SOLUTION TO PROBLEM JP.059 FROM ROMANIAN MATHEMATICAL MAGAZINE, NUMBER 4, SPRING 2017

MARIN CHIRCIU

JP.059. Let a, b, c be the side lengths of a triangle ΔABC with inradius r. Prove that:

$$\frac{1}{a^3}\tan\frac{A}{2}+\frac{1}{b^3}\tan\frac{B}{2}+\frac{1}{c^3}\tan\frac{C}{2}\le\frac{R}{48r^4}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.
The triplets $\left(\frac{1}{a^2},\frac{1}{b^2},\frac{1}{c^2}\right)$ and $\left(\frac{1}{a}\tan\frac{A}{2},\frac{1}{b}\tan\frac{B}{2},\frac{1}{c}\tan\frac{C}{2}\right)$ are reversed ordered. With Cebyshev's inequality we obtain

$$\sum\frac{1}{a^3}\tan\frac{A}{2}=\sum\Big(\frac{1}{a^2}\cdot\frac{1}{a}\tan\frac{A}{2}\Big)\le\frac{1}{3}\cdot\sum\frac{1}{a^2}\cdot\sum\frac{1}{a}\tan\frac{A}{2}\le\frac{1}{3}\cdot\frac{1}{4r^2}\cdot\frac{p^2+(4R+r)^2}{4Rp^2}=$$

$$=\frac{p^2+(4R+r)^2}{48p^2R}\le\frac{R}{48r^4},$$

where the last inequality is equivalent with

$$r^2\Big[p^2+(4R+r)^2\Big]\le p^2R^2\Leftrightarrow p^2(R^2-r^2)\ge r^2(4R+r)^2,$$

which follows from Gerretsen's inequality $p^2\ge16Rr-5r^2$. It remains to prove that

$$(16Rr-5r^2)(R^2-r^2)\ge r^2(4R+r)^2\Leftrightarrow$$

$$\Leftrightarrow16R^3-21R^2r-24Rr^2+4r^3\ge0\Leftrightarrow(R-2r)(16R^2+11Rr-2r^2)\ge0,$$

obviously from Euler's inequality $R\ge2r$.
We used the known inequality in triangle $\sum\frac{1}{a^2}\le\frac{1}{4r^2}$.

□

Remark
The inequality can be strengthen.

1) Prove that in any triangle the following inequality holds:

$$\frac{1}{a^3}\tan\frac{A}{2}+\frac{1}{b^3}\tan\frac{B}{2}+\frac{1}{c^3}\tan\frac{C}{2}\le\frac{9R}{16S^2}.$$

Proof.
The triplets $\left(\frac{1}{a^2},\frac{1}{b^2},\frac{1}{c^2}\right)$ and $\left(\frac{1}{a}\tan\frac{A}{2},\frac{1}{b}\tan\frac{B}{2},\frac{1}{c}\tan\frac{C}{2}\right)$ are reversed ordered. With Cebyshev's inequality we obtain

$$\sum\frac{1}{a^3}\tan\frac{A}{2}=\sum\Big(\frac{1}{a^2}\cdot\frac{1}{a}\tan\frac{A}{2}\Big)\le\frac{1}{3}\cdot\sum\frac{1}{a^2}\cdot\sum\frac{1}{a}\tan\frac{A}{2}\le$$

$$\le \frac{1}{3}\cdot\frac{1}{4r^2}\cdot\frac{p^2+(4R+r)^2}{4Rp^2}=\frac{p^2+(4R+r)^2}{48p^2R}\le\frac{9R}{16r^2p^2},$$

where the last inequality is equivalent with $p^2+(4R+r)^2\le 27R^2$, which follows from Gerretsen's inequality $p^2\le 4R^2+4Rr+3r^2$.
It remains to prove that

$$4R^2+4Rr+3r^2+(4R+r)^2\le 27R^2\Leftrightarrow$$

$$\Leftrightarrow 7R^2-12Rr-4r^2\ge 0\Leftrightarrow (R-2r)(7R+2r)\ge 0,$$

obviously form Euler's inequality $R\ge 2r$.
We used the known inequality in triangle $\sum\frac{1}{a^2}\le\frac{1}{4r^2}$.

□

Remark
Inequality **1)** is stronger then inequality **JP.059.**
Inequality **1)** can itself be strengthened.

2) Prove that in any triangle the following inequality holds:

$$\frac{1}{a^3}\tan\frac{A}{2}+\frac{1}{b^3}\tan\frac{B}{2}+\frac{1}{c^3}\tan\frac{C}{2}\le\frac{1}{12Rr^2}.$$

Marin Chirciu - Romania

Proof.
The triplets $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ and $\left(\frac{1}{a^2}\tan\frac{A}{2},\frac{1}{b^2}\tan\frac{B}{2},\frac{1}{c^2}\tan\frac{C}{2}\right)$ are reversed ordered. With Cebyshev's inequality we obtain

$$\sum\frac{1}{a^3}\tan\frac{A}{2}=\sum\left(\frac{1}{a}\cdot\frac{1}{a^2}\tan\frac{A}{2}\right)\le\frac{1}{3}\cdot\sum\frac{1}{a^2}\tan\frac{A}{2}\le\frac{1}{3}\cdot\frac{p}{3Rr}\cdot\frac{3}{4pr}=\frac{1}{12Rr^2},$$

where the last inequality follows from the known inequality in triangle $\sum\frac{1}{a}\le\frac{p}{3Rr}$ and $\sum\frac{1}{a^2}\tan\frac{A}{2}\le\frac{3}{3pr}$, true from: □

2a) Prove that in any triangle *ABC* the following inequality holds:

$$\frac{1}{a^2}\tan\frac{A}{2}+\frac{1}{b^2}\tan\frac{B}{2}+\frac{1}{c^2}\tan\frac{C}{2}\le\frac{\sqrt{3}}{6Rr}.$$

Proof.
The triplets $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ and $\left(\frac{1}{a}\tan\frac{A}{2},\frac{1}{b}\tan\frac{B}{2},\frac{1}{c}\tan\frac{C}{2}\right)$ are reversed ordered. With Cebyshev's inequality we obtain

$$\sum\frac{1}{a^2}\tan\frac{A}{2}=\sum\left(\frac{1}{a}\cdot\frac{1}{a}\tan\frac{A}{2}\right)\le\frac{1}{3}\cdot\sum\frac{1}{a}\cdot\sum\frac{1}{a}\tan\frac{A}{2}\le$$

$$\le\frac{1}{3}\cdot\frac{p}{3Rr}\cdot\frac{p^2+(4R+r)^2}{4Rp^2}=\frac{p^2+(4R+r)^2}{36pR^2r}\le\frac{\sqrt{3}}{6Rr},$$

where the last inequality is equivalent with $p^2+(4R+r)^2\le 6R\cdot p\sqrt{3}$, which follows from Gerretsen's inequality $p^2\le 4R^2+4Rr+3r^2$ and Doucet's inequality $4R+r\ge p\sqrt{3}$.
It remains to prove that

$$4R^2+4Rr+3r^2+(4R+r)^2\le 6R\cdot(4R+r)\Leftrightarrow$$

$$\Leftrightarrow 2R^2 - 3Rr - 2r^2 \geq 0 \Leftrightarrow (R - 2r)(2R + r) \geq 0,$$

obviously from Euler's inequality $R \geq 2r$.
We used the known inequality in triangle $\sum \frac{1}{a} \leq \frac{p}{3Rr}$. □

2b) Prove that in any triangle ABC the following inequalities holds

$$\frac{1}{a^2}\tan\frac{A}{2} + \frac{1}{b^2}\tan\frac{B}{2} + \frac{1}{c^2}\tan\frac{C}{2} \leq \frac{\sqrt{3}}{6Rr} \leq \frac{3}{4pr} \leq \frac{\sqrt{3}}{12r^2}.$$

Proof.
We use **2a)** and Mitrinovic's inequalities $3r\sqrt{3} \leq p \leq \frac{3R\sqrt{3}}{2}$.

□

Remark.
Inequality **2)** is stronger then inequality **1)**, which in turn is stronger then **JP.059.**

3) Prove that in any triangle the following inequality holds:

$$\frac{1}{a^3}\tan\frac{A}{2} + \frac{1}{b^3}\tan\frac{B}{2} + \frac{1}{c^3}\tan\frac{C}{2} \leq \frac{1}{12Rr^2} \leq \frac{9R}{16S^2} \leq \frac{R}{48r^2}.$$

Proof.
See **2)** and Mitrinovic's inequalities $27r^2 \leq p^2 \leq \frac{27R^2}{4}$.

□

To each of the above inequalities the equality holds if and only if the triangle is equilateral.

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.
E-mail address: dansitaru63@yahoo.com