+\frac{b^2}{\sqrt{c}}+\frac{c^2}{\sqrt{a}}\ge\sqrt{(a+b+c)(ab+bc+ac)}$$

$$(a+b+c)^4\ge\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\cdot(a+b+c)(ab+bc+ac)$$

$$(a+b+c)^3\ge\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2(ab+bc+ac)$$

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\le 3(a+b+c)$$

$$(a+b+c)^3\ge 3(a+b+c)(ab+bc+ac)$$

$$(a+b+c)^2\ge 3(ab+bc+ac)$$

93. Prove that in any triangle ABC:

$$\frac{h_a}{h_a+r_a}+\frac{h_b}{h_b+r_b}+\frac{h_c}{h_c+r_c}\le\frac{3}{2}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{h_a}{h_a+r_a}+\frac{h_b}{h_b+r_b}+\frac{h_c}{h_c+r_c}\le\frac{3}{2}$

$$\Rightarrow\frac{\frac{2S}{a}}{\frac{2S}{a}+\frac{2S}{2(s-a)}}+\frac{\frac{2S}{b}}{\frac{2S}{b}+\frac{2S}{2(s-b)}}+\frac{\frac{2S}{c}}{\frac{2S}{c}+\frac{2S}{2(s-c)}}\le\frac{3}{2}$$

$$\Rightarrow\frac{\frac{1}{a}}{\frac{b+c}{a(b+c-a)}}+\frac{\frac{1}{b}}{\frac{a+c}{b(a+c-b)}}+\frac{\frac{1}{c}}{\frac{a+b}{c(a+b-c)}}\le\frac{3}{2}$$

$$\Rightarrow\frac{b+c-a}{b+c}+\frac{a+c-b}{a+c}+\frac{a+b-c}{a+b}\le\frac{3}{2}$$

$$\Rightarrow\left(1-\frac{a}{b+c}\right)+\left(1-\frac{b}{a+c}\right)+\left(1-\frac{c}{a+b}\right)\le\frac{3}{2}$$

$$\Rightarrow \frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\ge\frac{3}{2}\Leftrightarrow$$ **(Desigualdad de Nesbitt)**

Solution 2 by Myagmarsuren Yadamsuren-Mongolia

$$\sum_{a,b,c}\frac{h_a}{h_a+r_a}=\sum\frac{1}{1+\frac{r_a}{h_a}}\overset{Cauchy}{\underbrace{\le}}$$

$$\le\frac{1}{2}\cdot\sum\frac{1}{\sqrt{1\cdot\frac{r_a}{h_a}}}=\frac{1}{2}\left(\sqrt{\frac{h_a}{z_a}}+\sqrt{\frac{h_b}{z_b}}+\sqrt{\frac{h_c}{z_c}}\right)\le$$

$$\overset{Cauchy-Schwarz}{\underbrace{\le}}\frac{1}{2}\cdot\sqrt{(1^2+1^2+1^2)\cdot\left(\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)}=\frac{1}{2}\cdot\sqrt{3\sum\frac{h_a}{r_a}}\le\frac{3}{2}$$

$$\sum\frac{h_a}{r_a}\le 3\Rightarrow \begin{matrix} h_a=\frac{2S}{a};h_b=\frac{2S}{b};h_c=\frac{2S}{c}\\ r_a=\frac{S}{p-a};\dots\end{matrix}$$

$$\frac{2S}{S}\cdot\left(\frac{p-a}{a}+\frac{p-b}{b}+\frac{p-c}{c}\right)\le 3;\ 2\cdot\left(p\cdot\frac{1}{a}+p\cdot\frac{1}{b}+p\cdot\frac{1}{c}-3\right)\le 3$$

$$2p\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le 9;\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{4R}\cdot\left(\frac{p}{r}+\frac{r}{p}\right)+\frac{1}{p}$$

$$2p\cdot\left(\frac{1}{4R}\cdot\left(\frac{p}{r}+\frac{r}{p}\right)+\frac{1}{p}\right)\le 9;\ \frac{1}{2R}\cdot\left(\frac{p^2}{r}+r\right)+2\le 9\Rightarrow p^2\le 14R\cdot r-r^2$$

$$14R\cdot r-r^2=12R\cdot r+2R\cdot r-r^2\overset{Euler}{\underbrace{\ge}}12R\cdot r+3r^2=$$

$$=8R\cdot r+4R\cdot r+3r^2\overset{Euler}{\underbrace{\ge}}4R^2+4R\cdot r+3r^2\ge p^2$$

(Gerretsen)

94. **In $\Delta\, ABC$ the following relationship holds:**

$$2R\sum w_a^3(w_a^4 - w_a^2 + 1) > (ab + bc + ca)\left(\frac{ab + bc + ca}{6R}\right)^4$$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren – Mongolia

In $\Delta\, ABC$: $2R\sum w_a^3(w_a^4 - w_a^2 + 1) > (ab + bc + ca)\left(\frac{ab+bc+ca}{6R}\right)^4$ ***(*)***

$$(*) \Rightarrow \sum w_a^3(w_a^4 + 1 - w_a^2) > \frac{1}{3^4}\cdot\left(\frac{ab+bc+ca}{2R}\right)^5$$

$$\sum w_a^3\,(w_a^4 + 1 - w_a^2) \overset{\substack{Cauchy\\ w_a^4+1\ge 2w_a^2}}{\geq} \sum w_a^5$$

$$w_a \ge h_a \Rightarrow \sum w_a^5 \ge \sum h_a^5 = h_a^5 + h_b^5 + h_c^5 \overset{CBC}{\geq}$$

$$\ge \frac{1}{3^4}(h_a + h_b + h_c)^5 = \frac{1}{3^4}\cdot\left(\frac{2S}{a} + \frac{2S}{b} + \frac{2S}{c}\right)^5 = \frac{1}{3^4}\cdot\left(\frac{ab + bc + ca}{2R}\right)^5$$

95. **In $\Delta\, ABC$ the following relationship holds:**

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2 + \frac{abc}{(a+b)(b+c)(c+a)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} < 2 + \frac{abc}{(a+b)(b+c)(a+c)}$ **... (M)**

En un triángulo ABC**, se cumple la siguientes desigualdades:**

$$2 > \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} > \frac{3}{2}$$

Como: a, b y c son lados de un triángulo tenemos:

$$a+b>c\to 2(a+b)>a+b+c\Rightarrow \frac{1}{a+b}<\frac{2}{a+b+c}\Rightarrow \frac{c}{a+b}<\frac{2c}{a+b+c}\ \text{... (A)}$$

Análogmente se obtiene: $\frac{b}{a+c}<\frac{2b}{a+b+c}$ **... (B);** $\frac{c}{a+b}<\frac{2c}{a+b+c}$ **... (C)**

Sumando: *(A)* + *(B)* + *(C)*: $2>\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}$ **... (LQQD)**

La desigualdad es equivalente en ... (M):

$$2-\frac{a}{b+c}-\frac{b}{a+c}-\frac{c}{a+b}+\frac{abc}{(a+b)(b+c)(a+c)}>\frac{abc}{(a+b)(b+c)(a+c)}>0\ \text{... (LQQD)}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$a(c+a)(a+b)+b(a+b)(b+c)+c(b+c)(c+a)<$$

$$<2(a+b)(b+c)(c+a)+abc$$

$$\Leftrightarrow a\left(\sum ab+a^2\right)+b\left(\sum ab+b^2\right)+c\left(\sum ab+c^2\right)<$$

$$<5abc+2\{ab(a+b)+bc(b+c)+ca(c+a)\}$$

$$\Leftrightarrow \left(\sum ab\right)(2s)+\sum a^3<$$

$$<5abc+2\{ab(2s-c)+bc(2s-a)+ca(2s-b)\}$$

$$\Leftrightarrow 2s\left(\sum ab\right)+\left(\sum a^3-3abc\right)<2abc-6abc+4s\left(\sum ab\right)$$

$$\Leftrightarrow 2s\left(\sum ab\right)>4abc+(2s)\left(\sum a^2-\sum ab\right)=$$

$$=4abc+2s\left(\sum a^2\right)-2s\left(\sum ab\right)\Leftrightarrow 4s\left(\sum ab\right)>4abc+2s\left(\sum a^2\right)$$

$$\Leftrightarrow 4s(s^2+4Rr+r^2)>16Rrs+4s(s^2-4Rr-r^2)$$

$$\Leftrightarrow s^2+4Rr+r^2>4Rr+s^2-4Rr-r^2\Leftrightarrow 2r^2+4Rr>0\to \textbf{\textit{true}}$$

(proved)

96. If in $\Delta ABC, \Delta\, A'B'C': a + a' = b + b' = c + c'$ then:

$$s(a'b' + b'c' + c'a') \le 6R'S'\left(\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Pal – Kolkata – India

Without loss of generality we can assume $a \ge b \ge c$ (1)

That implies $a' \le b' \le c'$ (Since $a + a' = b + b' = c + c'$)

$$\Rightarrow b'c' \ge c'a' \ge a'b' \quad (2)$$

(1) and (2) are two similarly oriented sequences: Applying Chebyshev,

$$(a + b + c)(b'c' + c'a' + a'b') \le 3(a \cdot b'c' + b \cdot c'a' + c \cdot a'b')$$

$$\Rightarrow 2s\left(\sum a'b'\right) \le 3a'b'c'\left(\sum \frac{a}{a'}\right)$$

$$\Rightarrow s\left(\sum a'b'\right) \le \frac{3}{2}a'b'c'\left(\sum \frac{a}{a'}\right) = 6R'\Delta'\left(\sum \frac{a}{a'}\right)$$

$$(a'b'c' = 4R'\Delta')$$

97. Let ABC be a triangle with circumradius R and inradius r. Prove that

$$4 \le \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \le \left(\frac{R}{r}\right)^2$$

Proposed by George Apostolopoulos –Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $4 \le \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \le \left(\frac{R}{r}\right)^2$

La desigualdad es equivalente:

$$4 \le \left(1 + \tan^2\frac{A}{2}\right) + \left(1 + \tan^2\frac{B}{2}\right) + \left(1 + \tan^2\frac{C}{2}\right) \le \left(\frac{R}{r}\right)^2$$

Desde que: $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\rightarrow \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{A}{2}\tan\frac{C}{2}=1$

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 1\Leftrightarrow\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2\geq 3\Leftrightarrow$$

$\Leftrightarrow \tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\geq\sqrt{3}$. **Además:** $\tan\frac{A}{2}=\frac{r_a}{p},\tan\frac{B}{2}=\frac{r_b}{p},\tan\frac{C}{2}=\frac{r_c}{p}$

$$\Rightarrow r_a+r_b+r_c=4R+r$$

$$\left(1+\tan^2\frac{A}{2}\right)+\left(1+\tan^2\frac{B}{2}\right)+\left(1+\tan^2\frac{C}{2}\right)=\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2+1\leq\left(\frac{R}{r}\right)^2$$

$$\Rightarrow\left(\frac{4R+r}{p}\right)^2+1\leq\left(\frac{R}{r}\right)^2\rightarrow(4R+r)^2r^2+p^2r^2\leq p^2R^2\Rightarrow$$

$\Rightarrow(4R+r)^2r^2\leq p^2(R^2-r^2)$. **Por la desigualdad de Gerretsen:**

$$p^2(R^2-r^2)\geq(16Rr-5r^2)(R^2-r^2)\geq(4R+r)^2r^2$$

$\Rightarrow$ **Solo es suficiente probar que:** $(16Rr-5r^2)(R^2-r^2)\geq(4R+r)^2r^2$

$$16R^3r-5R^2r^2-16Rr^3+5r^4\geq(16R^2+8Rr+r^2)r^2$$

$$\Rightarrow 16R^3r-21R^2r^2-24Rr^3+4r^4\geq 0$$

$$\Rightarrow r(16R^3-21R^2r-24Rr^2+4r^3)\geq 0$$

$$\Rightarrow r\left(16R^2(R-2r)+11Rr(R-2r)-2r^2(R-2r)\right)\geq 0$$

$\Rightarrow r(R-2r)(16R^2+11Rr-2r^2)\geq 0\Leftrightarrow R\geq 2r$ **... (Desigualdad e Euler)**

Además: $16R^2+11Rr-2r^2\geq 64r^2+22r^2-2r^2=84r^2>0$

Por lo tanto se ha demostrado: $4\leq\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\leq\left(\frac{R}{r}\right)^2$ **... (LQQD)**

Solution 2 by Martin Lukarevski-Skopje-Macedonia

The stronger inequality $5-\frac{2r}{R}\leq\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\leq 2+\frac{R}{r}$ **holds. We use the Garfunkel – Bankoff inequality [1]**

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

Which by the well – known identity $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}$ is equivalent to

$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 2-\frac{2r}{R}$. Then

$$\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}=3+\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 5-\frac{2r}{R},$$

which proves the LHS of the inequality. For the RHS we use the inequality

$(4R+r)^2\le s^2\left(1+\frac{R}{r}\right)$, which follows from Gerretsen's inequality

$s^2\ge 16Rr-5r^2$ and Euler's $R\ge 2r$, combined with the identity

$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=\frac{(4R+r)^2-2s^2}{s^2}$. We have

$\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}=1+\left(\frac{4R+r}{s}\right)^2\le 2+\frac{R}{r}$, and we are done.

References:

[1] Problem 825 (proposed by J. Garfunkel, solution by L. Bankoff) *Crux Math*. 9 (1983), 79 and 10 (1984), 168

98. Let ABC be a triangle and let P be any point in its plane. Prove that the following inequality holds for every positive real numbers x, y, z

$$(y+z)^2\frac{PA}{a}+(z+x)^2\frac{PB}{b}+(x+y)^2\frac{PC}{c}\ge 4\sqrt{xyz(x+y+z)}$$

where $a=bc, b=ca$, and $c=AB$.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sea ABC un triángulo y P sea cualquier punto en su plano.

Probar la siguiente desigualdad se mantiene para todos los números R^+ x, y, z:

$$(y+z)^2\frac{PA}{a}+(z+x)^2\frac{PB}{b}+(x+y)^2\frac{PC}{c}\geq 4\sqrt{xyz(x+y+z)}$$

1) De la desigualdad Ponderada: (Refinamiento de Ionescu-Weitzenbock)

Siendo a, b, c los lados de un triángulo ABC:

$ma^2+nb^2+pc^2\geq 4S\sqrt{mn+np+pm}$ ***... (A)*** $\Leftrightarrow (m,n,p)\geq 0$**, donde S es el área del triángulo. Realizando los siguientes cambios de variables:**

$$m=\frac{PA}{a}>0, n=\frac{PB}{b}>0, p=\frac{PC}{c}>0;$$

$$a=y+z>0, b=z+x>0, c=x+y>0$$

Luego: $S=\sqrt{(p)(p-a)(p-b)(p-c)}\to S=\sqrt{xyz(x+y+z)}$

2) De la desigualdad HAYASHY se llegó demonstrar lo siguiente:

$$mn+np+pm=\frac{PA}{a}\cdot\frac{PB}{b}+\frac{PB}{b}\cdot\frac{PC}{c}+\frac{PA}{a}\cdot\frac{PC}{c}\geq 1$$

Reemplazando en ... (A), se obtiene la desigualdad propuesta:

$$(y+z)^2\frac{PA}{a}+(z+x)^2\frac{PB}{b}+(x+y)^2\frac{PC}{c}\geq 4\sqrt{xyz(x+y+z)}\sqrt{mn+np+pm}\geq$$

$$\geq\sqrt{xyz(x+y+z)}$$ ***... (LQQD)***

99. Let ABC be a triangle and let P be any point in its plane. Prove that

$$\left(\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC}\right)(PA+PB+PC)^3\geq 3(a+b+c)^2$$

where $a=BC, b=CA$, and $c=AB$.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

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Solution by Kevin Soto Palacios – Huarmey – Peru

KLAMKIN INERTIAL MOMENT

Siendo a, b, c *los* lados de un triángulo ABC y PA, PB, PC son las distancias de un punto P en en plano ABC. Se cumple para todos los números R, x, y, z se tiene lo siguiente:

$(x + y + z)(xPA^2 + yPB^2 + zPC^2) \geq yza^2 + zxb^2 + xyc^2$ **... (A continuación lo demostaremos). La manera clásica es de la siguiente forma:**

$$(x\vec{PA} + y\vec{PB} + z\vec{PC})^2 \geq 0$$

$$\Rightarrow x^2PA^2 + y^2PB^2 + z^2PC^2 + 2xy\vec{PA}\vec{PB} + 2yz\vec{PB}\vec{PC} + 2zx\vec{PA}\vec{PC} \geq 0 \text{ ... (B)}$$

Desde que: $2\vec{PA}\vec{PB} = PA^2 + PB^2 - c^2$,

$$2\vec{PB}\vec{PC} = PB^2 + PC^2 - a^2, 2\vec{PA}\vec{PC} = PA^2 + PC^2 - b^2$$

Tenemos en ... (B)

$$\Rightarrow x^2PA^2 + y^2PB^2 + z^2PC^2 + xy(PA^2 + PB^2 - c^2) + yz(PB^2 + PC^2 - a^2) + zx(PA^2 + PC^2 - b^2) \geq 0$$

$$\Rightarrow (x^2PA^2 + xyPA^2 + xzPA^2) + (y^2PB^2 + yxPB^2 + yzPB^2) + (z^2PC^2 + zxPC^2 + zyPC^2) \geq yza^2 + zxb^2 + xyc^2$$

$$\Rightarrow (x + y + z)(xPA^2 + yPB^2 + zPC^2) \geq yza^2 + zxb^2 + xyc^2 \text{ ... (LQQD)}$$

Por la tanto, desde que:

$$(x + y + z)(xPA^2 + yPB^2 + zPC^2) \geq yza^2 + zxb^2 + xyc^2$$

Sean: $x = \frac{1}{PA} > 0, y = \frac{1}{PB} > 0, z = \frac{1}{PC} > 0$

$$\Rightarrow \left(\frac{1}{PA} + \frac{1}{PB} + \frac{1}{PC}\right)(PA + PB + PC) \geq \frac{a^2}{PBPC} + \frac{b^2}{PAPC} + \frac{c^2}{PAPB}$$

$$\Rightarrow \left(\frac{1}{PA} + \frac{1}{PB} + \frac{1}{PC}\right)(PA + PB + PC)^3 \geq \left(\frac{a^2}{PBPC} + \frac{b^2}{PAPC} + \frac{c^2}{PAPB}\right)(PA + PB + PC)^2$$

Por la desigualdad de Cauchy:

$$\left(\frac{a^2}{PBPC} + \frac{b^2}{PAPC} + \frac{c^2}{PAPB}\right)(PA + PB + PC)^2 \geq \left(\frac{a^2}{PBPC} + \frac{b^2}{PAPC} + \frac{c^2}{PAPB}\right)3(PBPC + PAPC + PAPB) \geq$$

Por transtividad: $\left(\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC}\right)(PA+PB+PC)^3\geq 3(a+b+c)^2$ **... (LQQD)**

100. Prove that in any triangle:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\sqrt{\frac{4(4R+r)^2}{p^2}-3}\geq 3$$

Proposed by Adil Abdullayev – Baku – Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC:\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\sqrt{\frac{4(4R+r)^2}{p^2}-3}\geq 3$

1) De la siguiente desigualdad demostrada anteriormente, siendo: $x,y,z>0$:

$\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)^2\geq(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$. **Sean:** $x=r_a, y=r_c, z=r_b$.

$$\Rightarrow\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)^2\geq(r_a+r_b+r_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\Rightarrow$$

$$\Rightarrow\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)^2\geq(4R+r)\left(\frac{1}{r}\right)$$

$\Rightarrow\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\sqrt{\frac{4R+r}{r}}\geq\sqrt{\frac{4(4R+r)^2}{p^2}-3}$. **Por lo cual es sufficiente probar que:**

$$\sqrt{\frac{4R+r}{r}}\geq\sqrt{\frac{4(4R+r)^2}{p^2}-3}\rightarrow\frac{4R+r}{r}\geq\frac{4(4R+r)^2}{p^2}-3\Rightarrow$$

$$\Rightarrow p^2(4R+4r)\geq 4r(4R+r)^2\ \text{... (A)}$$

2) Desigualdad de Gerretsen: $p^2\geq 16Rr-5r^2$

Utilizando la desigualdad de Gerretsen teneoms en ... (A)

$$p^2(4R+4r)\geq(16Rr-5r^2)(4R+4r)\geq 4r(4R+r)^2$$

Por último probaremos que: $(16Rr - 5r^2)(4R + 4r) \geq 4r(4R + r)^2$

$$\Rightarrow 64R^2r + 44Rr^2 - 20r^3 \geq 64R^2r + 32Rr^2 + 4r^3$$

$\Rightarrow 12r^2(R - 2r) \geq 0 \Leftrightarrow$ Siendo esto último válido por desigualdad de Euler:

$(R \geq 2r)$. Ahora demostraremos RHS

3) $\sqrt{\frac{4(4R+r)^2}{p^2} - 3} \geq 3$. Desde que:

$$4R + r = r_a + r_b + r_c = p\tan\frac{A}{2} + p\tan\frac{B}{2} + p\tan\frac{C}{2} \rightarrow$$

$$\rightarrow \frac{4R + r}{p} = \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq 3\tan\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3}\right) = \sqrt{3}$$

Por lo tanto: $\Rightarrow \sqrt{\frac{4(4R+r)^2}{p^2} - 3} \geq \sqrt{4(3) - 3} = 3$ ... (LQQD)

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Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru