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Murray Klamkin's Duality Principle for Triangle Inequalities

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Abstract

In this article we build inequalities using Murray Klamkin's duality principle. This is a way to solve inequalities in six variables, three of them being the lengths of sides in a triangle and the other three positive numbers.

1. Introduction

In 1979, M. S. Klamkin published in "Elements der Mathematik" [3] the article "Triangle Inequalities from Triangle Inequalities" about the duality principle. Famous inequalities such as Euler's, Leibniz's, Ionescu-Weitzenbock's, Mitrinovic's, Carlitz's, and Curry's can be restated using this principle. Using some notations in a new, constructed triangle, these inequalities are rediscovered in a classical form. We begin with Klamkin's theorem.

Theorem 1 *If* $P \in Int(\Delta ABC), let PA = x, PB = y; PC = z, AB = c, BC = a, and CA = b$. *Then* ax, by, cz *can be the lengths of the sides of a triangle.*

Proof. [3] [4] Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be such that the points A, B, C, and P. correspond to z_1, z_2, z_3 and z_4, respectively. We can easily prove the identity:

$$(z_1 - z_4)(z_2 - z_3) + (z_2 - z_4)(z_3 - z_1) + (z_3 - z_4)(z_1 - z_2) = 0$$

and hence

$$-(z_3 - z_4)(z_1 - z_2) = (z_1 - z_4)(z_2 - z_3) + (z_2 - z_4)(z_3 - z_1).$$

Then

$$|-(z_3-z_4)(z_1-z_2)| = |(z_1-z_4)(z_2-z_3)+(z_3-z_4)(z_3-z_1)|$$
$$\leq |z_1-z_4|\cdot|z_2-z_3|+|z_2-z_4|\cdot|z_3-z_1|$$

Hence: $cz < ax+by$. Analogously: $ax < by+cz$ and $by < ax+cz$. ■

Now let ΔMNP have sides of lengths ax, by, cz, and let s be the semiperimeter, S the area, R the exradius and r the inradius. Then:

$$s = \frac{ax+by+cz}{2}$$
$$S = \frac{1}{4}\sqrt{(ax+by+cz)(ax+by-cz)(ax-by+cz)(by+cz-ax)}$$
$$R = \frac{abcxyz}{\sqrt{(ax+by+cz)(ax+by-cz)(ax-by+cz)(by+cz-ax)}}$$
$$r = \frac{\sqrt{(ax+by+cz)(ax+by-cz)(ax-by+cz)(by+cz-ax)}}{2(ax+by+cz)}$$
$$= \frac{1}{2}\sqrt{\frac{(ax+by-cz)(ax-by+cz)(by+cz-ax)}{ax+by+cz}}.$$

2. Applications

Application 1 Let be $P \in Int(\Delta ABC)$ and let $PA = x, PB = y$, and $PC = z$.
Prove that:

$$(ax+by-cz)(ax-by+cz)(by+cz-ax) \leq abcxyz$$

Proof. The inequality can be written:

$$(ax+by-cz)(ax-by+cz)(by+cz-ax)(ax+by+cz)$$
$$\leq abcxyz(ax+by+cz)$$

or, equivalently,

$$\frac{abcxyz}{\sqrt{(ax+by-cz)(ax-by+cz)(by+cz-ax)(ax+by+cz)}}$$
$$\geq \frac{2\sqrt{(ax+by-cz)(ax-by+cz)(by+cz-ax)(ax+by+cz)}}{2(ax+by+cz).}$$

This is equivalent to $R \geq 2r$, which is Euler's Inequality for ΔMNP. ■

Application 2 Let be $P \in Int(\Delta ABC)$ and let $PA = x, PB = y$, and $PC = z$.
Prove that:

$$\frac{a^2x^2 + b^2y^2 + c^2z^2}{a^2b^2c^2x^2y^2z^2} \le \frac{9}{(ax+by+cz)(ax+by-cz)(ax-by+cz)(by+cz-ax)}$$

Proof. The inequality can be written:

$$(ax)^2 + (by)^2 + (cz)^2 \le \frac{9(abcxyz)^2}{(\sqrt{(ax+by+cz)^2(ax+by-cz)(ax-by+cz)(by+cz-ax)})^2}$$

which is equivalent to

$$(ax)^2 + (by)^2 + (cz)^2 \le 9R^2.$$

This is Leibniz's Inequality for ΔMNP. ■

Application 3 Let be $P \in Int(\Delta ABC)$ and let $PA = x, PB = y$, and $PC = z$.
Prove that:

$$(a^2x^2 + b^2y^2 + c^2z^2)^2 \ge 3(ax+by+cz)(ax+by-cz)(ax-by+cz)(by+cz-ax).$$

Proof. The inequality can be written:

$$(ax)^2 + (by)^2 + (cz)^2 \ge 4\sqrt{3}\cdot\frac{(ax+by+cz)(ax+by-cz)(ax-by+cz)(by+cz-ax)}{4}$$

or, equivalently,

$$(ax)^2 + (by)^2 + (cz)^2 \ge 4\sqrt{3}S,$$

which is Ionescu - Weitzenbock's Inequality for ΔMNP. ■

Application 4 Let be $P \in Int(\Delta ABC)$ and let $PA = x, PB = y$, and $PC = z$.
Prove that:

$$27(by+cz-ax)(ax+cz-by)(ax+by-cz) \le (ax+by+cz)^3$$

Proof. The inequality can be written:

$$27(by+cz-ax)(ax+cz-by)(ax+by-cz)(ax+by+cz) \le (ax+by+cz)^4$$

or, equivalently,

$$27 \cdot 16 \cdot S^2 \leq (ax + by + cz)^4.$$

Then

$$12\sqrt{3} \cdot S \leq (ax + by + cz)^2$$

so that

$$6\sqrt{3} \cdot r \cdot (ax + by + cz) \leq (ax + by + cz)^2.$$

It follows that

$$6\sqrt{3} \cdot r \leq ax + by + cz$$

or

$$6\sqrt{3} \cdot r \leq 2s$$

and hence that

$$3\sqrt{3}r \leq s.$$

This last is Mitrinovic's Inequality for ΔMNP. ■

Application 5 Let be $P \in Int(\Delta ABC)$ and let $PA = x, PB = y$, and $PC = z$.
Prove that:

$$(ax+by+cz)^3(ax+by-cz)^3(ax-by+cz)^3(by+cz-ax)^3 \leq 27(abcxyz)^4$$

Proof. The inequality can be written:

$$\sqrt{(ax + by + cz)(ax + by - cz)(ax - by + cz)(by + cz - ax)} \leq \sqrt{3}(abcxyz)^{\frac{2}{3}}$$

or, equivalently,

$$4S \leq \sqrt{3} \cdot \sqrt[3]{(ax)^2 \cdot (by)^2 \cdot (cz)^2}.$$

Then

$$\frac{4S}{\sqrt{3}} \leq \sqrt[3]{(ax)^2 \cdot (by)^2 \cdot (cz)^2},$$

which is Carlitz's Inequality [1] for ΔMNP. ■

Application 6 Let be $P \in Int(\Delta ABC)$ and let $PA = x, PB = y$, and $PC = z$.
Prove that:

$$(ax+by+cz)^3(ax+by-cz)(ax-by+cz)(by+cz-ax) \leq 27(abcxyz)^2$$

Proof. The inequality can be written:

$$(ax+by+cz)(ax+by-cz)(ax-by+cz)(by+cz-ax) \leq \frac{27(abcxyz)^2}{(ax + by + cz)^2}$$

or, equivalently,

$$16(S)^2 \leq \frac{27(abcxyz)^2}{(ax + by + cz)^2}.$$

Then

$$4S \le \frac{3\sqrt{3}abcxyz}{ax + by + cz}$$

and hence

$$4\sqrt{3}S \le \frac{9(ax)(by)(cz)}{(ax) + (by) + (cz)},$$

which is Curry's Inequality [2] for ΔMNP. ■

References

[1] L. Carlitz and F. Leuenberger, Problem E 1454, *American Mathematical Monthly,* Vol. 68, pg. 177 and 805–806, 1961.

[2] T. R. Curry, Problem E 1861, *American Mathematical Monthly,* Vol. 75, pg. 199, 1966.

[3] M. S. Klamkin, *Triangle Inequalities from Triangle Inequalities.* Elements der Mathematik, 1979, vol. 34, pp 49–55.

[4] Daniel Sitaru, *Math Phenomenon.* Paralela 45, Publishing House, Pitesti, Romania, 2016.