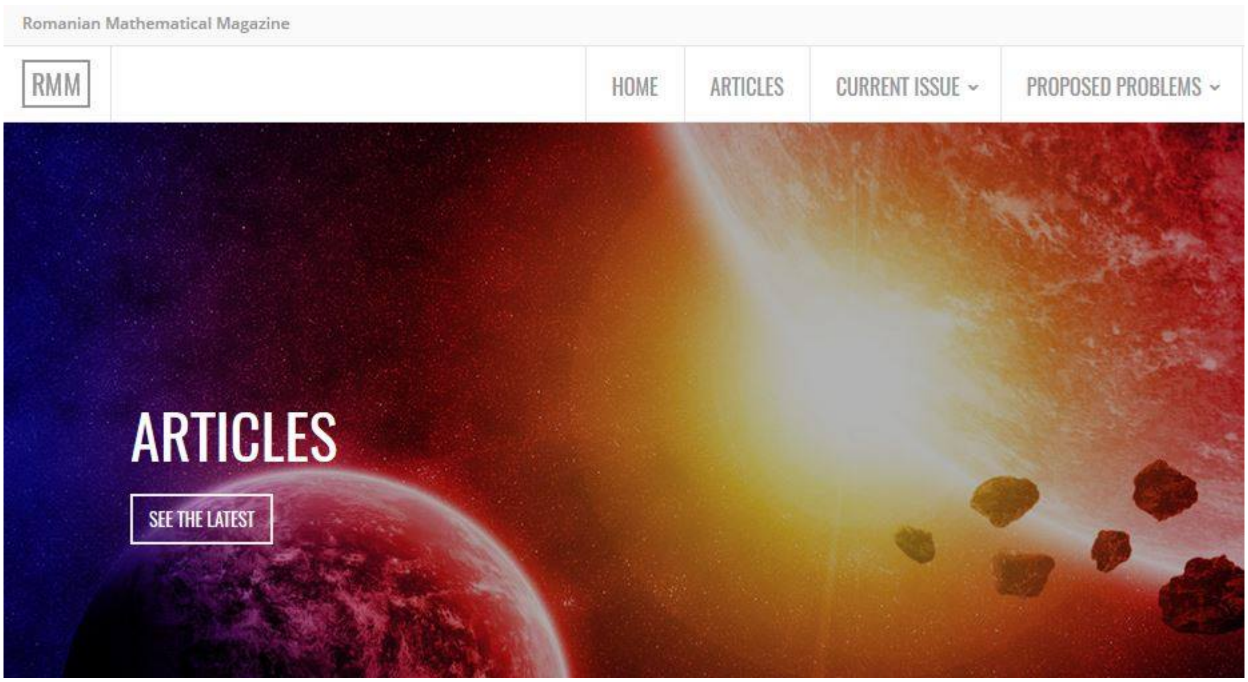

Romanian Mathmatical Magazine

A SPECIAL CLASS OF INEQUALITIES

By

Lê Khánh Sỹ- Long An- Việt Nam

Problem: Let a, b, c be nonnegative real numbers such that $c = \min\{a, b, c\}$.
Prove

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2 + \frac{2c(a-b)^2}{3(a+b)(b+c)(c+a)}$$

Solution.

lemma:

If $a, b, c \geq 0$ and $c = \min\{a, b, c\}$ then.

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq \frac{2ab+c^2}{ab+bc+ca} + \frac{2(a+b)c}{(b+c)(c+a)} + \frac{2c(a-b)^2}{3(a+b)(b+c)(c+a)}$$

Write the inequality as follows.

$$\frac{(a-b)^2}{ab+bc+ca} \geq \frac{8(a-b)^2c}{3(a+b)(b+c)(c+a)}$$

or

$$3(a+b)(b+c)(c+a) \geq 8c(ab+bc+ca).$$

We make the substitutions

$$\begin{cases} a = c + x \\ b = c + y \\ x, y \geq 0 \end{cases}$$

The inequlity becomes as follows.

$$8(x+y)c^2 + 2(3x^2 + 5xy + 3y^2)c + 3xy(x+y) \geq 0$$

Thus, the proof is completed. Therefore, it suffices to show that

$$\frac{2ab+c^2}{ab+bc+ca} + \frac{2(a+b)c}{(b+c)(c+a)} \geq 2.$$

Or

$$\frac{c^2(a-c)(b-c)}{(a+c)(b+c)(ab+bc+ca)} \geq 0.$$

The equality occurs for $a = b = c$, and for $c = 0$ and $b = c$ ■

Problem: Let a, b, c be nonnegative real numbers such that $a+b+c=3$ and $c=\min\{a,b,c\}$. Prove that

$$\sum_{cyc} \frac{a^2}{b+c} + 2(ab+bc+ca) \geq \frac{15}{2} + \left(\frac{a-b}{4}\right)^2.$$

Solution.

lemma:

If $a, b, c \geq 0$ and $c = \min\{a,b,c\}$ then.

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + 2ab \geq \frac{(a+b)^2}{a+b+2c} + \frac{(a+b)^2}{2} + \frac{(a-b)^2}{16}.$$

Or

$$\frac{(a-b)^2(a+b+c)^2}{(a+c)(b+c)(a+b+2c)} \geq \frac{9(a-b)^2}{16}.$$

By the AG inequality, we have

$$(a+c)(b+c)(a+b+2c) \leq \frac{(3+c)^3}{4} \leq 16.$$

Thus, the proof is completed. Therefore, it suffices to show

$$\frac{(3-c)^2}{3+c} + \frac{(a+b)^2}{2} + \frac{c^2}{3-c} + 2c(3-c) \geq \frac{15}{2}.$$

Or

$$\frac{3c^2(c-1)^2}{2(9-c^2)} \geq 0.$$

The equality occurs for $a=b=c$, and for $c=0$ and $b=c$ ■

Problem: Let a, b, c be nonnegative real numbers such that $a+b+c=3$ and $c=\min\{a,b,c\}$. Prove

$$3(a^4+b^4+c^4)+33 \geq 14(a^2+b^2+c^2)+2(a-b)^2$$

Solution.

lemma:

If $a, b, c \geq 0$ and $c=\min\{a,b,c\} \rightarrow a+b \geq 2$, then.

$$3(a^4+b^4)-14(a^2+b^2) \geq \frac{3(a+b)^4}{8}-7(a+b)^2+2(a-b)^2.$$

Or

$$\frac{3(a-b)^2(7a^2+ab+7b^2)}{8} \geq 9(a-b)^2.$$

We have

$$\frac{3(7a^2+ab+7b^2)}{8}=\frac{3\left[(a-b)^2+6(a+b)^2\right]}{8} \geq 9.$$

Thus, the proof is completed. Therefore, it suffices to

$$\frac{3(3-c)^4}{8}+3c^4+33 \geq 14c^2+7(3-c)^2,$$

Which is

$$\frac{3(c-1)^2(3c+1)^2}{8} \geq 0.$$

The equality occurs for $a=b=c=1$, ■

Problem: Let a, b, c be nonnegative real numbers such that $a + b + c = 3$ and $c = \max\{a, b, c\}$. Prove

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{2(ab+bc+ca)}{9} \geq \frac{13}{6} + \left(\frac{a-b}{12}\right)^2$$

Solution.

lemma:

If $a, b, c \geq 0$ and $c = \max\{a, b, c\} \Rightarrow a + b \leq 2$

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{2ab}{9} \geq \frac{4}{a+b+2} + \frac{(a+b)^2}{18} + \frac{(a-b)^2}{144}.$$

Or

$$\frac{(a-b)^2}{(a+1)(b+1)(a+b+2)} \geq \frac{(a-b)^2}{16}.$$

Or

$$(a+1)(b+1)(a+b+2) \leq \frac{(a+b+2)^3}{4} \leq 16.$$

Thus, the proof is completed. Therefore, it suffices to

$$\frac{4}{5-c} + \frac{(3-c)^2}{18} + \frac{1}{c+1} + \frac{2c(3-c)}{9} \geq \frac{13}{6}.$$

Or

$$\frac{(c-1)^2(c-2)^2}{6(5-c)(c+1)} \geq 0.$$

The equality occurs for $a = b = c = 1$, and for $c = 2$ and $a = b = \frac{1}{2}$. ■

Problem: Let a, b, c be positive real numbers such that $a+b+c=1$ and $c=\max\{a,b,c\}$. Prove

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+48(ab+bc+ca)\geq 25+\frac{3(a-b)^2}{2}$$

lemma:
If $a,b,c>0$ and $c=\max\{a,b,c\}\Rightarrow a+b\leq\frac{2}{3}$ then

$$\frac{1}{a}+\frac{1}{b}+48ab\geq\frac{4}{a+b}+12(a+b)^2+\frac{3(a-b)^2}{2}.$$

Or

$$\frac{(a-b)^2}{ab(a+b)}\geq\frac{27(a-b)^2}{2}.$$

Thus, the proof is completed, because

$$ab(a+b)\leq\frac{(a+b)^3}{4}\leq\frac{2}{27}.$$

T herefore, it suffices to

$$\frac{4}{1-c}+12(1-c)^2+\frac{1}{c}+48c(1-c)\geq 25.$$

Or

$$\frac{(3c-1)^2(2c-1)^2}{c(1-c)}\geq 0.$$

The equality occurs for $a=b=c=\frac{1}{3}$, and for $c=\frac{1}{2}$ and $a=b=\frac{1}{4}$. ■

Example: Schur. Let a,b,c be nonnegative real numbers, then

$$a^3+b^3+c^3+3abc\geq ab(a+b)+bc(b+c)+ca(c+a)$$

Example: khanhsy. Let a,b,c,m be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\sum_{cyc}\frac{a^2}{a+2}+\frac{64(ab+bc+ca)}{243}\geq\frac{145}{81}.$$

Example: khanhsy. Let a, b, c, m be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\sum_{cyc} \frac{1}{a+4} + \frac{16(ab+bc+ca)}{1125} \geq \frac{241}{375}.$$

Example: khanhsy. Let a, b, c, m be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\sum_{cyc} \frac{a^2}{a+m} + \frac{48m^2}{(3m+3)^3}(ab+bc+ca) \geq \frac{25m^2+18m+9}{3(m+1)^3}.$$

Example: Jack Garfunkel. Let $a, b, c,$ be nonnegative real numbers, then

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2.$$

Example: AoPs. Let $a, b, c,$ be be positive real numbers such that $abc=1$ and. Prove that

$$a^2+b^2+c^2+6 \geq \frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Example: Vasile Cirtoaje. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$8\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+9 \geq 10(a^2+b^2+c^2)$$

Problem: Let a, b, c be positive real numbers. Prove that

$$\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+a}\right)^2 \geq \frac{1}{2}\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right)$$

Solution.
Let

$$\begin{cases} \dfrac{a}{a+b} = \dfrac{x+1}{2} \\ \dfrac{b}{b+c} = \dfrac{y+1}{2} \\ \dfrac{c}{c+a} = \dfrac{z+1}{2} \end{cases}$$

Where $x, y, z \in (-1; 1)$ and$x+y+z+xyz=0$This inequality is equivalent to

$$x^2+y^2+z^2+3(x+y+z) \geq 0$$

Or

$$x^2+y^2+z^2-3xyz \geq 0$$

By virtue of the Am-GM, we have

$$x^2+y^2+z^2+-xyz \geq 3\sqrt[3]{(xyz)^2}-3xyz \geq 3|xyz|-3xyz \geq 0$$

The equation occurs $x=y=z=0$ or $a=b=c$ ■. **Problem:** Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{3a^2+a}{(1+a)^2}+\frac{3b^2+b}{(1+b)^2}+\frac{3c^2+c}{(1+c)^2} \geq 3$$

Solution.
Let

$$\begin{cases} a = \dfrac{1-x}{1+x} \\ b = \dfrac{1-y}{1+y} \\ c = \dfrac{1-z}{1+z} \end{cases}$$

Where $x, y, z \in (-1; 1)$. Since $abc=1$ involves $x+y+z+xyz=0$ This inequality is equivalent to

$$\frac{x^2-3x+2}{2}+\frac{y^2-3y+2}{2}+\frac{z^2-3z+2}{2} \geq 3$$

Or

$$x^2+y^2+z^2+3xyz \geq 0$$

By virtue of the Am-GM, we have

$$x^2+y^2+z^2+3xyz \geq 3\sqrt[3]{(xyz)^2}+3xyz \geq 3|xyz|+3xyz \geq 0$$

The equation occurs $x=y=z=0$ or $a=b=c$. ■

Example: From Mathematics and Youth Magazine in VietNam. If a, b, c are positive real numbers, then

$$\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+a}\right)^2 + 3 \geq \frac{5}{2}\left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}\right)$$

Example: Vasile Cirtoaje. If a, b, c are positive real numbers such that $abc = 1$. Prove that

$$\left(1 + \frac{4a}{a+b}\right)^2 + \left(1 + \frac{4b}{b+c}\right)^2 + \left(1 + \frac{4c}{c+a}\right)^2 \geq 27$$

Example: Pham Van Thuan. If a, b, c are positive real numbers, then

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \geq 1$$