SPECIAL APPLICATIONS

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1. Show that

$$\int_0^\infty \frac{e^{-x^p} - e^{-x^q}}{x} dx = \frac{p-q}{pq}\gamma$$

for $p, q > 0$, where γ is the Euler - Mascheroni constant.

Proof.

$$I = \int_0^\infty (e^{-x^p} - e^{-x^q}) d(\ln x)$$

$$I = \Big[(e^{-x^p} - e^{-x^q})\ln\Big]_0^\infty + p\int_0^\infty e^{-x^p} x^{p-1}\ln x - q\int_0^\infty e^{-x^q} x^{q-1} \ln x dx$$

The substitutions $x^p = u, x^q = v$ makes the last two integrals as,

$$p\int_0^\infty e^{-x^p} x^{p-1}\ln x - q\int_0^\infty e^{-x^q} x^{q-1}\ln x dx = \int_0^\infty e^{-x} x \ln x dx\Big(\frac{q-p}{qp}\Big)$$

$$= \int_0^\infty e^{-x}\ln x dx\Big(\frac{q-p}{qp}\Big) = \gamma\Big(\frac{p-q}{qp}\Big)$$

$$\text{Now } \Big[(e^{-x^p} - e^{-x^q})\ln x\Big]_0^\infty = \lim_{x\to\infty}\frac{\ln x}{\frac{1}{e^{-x^p}-e^{-x^q}}} - \lim_{x\to 0}\frac{\ln x}{\frac{1}{e^{-x^p}-e^{-x^q}}}$$

Both of these can be evaluated by applying L-Hospital's rule successively and both of them are 0

$$\text{Hence } \int_0^\infty \frac{(e^{-x^p} - e^{-x^q})}{x} dx = \gamma\Big(\frac{p-q}{pq}\Big)$$

□

2.

$$\int_0^1 \frac{t^{\alpha-1} - t^{\beta-1}}{(1+t)\ln t} dx = \ln\left(\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\beta)}{\Gamma(\frac{1}{2}+\frac{1}{2}\beta)\Gamma(\frac{1}{2}\alpha)}\right)$$

Proof.

$$\text{Let } F(a) = \int_0^1 \frac{x^{a-1}}{(1+x)\ln x} dx \text{ then by differentiating w.r.t to } a$$

$$F'(a) = \int_0^1 \frac{x^{a-1}}{1+x} dx = \sum_{n\geq 0}(-1)^n \frac{1}{n+a}$$

$$\text{Now, } \sum_{n\geq 0}(-1)^n\frac{1}{n+a} = \frac{1}{2}\Big(\sum_{n\geq 0}\frac{1}{n+\frac{a}{2}} - \sum_{n\geq 0}\frac{1}{n+\frac{a+1}{2}}\Big) = \frac{1}{2}\Big(\psi\Big(\frac{a+1}{2}\Big) - \psi\Big(\frac{a}{2}\Big)\Big)$$

$$\text{So, } F'(\alpha) - F'(\beta) = \frac{1}{2}\Big(\psi\Big(\frac{\alpha+1}{2}\Big) - \psi\Big(\frac{\alpha}{2}\Big)\Big) - \frac{1}{2}\Big(\psi\Big(\frac{\beta+1}{2}\Big) - \psi\Big(\frac{\beta}{2}\Big)\Big)$$

$$\text{Integrating we have, } F(\alpha) - F(\beta) = \left(\ln\Big(\frac{\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{1+\beta}{2})\Gamma(\frac{\alpha}{2})}\Big)\right) + C$$

$$\text{Now since, } F(1) = 0 \Rightarrow C = 0$$

$$\text{So } F(\alpha) - F(\beta) = \left(\ln \frac{\Gamma(\frac{1+\alpha)}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{1+\beta}{2})\Gamma(\frac{\alpha}{2})}\right)$$

□

3. Find a closed form for

$$\int_0^\infty \frac{\sin^{2n+1} x}{x} dx$$

for $n \geq 0$

Proof.

$$I = \int_0^\infty \frac{\sin^{2n+1} x}{x} dx = \int_0^\infty \int_0^\infty e^{-xy} \sin^{2n+1} dxdy$$

Now by IBP two times we can create a recurrence relation and thus evaluate,

$$J_n = \int_0^\infty e^{-ax} \sin^{2n+1} x dx = \frac{(2n+1)!}{\prod_{r=1}^n (a^2 + (2r+1)^2)}$$

$$\text{Thus, } I = (2n+1)! \int_0^\infty \frac{dy}{\prod_{r=1}^n (a^2 + (2r+1)^2)}$$

$$\text{The expression } \frac{1}{\prod_{r=1}^n (a^2 + (2r+1)^2)} = \sum_{r=0}^n \frac{A_r}{y^2 + (2r+1)^2}$$

where the coefficients can be determined by the cover-up rule as

$$A_k = \frac{(-1)^k}{2^{2n}} \cdot \frac{2k+1}{(n-k)!(n+k+1)!}$$

$$\text{So, } I = (2n+1)! \sum_{k=0}^n \int_0^\infty \frac{A_k}{y^2 + (2k+1)^2} dy = \frac{\pi}{2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n-k} = \frac{\pi}{2^{2n+1}} \binom{2n}{n}$$

□

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