Math Adventures on CutTheKnot Math 51-100

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MATH ADVENTURES
ON
CutTheKnotMath

51 - 100

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51. An Inequality from RMM with a Generic 5

Prove that, for $a, b, c > 0$,

$$\sqrt[3]{(2a+5)(2b+5)(2c+5)} \geq \frac{6abc}{ab+bc+ca} + 5$$

Proposed by Daniel Sitaru - Romania

Proof:
By *Hölder's inequality,*

$$\begin{aligned} 2\sqrt[3]{abc} + 5 &= \sqrt[3]{8abc} + \sqrt[3]{125} \leq \sqrt[3]{2a+5}\sqrt[5]{2b+5}\sqrt[5]{2c+5} \\ &= \sqrt[3]{(2a+5)(2b+5)(2c+5)} \end{aligned}$$

On the other hand, $2\sqrt[3]{abc} \geq \frac{6abc}{ab+bc+ca}$. Indeed, by the ***AM - GM inequality***, the latter is equivalent to

$$2\sqrt[3]{abc}(ab+bc+ca) \geq 2\sqrt[3]{abc}\left(3\sqrt[3]{(abc)^2}\right) = 6abc$$

$\square$

Acknowledgment (by Alexander Bogomolny - USA)

Daniel Sitaru has kindly posted the above problem from the ***Romanian Mathematical Magazine***, with two practically identical proofs - one by Kevin Soto Palacios (Peru), the other Pham Quy (Vietnam), at the ***CutTheKnotMath page***.

52. An Inequality from RMM with Powers of 2

Prove that, for $x, y, z > 0$

$$2^x + 2^y + 2^z + 2^{x+y+z} > \sqrt[x+y]{16^{xy}} + \sqrt[y+z]{16^{yz}} + \sqrt[z+x]{16^{zx}} + 1$$

Proposed by Daniel Sitaru - Romanian

Proof (by Ravi Prakash - India):
Note that

$$\begin{aligned} &2^x + 2^y + 2^z + 2^{x+y+z} - 2^{x+y} - 2^{y+z} - 2^{z+x} - 1 \\ &= (2^x - 1)(2^y - 1)(2^z - 1) > 0, \end{aligned}$$

implying that

$$2^x + 2^y + 2^z + 2^{x+y+z} > 2^{x+y} + 2^{y+z} + 2^{z+x} + 1.$$

But

$$2^{x+y} = 4^{\frac{x+y}{2}} \geq 4^{\frac{2xy}{x+y}} = \sqrt[x+y]{16^{xy}}$$

The required inequality follows from the above by cycling through the pairs (y, z) and (z, x) then adding. $\square$

Acknowledgment (by Alexander Bogomolny - USA)
Daniel Sitaru has kindly posted the above problem from the **Romanian Mathematical Magazine**, with a proof by Ravi Prakash (India), at the ***CutTheKnotMath page***.

53. An Inequality in Acute Triangle, Courtesy of Ceva's Theorema

Let, in $\Delta ABC, a, b, c, AA', BB', CC'$ be the altitudes; AA'', BB'', CC'' the angle bisectors, and AA''', BB''', CC'' the symmedians.
Then

$$AB' \cdot BC' \cdot CA' + AB'' \cdot BC'' \cdot CA'' + AB''' \cdot BC''' \cdot CA''' \le \frac{3}{8}abc.$$

Proposed by Daniel Sitaru - Romania

Proof (by Daniel Sitaru - Romania):
Lemma:

*Assume that in an acute $\Delta ABC, AA_0, BB_0, CC_0$ are concurrent **cevians.** Then*

$$8 \cdot BA_0 \cdot CB_0 \cdot AC_0 \le abc.$$

For convenience, denote $BA_0 = x_1, A_0C = y_1, CB_0 = x_2, B_0A = y_2, AC_0 = x_3$, $C_0B = y_3$. Then by ***Ceva's theorem***,

$$\frac{x_1}{y_1} \cdot \frac{x_2}{y_2} \cdot \frac{x_3}{y_3} = 1.$$

We have to prove that $8x_1x_2x_3 \le (x_1 + y_1)(x_2 + y_2)(x_3 + y_3)$. In other words, we need to show that

$$\frac{x_1 + y_1}{x_1} \cdot \frac{x_2 + y_2}{x_2} \cdot \frac{x_3 + y_3}{x_3} \ge 8,$$

or,

$$\left(1 + \frac{y_1}{x_1}\right) \cdot \left(1 + \frac{y_2}{x_2}\right) \cdot \left(1 + \frac{y_3}{x_3}\right) \ge 8,$$

Multiplying out, this is reduced to

$$1 + \frac{y_1}{x_1} + \frac{y_2}{x_2} + \frac{y_3}{x_3} + \frac{y_1y_2}{x_1x_2} + \frac{y_2y_3}{x_2x_3} + \frac{y_3y_1}{x_3x_1} + \frac{y_1y_2y_3}{x_1x_2x_3} \ge 8$$

Making multiple uses of Ceva's theorem, this is equivalent to

$$1 + \frac{y_1}{x_1} + \frac{y_2}{x_2} + \frac{y_3}{x_3} + \frac{x_3}{y_3} + \frac{x_1}{y_1} + \frac{x_2}{y_2} + 1 \ge 8$$

and, in turn, to

$$\left(\frac{y_1}{x_1} + \frac{x_1}{y_1}\right) + \left(\frac{y_2}{x_2} + \frac{x_2}{y_2}\right) + \left(\frac{y_3}{x_3} + \frac{x_3}{y_3}\right) \ge 6$$

which is true by the ***AM - GM inequality*** applied thrice.
Since the triples of angle bisectors, altitudes, and symmedians are all ***concurrent cevians,*** we may apply the lemma to each triple:

$$8 \cdot AB' \cdot BC' \cdot CA' \le abc$$
$$8 \cdot AB'' \cdot BC'' \cdot CA'' \le abc$$
$$8 \cdot AB''' \cdot BC''' \cdot CA''' \le abc$$

Adding up give the desired result. □

Aknowledgment (by Alexander Bogomolny - USA)
The inequality with the solution has been posted by Daniel Sitaru at the ***CutTheKnotMath page***.

54. Power and Fractions Inequality

Prove that, for $a, b, c > 0$,

$$\sum_{cycl} \frac{a^3b^3}{c^5} \geq \sum_{cycl} \frac{ab}{c}$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Ravi Prakash - India):
By the ***AM - GM inequality***,

$$\frac{a^3c^3}{b^5} + \frac{ba}{c} + \frac{cb}{a} \geq \left(\frac{a^3c^3}{b^5} \cdot \frac{ba}{c} \cdot \frac{cb}{a}\right)^{\frac{1}{3}} = \frac{3ac}{b}.$$

Similarly, $\frac{b^3a^3}{c^5} + \frac{ac}{b} + \frac{cb}{a} \geq \frac{3ba}{c}$ and $\frac{c^3b^3}{a^5} + \frac{ac}{b} + \frac{ba}{c} \geq \frac{3cb}{a}$. Adding up,

$$\sum_{cycl} \frac{a^3c^3}{b^5} + 2\sum_{cycl} \frac{ba}{c} \geq 3\sum_{cycl} \frac{ba}{c}$$

which directly proves the required inequality. □

Proof 2 (by Lâm Phan - Vietnam).
By the ***AM - GM inequality***,

$$2\sum_{cycl} \frac{a^3c^3}{b^5} = \left(\frac{a^3c^3}{b^5} + \frac{b^3a^3}{c^5}\right) + \left(\frac{b^3a^3}{c^5} + \frac{c^3b^3}{a^5}\right) + \left(\frac{c^3b^3}{a^5} + \frac{a^3c^3}{b^5}\right)$$

$$\geq 2\frac{a^3}{bc} + 2\frac{b^3}{ca} + 2\frac{c^3}{ab} =$$

$$= \left(\frac{a^3}{bc} + \frac{b^3}{ca}\right) + \left(\frac{b^3}{ca} + \frac{c^3}{ab}\right) + \left(\frac{c^3}{ab} + \frac{a^3}{bc}\right) \geq 2\frac{ab}{c} + 2\frac{bc}{a} + 2\frac{ca}{2} = 2\sum_{cycl} \frac{ac}{b}$$

□

Proof 3 (by Soumava Chakraborty - India).
The given inequality is equivalent to

$$\sum_{cycl} a^8b^8 \geq a^4b^4c^4 \sum_{cycl} a^2b^2.$$

Let $a^2b^2 = x, b^2c^2 = y, c^2a^2 = z, x, y, z > 0$. We need to prove that

(1) $$x^4 + y^4 + z^4 \geq xyz(x + y + z)$$

Schur's inequality for $t = 2$ gives

(a) $$x^4 + y^4 + z^4 + xyz(x + y + z) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2)$$

Now, $x^2 + y^2 \geq 2xy$, etc., so that

(2) $$xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$$

Further,

$$2(x^2y^2 + y^2z^2 + z^2x^2) - 2xyz(x + y + z) =$$
$$= (xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2 \geq 0,$$

implying

(3) $$2(x^2y^2 + y^2z^2 + z^2x^2) \geq 2xyz(x + y + z)$$

(2) and (3) give

(b) $$xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\ge 2xyz(x+y+z)$$

(a) and (b) add up to the required (1) □

Proof 4 (by Rory Tarnow - Mordi).

$$4\sum_{cycl}\frac{a^3b^3}{c^5}=\sum_{cycl}\Big(2\frac{a^3b^3}{c^5}+\frac{b^3c^3}{a^5}+\frac{c^3a^3}{b^5}\Big)\ge$$
$$\ge\sum_{cycl}4\Big(\frac{a^3b^3}{c^5}\cdot\frac{a^3b^3}{c^5}\cdot\frac{b^3c^3}{a^5}\cdot\frac{c^3a^3}{b^5}\Big)^{\frac{1}{4}}=4\sum_{cycl}\frac{ab}{c}.$$

□

Proof 5 (by Alexander Bogomolny - USA).
As in Proof 3, the required inequality is reduced to

$$\sum_{cycl}a^8b^8\ge a^4b^4c^4\sum_{cycl}a^2b^2.$$

By ***Chebysev's inequality***,

$$3\sum_{cycl}a^8b^8\ge\sum_{cycl}a^6b^6\sum_{cycl}a^2b^2.$$

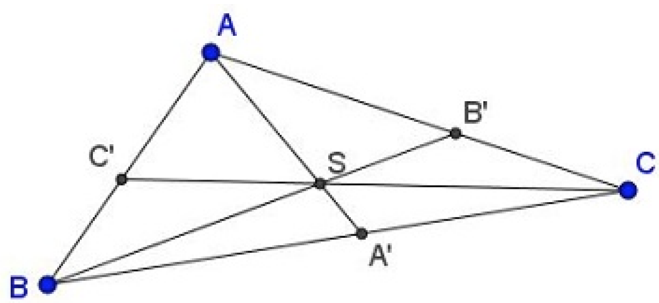
But, by the ***AM - GM inequality***,

$$\sum_{cycl}a^6b^6\ge 3(a^{12}b^{12}c^{12})^{\frac{1}{3}}=3a^4b^4c^4.$$

Acknowledgment (by Alexander Bogomolny - USA):
Daniel Sitaru has kindly posted the above problem at the ***CutTheKnotMath page***, along with several solutions. The problems comes from his book *Math Accent.*
□

55. An Inequality for the Cevians through Spieker Point via Brocard Angle

Let AA', BB', CC' be the cevians through the Spieker point in ΔABC.



Then

$$a^2b^2+b^2c^2+c^2a^2\ge 2s(AC'\cdot BA'\cdot CB'+AB'\cdot BC'\cdot CA')$$

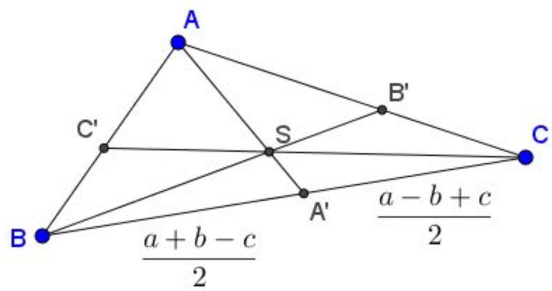
where s is the semiperimeter of ΔABC and a,b,c are its side lengths.

Proposed by Daniel Sitaru - Romania

Proof (by Alexander Bogomolny - USA).
The Spieker point can be characterised in several ways. One of these is a ***point where*** the three ***triangle's cleavers*** intersect. Thus, in particular,

$$AB + BA' = AC + CA' = \frac{a+b+c}{2} = s,$$

i.e., $BA' = \frac{a+b-c}{2}$ and $A'C = \frac{a-b+c}{2}$.



Similarly we can calculate the remaining four segments. To sum up,

$$A'B = B'A = \frac{a+b-c}{2},$$

$$A'C = C'A = \frac{a-b+c}{2},$$

$$B'C = C'B = \frac{-a+b+c}{2}.$$

There is a well known expression involving the ***Brocard angle*** ω of ΔABC:

$$\sin^2\omega = \frac{(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}{4(a^2b^2+b^2c^2+c^2a^2)}.$$

Using that and $1 > \sin^2\omega$, we obtain

$$1 > \frac{2B'C \cdot 2C'A \cdot 2A'B \cdot 2s}{4(a^2b^2+b^2c^2+c^2a^2)} = \frac{4s \cdot B'C \cdot C'A \cdot A'B}{a^2b^2+b^2c^2+c^2a^2},$$

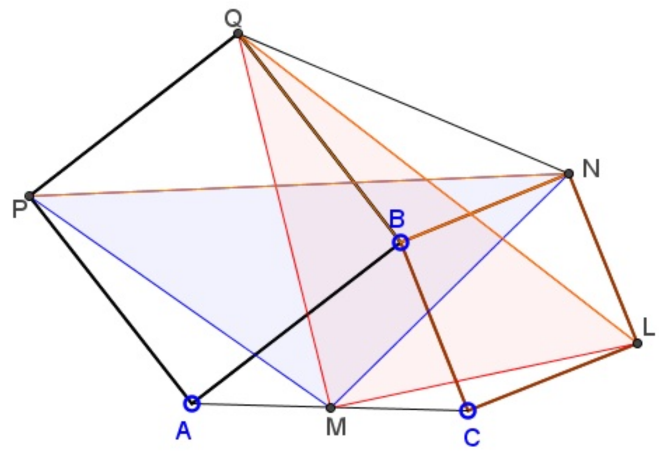
$$1 > \frac{4s \cdot A'C \cdot C'B \cdot B'A}{a^2b^2+b^2c^2+c^2a^2}.$$

Summing up leads to the desired result.

□

56. Sanchez's Areas in Bottema's Configuration

$ABQP$ and $CBNL$ are two squares sharing a vertex. M is the midpoint of AC.

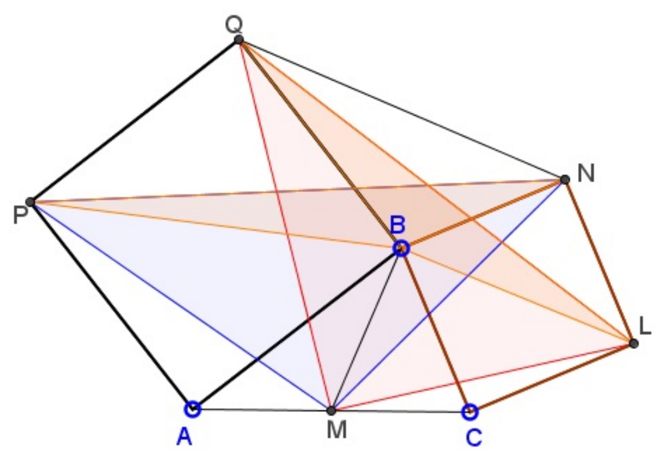


Prove that $[\Delta MPN] = [\Delta MLQ]$, where $[F]$ denotes the area of shape F.

Proposed by Miguel Ochoa Sanchez - Peru

Proof (by Leonard Giugiuc - Romania).
Note that $\Delta MPN = \Delta BPN \cup \Delta MBP \cup \Delta MBN$ and

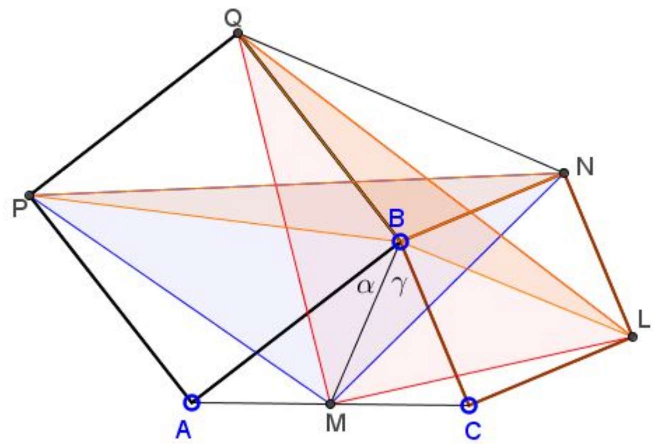
$$\Delta MLQ = \Delta BLQ \cup \Delta MBL \cup \Delta MBQ :$$



Let's set $BC = c, AB = a$. Observe that $\angle PBN = QBN + 45^\circ = \angle QBL$. Denote this angle as ω. Then

$$[\Delta BPN] = \frac{1}{2}(a\sqrt{2}) \cdot c \cdot \sin\omega = \frac{1}{2}a \cdot (c\sqrt{2}) \cdot \sin\omega = [\Delta BLQ].$$

Introduce angles α and γ as shown:



Then, since $[\Delta ABM] = [\Delta CBM]$,

$$a \sin \alpha = c \sin \gamma$$

Using that,

$$2([\Delta MBP] + [\Delta MBN]) = BM \cdot (BP \cdot \sin \angle MBP + BN \cdot \sin \angle MBN)$$
$$= BM \cdot \left(a\sqrt{2} \sin(\alpha + 45^\circ) + c \cdot \sin(\gamma + 90^\circ) \right)$$
$$= BM \cdot (a \sin \alpha + a \cos \alpha + c \cos \gamma) = BM \cdot (c \sin \gamma + a \cos \alpha + c \cos \gamma)$$
$$= BM \cdot (\cos \alpha + c \sin \gamma + c \cos \gamma) = BM \cdot \left(a \sin(\alpha + 90^\circ) + c\sqrt{2} \sin(\gamma + 45^\circ) \right)$$
$$= BM \cdot (BQ \cdot \sin \angle MBQ + BL \cdot \sin \angle MBL) = 2([\Delta MBQ] + [\Delta MBL]),$$

which proves the require $[\Delta MPN] = [\Delta MLQ]$. $\square$

57. Non square Matrix as a Tool for Proving an Inequality

Let a, b, c be non-negative. Prove that

$$2(a + b + c)(a + 3b + 3c) \geq \left(\sqrt{b(a + b)} + 2\sqrt{c(b + c)} + \sqrt{a(c + a)} \right)^2$$

Proposed by Daniel Sitaru, Leonard Giugiuc - Romania

Proof (by Daniel Sitaru, Leonard Giugiuc - Romania).
Define matrix

$$A = \begin{pmatrix} \sqrt{a+b} & \sqrt{b+c} & \sqrt{a} & \sqrt{c} \\ \sqrt{b} & \sqrt{c} & \sqrt{a+c} & \sqrt{b+c} \end{pmatrix}. \text{ We have } A \in M_{4,2}(\mathbb{R}).$$

Further

$$AA^t = \begin{pmatrix} a+b+b+c+a+c & \sqrt{a(a+b)} + 2\sqrt{c(b+c)} + \sqrt{a(a+c)} \\ \sqrt{a(a+b)} + 2\sqrt{c(b+c)} + \sqrt{a(a+c)} & a+b+b+c+a+c \end{pmatrix}$$

$AA^t \in M_2(\mathbb{R})$. By the ***Cauchy - Binet theorem***, $\det(AA^t) \geq 0$. More explicitly,

$$AA^t = \begin{pmatrix} 2a+2b+2c & \sqrt{a(a+b)} + 2\sqrt{c(b+c)} + \sqrt{a(a+c)} \\ \sqrt{a(a+b)} + 2\sqrt{c(b+c)} + \sqrt{a(a+c)} & 2a+2b+2c \end{pmatrix}$$

whereas,

$$\det(AA^t) = (2a+2b+2c)(a+2b+3c) - \left(\sqrt{b(a+b)} + 2\sqrt{c(b+c)} + \sqrt{a(c+a)}\right)^2.$$

Or, else,

$$\det(AA^t) = 2(a+b+c)\left((a+2b+3c) - \left(\sqrt{b(a+b)} + 2\sqrt{c(b+c)} + \sqrt{a(c+a)}\right)\right)^2 \geq 0$$

□

58. An Inequality with Determinants V

With a, b, c the sides and s the semiperimeter of ΔABC, prove that

$$\Delta = \begin{vmatrix} s & \frac{a^2b}{a^3+b} & \frac{b^2c}{b^3+c} & \frac{c^2a}{c^3+a} \\ \frac{a^2b}{a^3b} & s & \frac{c^2a}{c^3+a} & \frac{b^2c}{b^3+c} \\ \frac{b^2c}{b^3+c} & \frac{c^2a}{c^3+a} & s & \frac{a^2b}{a^3+b} \\ \frac{c^2a}{c^3+a} & \frac{b^2c}{b^3+c} & \frac{a^2b}{a^3+b} & s \end{vmatrix} \geq 0$$

Equality is only achieved for $a = b = c = 1$.

Proposed by Daniel Sitaru - Romania

Lemma:

For $x, y, z, t \in \mathbb{R}$,

$$\Delta' = \begin{vmatrix} x & y & z & t \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{vmatrix}$$

$$= (x+y+z+t)(x-y+z-t)(x+y-z-t)(x-y-z+t)$$

Proof 1 of Lemma (by Leonard Giugiuc - Romania).

We shall compute the determinant first using row and/or column operations, describing each step symbolically next to the determinant the operation applied to:

$$\Delta' = \begin{vmatrix} x & y & z & t \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{vmatrix}^{(r_1 =: r_1+r_2+r_3+r_4)}$$

$$= (x+y+z+t)\begin{vmatrix} 1 & 1 & 1 & 1 \\ y & x & t & z \\ z & t & x & y \\ t & z & y & z \end{vmatrix}^{(c_4 =: c_4-c_3+c_2-c_1)}$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} 1 & 1 & 1 & 0 \\ y & x & t & 1 \\ z & t & x & -1 \\ t & z & y & 1 \end{vmatrix}^{(c_2 =: c_2-c_1, c_3 =: c_3-c_1)}$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} 1 & 0 & 0 & 0 \\ y & x-y & t-y & 1 \\ z & t-z & x-z & -1 \\ t & z-t & y-t & 1 \end{vmatrix}$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} x-y & t-y & 1 \\ t-z & x-z & -1 \\ z-t & y-t & 1 \end{vmatrix}^{(r_1=:r_1+r_2, r_3=:r_3+r_2)}$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} x-y+t-z & x-y+t-z & 0 \\ t-z & x-z & -1 \\ 0 & x+y-z-t & 0 \end{vmatrix}$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} x-y+t-z & x-y+t-z \\ 0 & x+y-z-t \end{vmatrix}$$

$$= (x+y+z+t)(x-y+z-t)(x-y+t-z)(x+y-z-t)$$

□

Proof 2 of Lemma (by Alexander Bogomolny - USA).
Note that the matrix in the lemma is defined block-wise, say

$$S = \begin{pmatrix} x & z & z & t \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where $A = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$ and $B = \begin{pmatrix} t & z \\ z & t \end{pmatrix}$. It is easily verifiable that the matrices commute: $AB = BA$, which allows for an application of Silvester's theorem, concerning the determinants of block - matrices. In our case,

$$\det(S) = \det(A^2 - B^2) = (x^2+y^2-t^2-z^2)^2 - [2(xy-tz)]^2$$
$$= (x^2+y^2-t^2-2xy+2tz)(x^2+y^2-t^2-z^2+2xy-2tz)$$
$$= \Big((x-y)^2-(t-z)^2\Big)\Big((x+y)^2-(t+z)^2\Big)$$

which is exactly the same expression as above.

Reference:

1. John R. Silvester, Determinants of Block Matrices, *The Mathematical Gazzete*, Vol. 84, No. 501 (Nov, 2000), pp. 460 - 467 □

Proof (by Daniel Sitaru, Leonard Giugiuc - Romania):
In the problem, let $x = s, y = \frac{a^2b}{a^3+b}, z = \frac{b^2c}{b^3+c}, t = \frac{c^2a}{c^3+a}$. We have, for example, by the ***AM - GM inequality***,

$$\frac{a^2b}{a^3+b} \le \frac{a^2b}{2a\sqrt{ab}} = \frac{ab}{2\sqrt{ab}} = \frac{\sqrt{ab}}{2} \le \frac{a+b}{4}.$$

For equality, we need $a^3 = b$ and also $a = b$, with the only feasible solution $a = b = 1$.
Similarly to the above, $\frac{b^2c}{b^3+c} \le \frac{b+c}{4}$ and $\frac{c^2a}{c^3+a} \le \frac{c+a}{4}$. From these we conclude that $y+z+t \le x$ which guarantees that all four factors in Lemma are nonnegative, making $\Delta' \ge 0$ and also $\Delta \ge 0$. The equality in $y+z+t \le x$ is achieved when $a = b = c = 1$. Otherwise, $\Delta > 0$. □

Acknowledgment (by Alexander Bogomolny - USA)
The inequality from the ***Romanian Mathematical Magazine*** has been shared at the ***CutTheKnotMath page*** by Daniel Sitaru. The problem and the solution

above are due to Daniel Sitaru and Leo Giugiuc. I reproduce that part here because of a lemma of a general character, interesting in its own right.

59. An Inequality with Determinants VI

Let $a, b, c > 0$, all distinct, and

$$\Delta_1 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a^2 & b^2 & c^2 \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \\ bc & ca & ab \end{vmatrix}$$

Prove that

$$\frac{\Delta_1 - \Delta_2}{(b-a)(a-c)(b-c)} \geq 12\sqrt[6]{(abc)^5}.$$

Proposed by Daniel Sitaru - Romania

Proof (by Ravi Prakash - India):

$$\Delta_1 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}^{r_1:r_1-r_2, r_2=r_2-r_3} = \begin{vmatrix} 0 & a-b & a^3-b^3 \\ 0 & b-c & b^3-c^3 \\ 1 & c & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a^2+ab+b^2 \\ 1 & b^2+bc+c^2 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

$$\Delta_2 = \begin{vmatrix} a^2 & b^2 & c^2 \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \\ bc & ca & ab \end{vmatrix}^{r_2:=r_1+r_2} = (a^2+b^2+c^2)\begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ bc & ca & ab \end{vmatrix}$$

$$= \frac{a^2+b^2+c^2}{abc}\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ abc & abc & abc \end{vmatrix} = (a^2+b^2+c^2)\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} =$$

$$= (a^2+b^2+c^2)\Delta_1$$

Thus, we have

$$\frac{\Delta_1 - \Delta_2}{(a-b)(b-c)(c-a)} = (a+b+c) + (a+b+c)(a^2+b^2+c^2)$$

$$3(abc)^{\frac{1}{3}} + 3(abc)^{\frac{1}{3}}(a^2+b^2+c^2) \geq 3(abc)^{\frac{1}{3}} \cdot 4(a^2b^2c^2)^{\frac{1}{4}} = 12(abc)^{\frac{5}{6}}$$

□

Acknowledgment (by Alexander Bogomolny - USA)
The inequality from his book "Ice Math" (Problem 026) has been kindly shared at the ***CutTheKnotMath page*** by Daniel Sitaru, along with a solution by Ravi Prakash.

60. Inequality in Quadrilateral

In a quadrilateral $ABCD$, with sides $AB = a, BC = b, CD = c, DA = d$, the following inequality holds

$$\sum_{cycl} \sqrt{a^2+b^2+c^2} > 2\sqrt{3 \cdot AC \cdot BD}.$$

Proposed by Daniel Sitaru - Romania

Proof (by Alexander Bogomolny- USA).
The ***Arithmetic Mean - Quadratic Mean inequality*** gives:

$$\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3},$$

implying $\sqrt{a^2+b^2+c^2} \geq \frac{a+b+c}{\sqrt{3}}$ and similarly, for other triples of the sides, such that, on adding up, we obtain

$$\sum_{cycl} \sqrt{a^2+b^2+c^2} \geq \frac{3(a+b+c+d)}{\sqrt{3}} = \sqrt{3}(a+b+c+d).$$

By the ***AM - GM inequality***, $a+c \geq 2\sqrt{ac}$ and $b+d \geq 2\sqrt{bd}$. Now, by the ***Ptolemy's inequality***,

$$(\sqrt{ac}+\sqrt{bc})^2 > ac+bd \geq AC \cdot BD,$$

so that $\sqrt{ac}+\sqrt{bd} > \sqrt{AC \cdot BD}$. Putting everything together shows that

$$\sum_{cycl} \sqrt{a^2+b^2+c^2} \geq \sqrt{3}(a+b+c+d) \geq 2\sqrt{3}(\sqrt{ac}+\sqrt{bd}) > 2\sqrt{3}\sqrt{AC \cdot BD}.$$

□

Acknowledgment (by Alexander Bogomolny - USA)
The problem, due to Daniel Sitaru, has been published in the ***Romanian Mathematical Magazine*** where more solutions can be found.

61. Cyclic Inequality with Logarithms

Let $a,b,c > 1$. Prove that

$$\ln(a^b \cdot b^c \cdot c^a) + 6\sum_{cycl} \frac{b(1+2a)}{1+4a+a^2} \geq 3(a+b+c).$$

Proposed by Daniel Sitaru - Romania

Solution 1 (by Leonard Giugiuc - Romania).
First we prove
Lemma

For $a,b \geq 1$,

$$b\ln a + \frac{6b(1+2a)}{a^2+4a+1} \geq 3b.$$

Indeed, set $f(b) = b\ln a + \frac{6b(1+2a)}{a^2+4a+1} - 3b$ on $[1,\infty)$.

$$f'(b) = \ln a + \frac{6(1+2a)}{a^2+4a+1} - 3.$$

Now let $g(a) = (a^2+4a+1)f'(b) = (a^2+4a+1)\ln a - 3a^2+3$, on $[1,\infty)$. We have

$$g'(a) = 2(a+2)\ln a - 5a + 4 + \frac{1}{a}.$$

$$g''(a) = 2\ln a - 3 + \frac{4}{a} - \frac{1}{a^2},$$

$$g'''(a) = \frac{2(a-1)^2}{a^3} \geq 0, a \geq 1$$

We deduce that $g''(a), g'(a), g(a)$ are all increasing for $a \geq 1$, implying that so is f and since $f(1) = \frac{g(a)}{a^2+4a+1} \geq 0$, the conclusion follows.
The other two inequalities are treated in a similar manner and then added to obtain the required inequality.

$$f'(x) < 0, \text{ for } x < 1, \text{ and } f'(x) > 0, \text{ for } x > 1.$$

$$\text{Since } f(1) = 0, f(x) \geq 0, \text{ for } x > 0.$$

□

Solution 2 (by Leonard Giugiuc -Romania).
First we prove
Lemma

$$\textit{Function } f(x) = \ln x + \frac{6(1+2x)}{x^2+4x+1} \textit{ is strictly increasing on } [0, \infty).$$

Indeed, $f'(x) = \frac{1}{x} - \frac{12(x^2+x+1)}{(x^2+4x+1)^2}$. Further,

$$f'(x) \geq 0 \Leftrightarrow [(x^2+x+1)+3x]^2 \geq 12x(x^2+x+1),$$

which is true by the ***AM - GM inequality*** for $u = x^2 + x + 1$ and $v = 3x$. Back to the problem: by the lemma, $f(x) \geq f(1) = 3, x \geq 1$.
Thus, $bf(a) + cf(b) + af(c) \geq 3(a+b+c)$, implying the required inequality. □

Acknowledgment (by Alexander Bogomolny - USA)
The problem above has been posted on the ***CutTheKnotMath page*** by Daniel Sitaru. Leonard Giugiuc submitted two solutions (Solution 1 and Solution 2); Soumitra Mandal submitted a solution, practically the same as Solution 2.

62. Beatty Sequences II

Assume r and s are two (strictly) irrational numbers that satisfy $\frac{1}{r} + \frac{1}{s} = 1$. Then the sequences $\{a_n\} = \{\lfloor nr \rfloor : n \in \mathbb{N}\}$ and $\{b_n\} = \{\lfloor ns \rfloor : n \in \mathbb{R}\}$ are complementary. In other words,

$$\{a_n\} \cup \{b_n\} = \mathbb{N} \textit{ and } \{a_n\} \cap \{b_n\} = \emptyset$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

This statement of Beatty's theorem (1926) one proof of which was published in 1927 and has been reproduced elsewhere at this site.

Below is a slight modification of the proof posted by Daniel Sitaru at the CutTheKnotMath page.

Proof (by Alexander Bogomolny - USA).
Assume to the contrary that there are integers n, m, q such that

$$q < mr < q+1$$

$$q < ns < q+1,$$

which is the same as

$$\frac{m}{q+1} < \frac{1}{r} < \frac{m}{q}$$

$$\frac{n}{q+1} < \frac{1}{s} < \frac{n}{q}.$$

Adding up we obtain $\frac{m+n}{q+1} < \frac{1}{r} + \frac{1}{s} < \frac{m+n}{q}$, or

$$\frac{m+n}{q+1} < 1 < \frac{m+n}{q},$$

so that $q < m+n < q+1$, which is impossible since both $m+n$ and q have been assumed to be integers. This immediately implies that $\{a_n\} \cap \{b_n\} = \emptyset$.

Below any integer N the two sequence have between them $\left\lfloor \frac{N}{r} \right\rfloor + \left\lfloor \frac{N}{s} \right\rfloor$ terms.

Let's denote the two numbers as, say $a(N)$ and $b(N)$. We have

$a(N) < \frac{N}{r} < a(N)+1$ and $b(N) < \frac{N}{s} < b(N)+1$

so that

$$\frac{a(N)}{N} < \frac{1}{r} < \frac{a(N)+1}{N}$$
$$\frac{b(N)}{N} < \frac{1}{s} < \frac{b(N)+1}{N}$$

Adding up gives $\frac{a(N)+b(N)}{N} < 1 < \frac{a(N)+b(N)+2}{N}$, or

$a(N)+b(N) < N < a(N)+b(N)+2$. Since all the quantities involved are integers, it follows that, it follows that $N = a(N)+b(N)+1$, or $a(N)+b(N) = N-1$, the exact number of integer intervals up to and including N. Thus every interval of with successive integer endpoints, say $[u, u+1]$, contains exactly one term of the union $\{a_n\} \cup \{b_n\}$ so that, indeed, $\{a_n\} cup \{b_n\} = \mathbb{N}$. □

63. An Inequality in Cyclic Quadrilateral IV

Prove that in quadrilateral $ABCD$, with sides $AB = a, BC = b, CD = c$, $DA = d$, and the area $= [ABCD]$, the following inequality holds

$$a^2 - b^2 - c^2 + d^2 + 4S \leq 2\sqrt{2}(ad+bc)$$

Proposed by Daniel Sitaru - Romania

Proof (by Daniel Sitaru - Romania).

In $\Delta ABD : BD^2 = a^2 + d^2 - 2ad\cos A$,

In $\Delta BCD : BD^2 = b^2 + c^2 - 2bc\cos(\pi - A)$.

It follows that $a^2 + d^2 - 2ad\cos A = b^2 + c^2 + 2bc\cos A$, or,

$S = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin A$, or, $\sin A = \frac{2S}{ad+bc}$.

Let $f : (0, 2\pi) \to \mathbb{R}, f(x) = \sin x + \cos x = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$.

Thus $\max f(x) = \sqrt{2}$. We now have $\sin x + \cos x \leq \sqrt{2}$, i.e.

$\frac{a^2-b^2-c^2+d^2}{2(ad+bc)} + \frac{2S}{ad+bc} \leq \sqrt{2}$, which is $a^2 - b^2 - c^2 + d^2 + 4S \leq 2\sqrt{2}(ad+bc)$. □

Acknowledgment (by Alexander Bogomolny - USA)

The problem from his book *Math Accent* has been posted at ***CutTheKnotMath page*** *by Daniel Sitaru, with his solution.*

64. Algebraic - Geometric Inequality

Let $x, y, z > 0$. Prove that

$$\sqrt{x^2-\sqrt{3}xy+y^2}+\sqrt{y^2-\sqrt{2}yz+z^2}\geq\sqrt{z^2-zx+x^2}$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Leonard Giugiuc - Romania).
In ***complex numbers***, let $u=e^{\frac{i\pi}{6}}, v=e^{\frac{i\pi}{4}}$, and $w=e^{\frac{5i\pi}{12}}$. We have

$$\sqrt{x^2-\sqrt{3}xy+y^2}=|x-yu|, \sqrt{y^2-\sqrt{2}yz+z^2}=|y-zv|=|u||y-zw|=$$

$=|yu-zw|$. It follows that

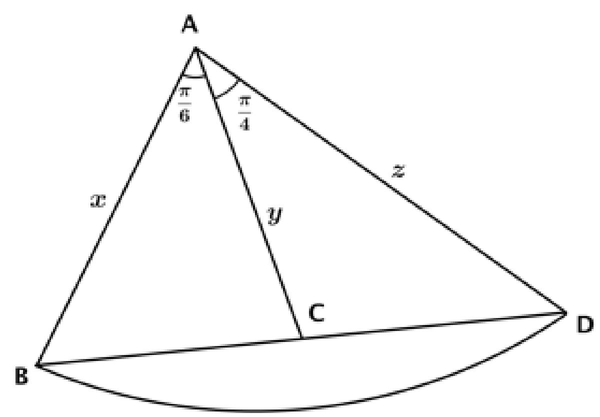
$$\sqrt{x^2-\sqrt{3}xy+y^2}+\sqrt{y^2-\sqrt{2}yz+z^2}=|x-yu|+|yu-zw|$$

$$\geq|x-yu+yu-zw|=|x-zw|=\sqrt{x^2-xz\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)+z^2}$$

$$\geq\sqrt{x^2-xz+z^2}.$$

□

Proof 2.
Consider triangles ABC and ACD such that $AB=x, AC=y, AD=z$, $\angle BAC=\frac{\pi}{6}, \angle CAD=\frac{\pi}{4}$. Then $BC=\sqrt{x^2-\sqrt{3}xy+y^2}$ and $CD=\sqrt{y^2-\sqrt{2}+z^2}$. Also, $\angle BAD=75^\circ, BD=\sqrt{z^2-zx\cos 75^\circ+x^2}$.



Since $\cos 75^\circ<\cos 60^\circ=\frac{1}{2}, z^2-zx\cos 75^\circ+x^2>z^2-zx+x^2$. Now,

$$BC+CD\geq BD>\sqrt{z^2-zx+x^2}$$

which proves the required inequality. □

Proof 3.

$$\sqrt{x^2-\sqrt{3}xy+y^2}+\sqrt{y^2-\sqrt{2}yz+z^2}$$

$$=\sqrt{\left(\frac{\sqrt{3}}{2}x-y\right)^2+\left(\frac{x}{2}\right)^2}+\sqrt{\left(y-\frac{z}{\sqrt{2}}\right)^2+\left(\frac{z}{\sqrt{2}}\right)^2}$$

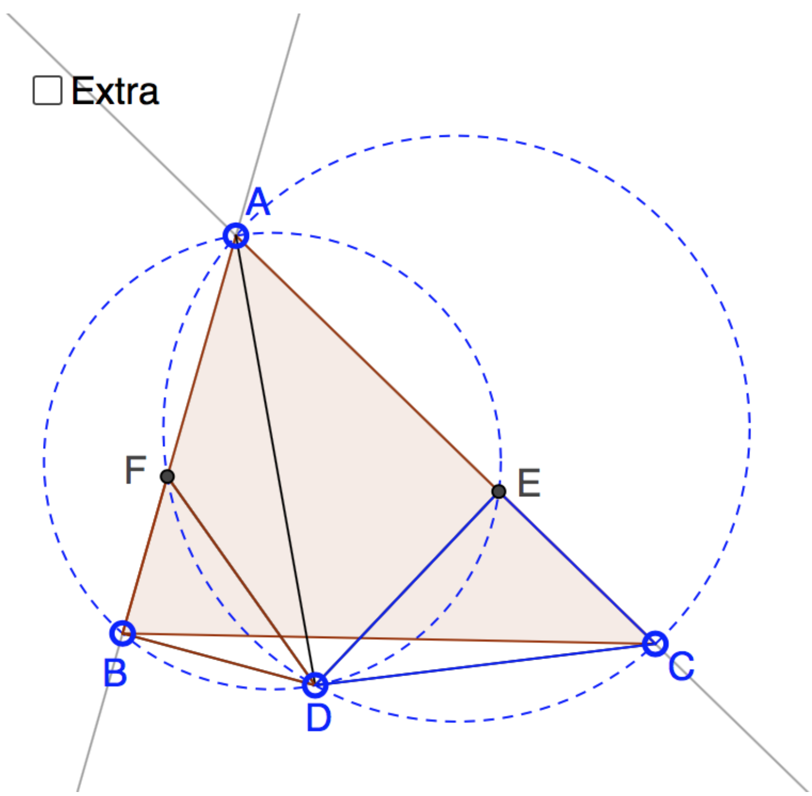
$$\geq \sqrt{\left(\frac{\sqrt{3}}{2}x - \frac{z}{\sqrt{2}}\right)^2 + \left(\frac{x}{2} + \frac{z}{\sqrt{2}}\right)^2} = \sqrt{x^2 + z^2 - \frac{\sqrt{3}-1}{\sqrt{2}}zx} > \sqrt{z^2 - xz + x^2}$$

since $\sqrt{2} + 1 > \sqrt{3}$. This completes the proof. □

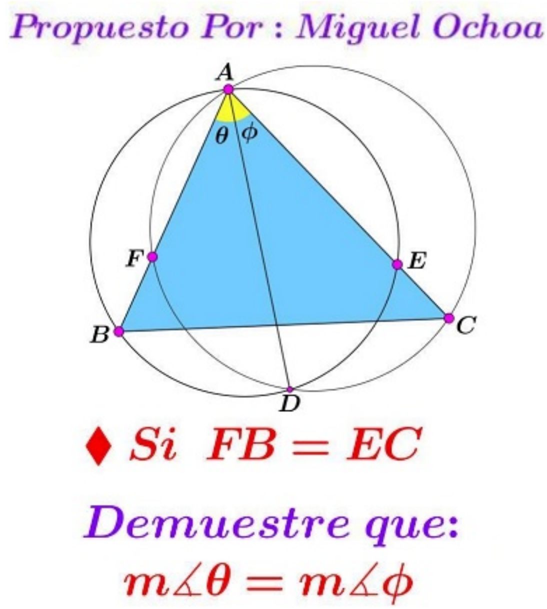
Acknowledgment (by Alexander Bogomolny - USA)
Daniel Sitaru has kindly posted the above problem (from his book "Math Accent") at the ***CutTheKnotMath page***. Solution 1 is by Leo Giugiuc; Solution 2 is by Ravi Prakash and, independently, by Chris Kyriazis; Solution 3 is by Nguyen Minh Triet and, independently, by Soumitra Mandal.

65. For Equality Choose Angle Bisector

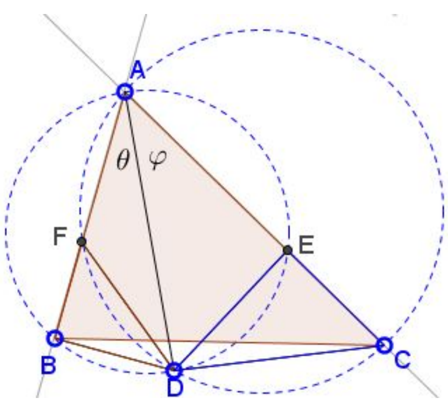
What Might This Be About?



Source:



We'll prove a little more: the condition $BF = CE$ is not sufficient for $\theta = \varphi$, it is also necessary:
Given ΔABC and point D, on neither AB or AC. From circles (ABD) and (ACD) intersect AC in E; (ACD) intersect AB in F:



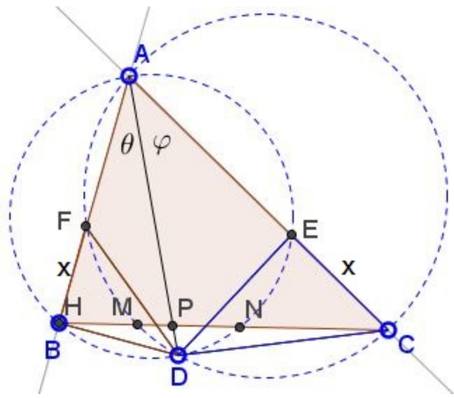
Let $\angle BAD = \theta, \angle CAD = \varphi$. Prove that $\theta = \varphi$ if $BF = CE$.

Proposed by Miguel Ochoa Sanchez - USA

Solution 1 (by Alexander Bogomolny - USA).
Angles θ and φ are subtended by the chords BD, DE in circle (ABD) and by the chords DF, CD in circle (ACD), implying $\frac{DF}{CD} = \frac{BD}{DE}$. In addition, $\angle BDF = \angle BDE - \angle FDE$, whereas $\angle CDE = \angle CDF - \angle FDE$. Both angles BDE and CDF are ***supplementary*** to $\angle BAC$ and are thus equal. It follows that $\angle BDF = \angle CDE$, and, consequently, triangles BDF and CDE are similar. They are equal when, say, the two chords BD and DE in circle (ABD) are equal. This

only happens when $\theta = \varphi$. Thus $\Delta BDF = \Delta CDE$ and then also $BF = CE$ if AD bisects $\angle BAC$. □

Solution 2 (by Leonard Giugiuc - Romania).
Here we establish only the sufficiency of the condition $BF = CE$ for $\theta = \varphi$. Introduce M, N, P and x as below.



Denote circle (ABE) as w and circle (ACF) as q. By the power of B relative to $q, BM \cdot BC = BF \cdot AB$, i.e., $a \cdot BM = xc$. Similarly, by the ***power*** of C with respect to $w, a \cdot CN = x \cdot b$. From here, $\frac{BM}{CN} = \frac{c}{b}$.

P belongs to the ***radical axis*** w and q, hence, it has the same power relative to both w and q. It follows that $PB \cdot PN = PC \cdot PM$, or $PB(PC - CN) = PC(PB - BM)$, which is equivalent to $PB \cdot CN = PC \cdot BM$, implying $\frac{PB}{PC} = \frac{BM}{CN} = \frac{c}{b}$. By the inverse of the ***Internal Bisector theorem***, AP is the angle bisector of $\angle BAC$. □

Acknowledgment (by Alexander Bogomolny - USA)
The problem that is due to Miguel Ochoa Sanchez has been posted by Leonard Giugiuc at the ***CutTheKnotMath page*** along with a ***solution*** (Solution 2).

66. A Cyclic Inequality in Three Variables XX

Prove that, for $a, b, c > 0$, with $a + b + c = 1$.

$$5\sum_{cycl}\sqrt{ab} \le \sum_{cycl}\sqrt[4]{(a+4b)(2a+3b)(3b+2a)(4a+b)} \le 5$$

Proposed by Daniel Sitaru - Romania

Proof 1.
By the ***AM - GM inequality***,

$$\sum_{cycl}\sqrt[4]{(a+4b)(2a+3b)(3b+2a)(4a+b)}$$

$$\le \sum_{cycl}\frac{(a+4b)(2a+3b)(3b+2a)(4a+b)}{4}$$

Again, by the AM - GM inequality,

$$\sum_{cycl}\sqrt[4]{(a+4b)(2a+3b)(3b+2a)(4a+b)} \le$$

$$\geq \sum_{cycl} \sqrt[4]{\sqrt[5]{ab^4 \cdot a^2b^3 \cdot a^3b^2 \cdot a^4b}} = 5\sum_{cycl} \sqrt{ab}.$$

This completes the proof. □

Proof 2.

$$\sqrt[4]{(a+4b)(2a+3b)(3b+2a)(4a+b)}$$
$$= \sqrt[4]{(4a^2+4b^2+17ab)(6a^2+6b^2+13ab)} \geq \sqrt[4]{(25ab)(25ab)} = 5\sqrt{ab}.$$

Further

$$\sqrt[4]{(a+4b)(2a+3b)(3b+2a)(4a+b)}$$
$$\leq \sum_{cycl} \sqrt[4]{\frac{(a+4b)(2a+3b)(3b+2a)(4a+b)}{4}} = \frac{10}{2}a + b + c = 5.$$

□

Acknowledgment (by Alexander Bogomolny - USA)

This is a problem from the ***Romanian Mathematical Magazine***, posted by Daniel Sitaru at the ***CutTheKnotMath page***. Solution 1 is by Anas Adlany and independently by Diego Alvariz and also by Dang Thanh Tùng; Solution 2 is by Kevin Soto Palacios and independently by Soumava Chakraborty.

67. An Inequality from Gazeta Matematica, March 2016 III

Several inequalities with solution by Daniel Sitaru and Leonard Giugiuc have been just published in Gazeta Matematica (March 2016). Here is one with two of its applications and a proof (Proof 1) from the article. Along the way several additional proofs have been added. Proof 2 is by Imad Zak; Proof 3 is by Emil Stoyanov; Proof 4 is by Grégoire Nicollier.

Let a, b, c be real numbers. Prove that:

$$a^2 + b^2 + 1 \geq a + ab + b$$

Proposed by Daniel Sitaru, Leonard Giugiuc - Romania

Proof 1 (by Daniel Sitaru, Leonard Giugiuc - Romania).

Define $A = \begin{pmatrix} 1 & a & b \\ a & b & 1 \end{pmatrix}$. By the ***Binet - Cauchy theorem***, $\det(AA^T) \geq 0$. But

$$\det(AA^T) = (a^2 + b^2 + 1)^2 - (a + ab + b)^2$$

proving the inequality at hand. □

Proof 2 (by Imad Zak - Lebanon).

Let $S = a + b$ and $P = ab$, by the ***AM - GM inequality***, we have $P \leq \frac{S^2}{4}$ and the required inequality is equivalent to $S^2 - S + 1 \geq 3P$, so suffice it to prove that $S^2 - S + 1 \geq \frac{3S^2}{4}$ which is equivalent to $\frac{S^2}{4} - S + 1 \geq 0$, or $\left(\frac{S}{2} - 1\right)^2 \geq 0$ which is clearly true. The equality holds when $S = 2$ and $P = 1$, i.e., when $a = b = 1$. □

Proof 3 (by Emil Stoyanov).
The required inequality is equivalent to $a^2-(b+1)a+(b^2-b+1)\geq 0$. Consider the quadric function $f(x)=x^2-(b+1)x+(b^2-b+1)\geq 0$. Its discriminant $D=(b+1)^2-4b^2+4b-4=-3(b-1)^2$ is never positive, implying that function f is never negative. $\square$

Proof 4 (by Grégoire Nicollier).
The inequality reduces to $(a-1)^2+(b-1)^2\geq(a-1)(b-1)$ which could be strengthened to $(a-1)^2+(b-1)^2\geq 2(a-1)(b-1)$. $\square$

Proof 5 (by Alexander Bogomolny - USA).
By the ***AM - QM inequality***,

$$a^2+b^2+1\geq\frac{1}{2}(a+b)^2+1.$$

Suffice it to prove that

$$(a+b)^2+2\geq 2a+2ab+2b.$$

But this is equivalent to $(a-1)^2+(b-1)^2\geq 0$, which is obvious. $\square$

Application 1 (by Daniel Sitaru)

$$\prod_{1\leq i\leq j\leq n}(i^2+j^2+1)\geq n!\prod_{1\leq i\leq j\leq n}(2+\sqrt{ij}).$$

Observe that $a^2+b^2+1\geq a+b+ab\geq ab+2\sqrt{ab}=\sqrt{ab}(2+\sqrt{ab})$.
Using this,

$$\prod_{1\leq i\leq j\leq n}(i^2+j^2+1)\geq\prod_{1\leq i<j<n}\sqrt{ij}(2+\sqrt{ij})=$$
$$=\prod_{1\leq i\leq j\leq n}\sqrt{ij}\prod_{1\leq i\leq j\leq n}(2+\sqrt{ij})=n!\prod_{1\leq i\leq j\leq n}(2+\sqrt{ij}).$$

Obviously, the inequality can be strengthened.

Application 2 (by Daniel Sitaru)
Prove that

$$\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+\sin x\cos x+\cos x}dx\geq\frac{\pi}{4}.$$

Set $a=\sin x$ and $b=\cos x$.
Then $2\geq 1+\sin^2 x+\cos^2 x\geq\sin x+\sin x\cos x+\cos x$, implying

$$\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+\sin x\cos x+\cos x}dx\geq\frac{1}{2}\int_0^{\frac{\pi}{2}}dx\geq\frac{\pi}{4}$$

Note that, according to wolframalpha,

$$\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+\sin x\cos x+\cos x}dx\approx 1.02245.$$

68. An Inequality form Gazeta Matematica, March 2016 IV

Several inequalities with solution by Dan Sitaru and Leo Giugiuc have been just published in Gazeta Matematica (March 2016). Here is one of two exercises that lets you check your understanding of the technique. I have.

For real a, b, c such that $a^2 + b^2 + c^2 = 1$, prove the inequality

$$a + ac + b \leq 2.$$

Proposed by Daniel Sitaru, Leonard Giugiuc - Romania

Proof (by Alexander Bogomolny - USA).
Define $A = \begin{pmatrix} a & b & c \\ 1 & 1 & a \end{pmatrix}$. By the ***Binet - Cauchy theorem***, $\det(AA^T) \geq 0$. But

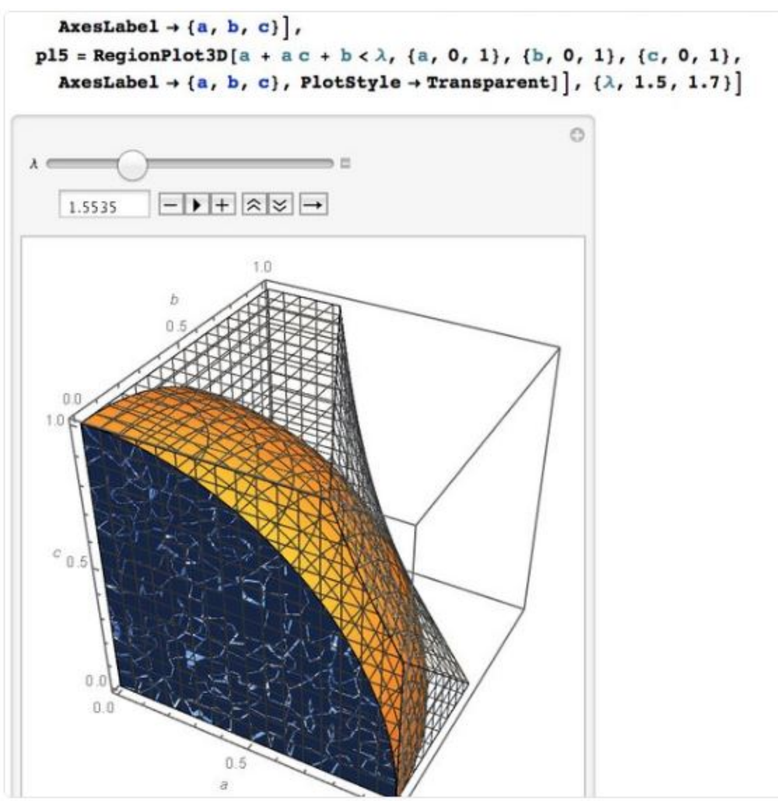
$$\det(AA^T) = (a^2 + b^2 + c^2)(2 + a^2) - (a + ac + b)^2 \geq 0,$$

which is $2 + a^2 \geq (a + ac + b^2)$. Given that $a^2 \leq a^2 + b^2 + c^2 = 1$, we conclude that

$$3 \geq 2 + a^2 \geq (a + ac + b)^2$$

i.e., $a + ac + b \leq \sqrt{3} - a$ somewhat stronger inequality than is required.
With the constraint $a^2 + b^2 + c^2 = 2$, we are led to $4 \geq 2 + a^2 \geq (a + ac + b)^2$, and $a + ac + b \leq 2$. Ought to be a typo. $\square$

Illustration (by Nassim Nicholas Taleb - USA)
Nassim Nicholas Taleb has kindly produced the following graphics:



What graphics tells us is that 1.5535 is closer to the smallest bound for $a+ac+b$ than $\sqrt{3}$.

Using Lagrange's multiplier to find $\max(a+ac+b)$ subject to $a^2+b^2+c^2=1$ produced an approximation, 1.576881.

Pradyumna Agashe found this estimate: $a+ac+b \le \frac{19}{12} = 1.58\overline{3}$. The proof stems from an equivalent inequality

$$\left(\frac{a}{2}-c\right)^2+\left(b-\frac{1}{2}\right)^2+\left(\frac{a\sqrt{3}}{2}-\frac{1}{\sqrt{3}}\right)^2 \ge 0.$$

69. Inequality with Roots, Squares and the Area

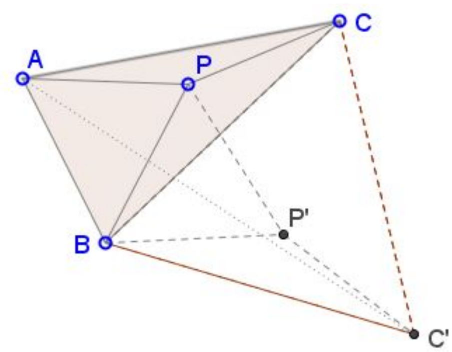
Let P be an interior point in ΔABC . Prove that:

$$\sqrt{2}(PA+PB+PC) \ge \sqrt{a^2+b^2+c^2+4\sqrt{3}S},$$

where $S=[\Delta ABC]$, the area of $\Delta ABC, a, b, c$ its side lengths. Equality is achieved when P is the Fermat - Torricelli point in ΔABC.

Proposed by Daniel Sitaru - Romania

Proof 1 (by Alexander Bogomolny - USA).
Rotate ΔCBP around B and away from A through $60°$ into position $C'BP'$. Observe that this creates equilateral triangles BCC' and BPP'.



This gives us $PB=PP'$, and $PC=P'C'$ so that

(1) $$PA+PB+PC=AP+PP'+P'C' \ge AC'.$$

The ***Law of Cosines*** in $\Delta ABC'$ gives (with $\angle ABC=\beta$)

$$AC'^2 = AB^2+BC'^2-2\cdot AB\cdot BC'\cos\angle ABC' = c^2+a^2-2ac\cos(\beta+60°)$$

$$= c^2+a^2-2ac(\cos 60°\cos\beta-\sin 60°\sin\beta) = c^2+a^2-2ac\left(\frac{1}{2}\cos\beta-\frac{\sqrt{3}}{2}\sin\beta\right)$$

$$= c^2+a^2-(ac\cos\beta+\sqrt{3}ac\sin\beta) = c^2+a^2-\frac{a^2+c^2-b^2}{2}+\sqrt{3}\cdot 2S$$

$$= \frac{2a^2+2c^2-a^2-c^2+b^2}{2}+2\sqrt{3}S = \frac{a^2+c^2+b^2}{2}+2\sqrt{3}S$$

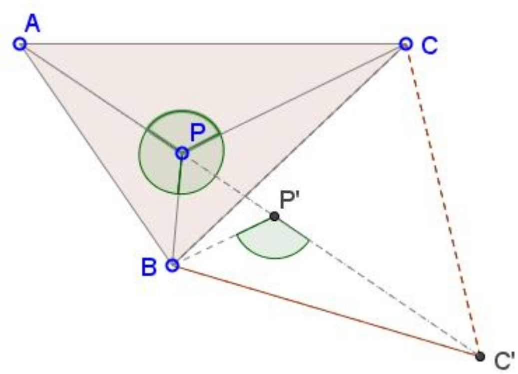
Thus $AC' = \sqrt{\frac{a^2+b^2+c^2}{2} + 2\sqrt{3}S}$. With (1), this implies

$$PA + PB + PC \geq \sqrt{\frac{a^2+b^2+c^2}{2} + 2\sqrt{3}S},$$

which is the same as the required

$$\sqrt{2}(PA + PB + PC) \geq \sqrt{a^2 + b^2 + c^2 + 4\sqrt{3}S}$$

For equality, we need $P, P' \in AC'$.



In such a case, $\angle BP'C' = 120°$, for, it's complementary to $\angle BP'P$. This makes $\angle BPC = 120°$. Also, $\angle BPA = 120°$, as complementary to $\angle BPP'$. Thus, $\angle ABC = 120°$ also, which makes P the ***Fermat - Torricelli*** point in ΔABC. Naturally, this argument does not work when $\angle ABC > 120°$. □

Solution 2 (by Leonard Giugiuc - Romania).
We'll consider the case in which $A, B, C < 120°$. Let T be the Fermat - Torricelli point. Denote $TA = x, TB = y$ and $TC = z$. Choose $T = 0, A = x, B = yu$ and $C = zu^2, u = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. We have:

$$a^2 = y^2 + yz + z^2,$$
$$b^2 = z^2 + zx + x^2,$$
$$a^2 = x^2 + xy + y^2,$$
$$4S\sqrt{3} = 3(xy + yz + zx),$$

so that $a^2 + b^2 + c^2 + 4S\sqrt{3} = 2(x + y + z)^2$. Thus, our inequality reduces to $PA + PB + PC \geq TA + TB + TC$, ***which is known***. Let's prove it, though.

Let $P = w$. We need to show that

$$|w - x| + |w - yu| + |w - zu^2| \geq x + y + z$$

which is equivalent to

$$|w - x| + |u^2||w - yu| + |u||w - zu^2| \geq x + y + z$$

But

$$|w - x| + |u^2||w - yu| + |u||w - zu^2| \geq |w(1 + u^2 + u) - (x + y + z)|$$
$$= x + y + z.$$

Naturally, equality holds iff $w = 0$, i.e., when $P = T$. □

70. Romano Norwegian Inequality

Here is a sample inequality from a recent book *300 Romanian Mathematical Challenges* by Professor Radu Gologan, Daniel Sitaru and Leonard Giugiuc. The problem is an invention of Lorian Nelu Saceanu, Norway - Romania. Solution below is by Leonard Giugiuc.

Let ABC be a triangle with no obtuse angles.
Prove that $\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}\geq 2$.

Proposed by Lorian Nelu Saceanu - Romania

Proof (by Leonard Giugiuc - Romania).
Denote $x=\cot A, y=\cot B, z=\cot C$. Then $x,y,z\geq 0$ and $xy+yz+zx=1$. We need to prove $\sqrt{x}+\sqrt{y}+\sqrt{z}\geq 2$.

WLOG, let's assume that $yz=\max\{xy,yz,xz\}$. As a consequences, $\frac{1}{3}\leq yz\leq 1$. Define $y+z=2s$ and $yz=p$; then, by the ***AM - GM inequality***, $s\geq p$ and also $\frac{1}{\sqrt{3}}\leq p\leq 1$. On the other hand, $x=\frac{1-xy}{x+y}=\frac{1-p^2}{2s}$. Further
$\sqrt{y}+\sqrt{z}=\sqrt{y+z+2\sqrt{yz}}=\sqrt{2s+2p}$. For any fixed $p\in\left[\frac{1}{\sqrt{3}},1\right]$ we consider the function $f_p:[p,\infty)\to\mathbb{R}$, defined by $f_p(t)=\sqrt{\frac{1-p^2}{2t}}+\sqrt{2t+2p}$.
First off, $f_p'(t)=-\frac{\sqrt{1-p^2}}{(2t)^{\frac{3}{2}}}+\frac{1}{\sqrt{2t+2p}}$. We'll prove that $f_p'(t)\geq 0$. This is equivalent to showing that $8t^3\geq(1-p^2)(2t+2p)$, i.e., $4t^3-(1-p^2)t-(1-p^2)p\geq 0$, for $t\geq p$.
Define function $g_p(t):[p,\infty)\to\mathbb{R}$, by $g_p(t)=4t^3-(1-p^2)t-(1-p^2)p$.
The only critical point of $g_p(t)$ in $[0,\infty)$ is $t=\sqrt{\frac{1-p^2}{12}}$, which is clearly less than p, implying $g_p(t)\geq g_p(p)=2p(3p^2-1)\geq 0$, for $t\geq p$, so that
$f_p(t)\geq f_p(p)=\sqrt{\frac{1-p^2}{2p}}+2\sqrt{p}$ for $t\geq p,s$, in particular.

Thus, suffice it to show that, for $p\in\left[\frac{1}{\sqrt{3}}\right]$, $\sqrt{\frac{1-p^2}{2p}}+2\sqrt{p}\geq 2$. This is equivalent to $\sqrt{\frac{(1-p)(1+p)}{2p}}\geq\frac{2(1-p)}{2+\sqrt{p}}$. Since $1-p\geq 0$, we just need to prove $\sqrt{\frac{1+p}{2p}}\geq\frac{2\sqrt{1-p}}{1+\sqrt{p}}$. Set $\sqrt{p}=u$. Then $u\in\left[\frac{1}{\sqrt[4]{3}},1\right]$ and we'll show that $\frac{1+u^2}{2u^2}\geq\frac{4(1-u^2)}{(1+u^2)}$ which is $9u^4+2u^3-6u^2+2u\geq 0$, or $(3u^2-1)^2+2u^3+2u\geq 0$. The latter is obviously true for $u\in\left[\frac{1}{\sqrt[4]{3}},1\right]$. The proof is complete. □

71. Radon's Inequality and Applications

Radon's Inequality (by Alexander Bogomolny - USA)
The content of the present page has been borrowed (at least in its initial form) from an article by Dorin Marghidanu ***Generalisations and Refinements for Bergström and Radon's Inequalities***.

If $x_k, a_k > 0, k \in \{1, 2, \ldots, n\}, p > 0$, *then:*

$$\frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} + \ldots + \frac{x_n^{p+1}}{a_n^p} \geq \frac{(x_1 + x_2 + \ldots + x_n)^{p+1}}{(a_1 + a_2 + \ldots + a_n)^p}.$$

The equality is only attained for

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}.$$

Clearly, for $p = 1$ the inequality becomes that of Bergström.

Proof of Radon's Inequality.
As a first step, we prove the inequality for $n = 2$, deriving it from the well-known Hölder's inequality:

$$\sum_{i=1}^{n} u_i v_i \leq \left(\sum_{i=1}^{n} u_i^s\right)^{\frac{1}{s}} \left(\sum_{i=1}^{n} v_i^t\right)^{\frac{1}{t}},$$

where $\frac{1}{s} + \frac{1}{t} = 1, s, t, > 1$, and all u_i and v_i are assumed positive. This is obviously a generalisation of the ***Cauchy - Schwarz inequality***. The same method will also work for larger n but I prefer to use Dorin Marghidanu's original derivation that depends on the case of $n = 2$.

Thus, we want to prove that, say,

$$\frac{x^{p+1}}{a^p} + \frac{y^{p+1}}{b^p} \geq \frac{(x+y)^{p+1}}{(a+b)^p}.$$

Setting $s = \frac{p+1}{p}$ and $t = p + 1$, we start with

$$x + y = a^{\frac{1}{s}}\Big(\frac{x}{a^{\frac{1}{s}}}\Big) + b^{\frac{1}{s}}\Big(\frac{y}{b^{\frac{1}{s}}}\Big) \leq (a^{\frac{s}{s}} + b^{\frac{s}{s}})^{\frac{1}{s}}\Big(\frac{x^t}{a^{\frac{t}{s}}} + \frac{y^t}{b^{\frac{t}{s}}}\Big)^{\frac{1}{t}}$$

$$= \left[(a+b)^p\Big(\frac{x^{p+1}}{a^p} + \frac{y^{p+1}}{b^p}\Big)\right]^{\frac{1}{(p+1)}}$$

This is equivalent to the required inequality. Now for the rest of n. Define

$$d_n = \frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{x_2^p} + \ldots + \frac{x_n^{p+1}}{x_n^p} - \frac{(x_1 + x_2 + \ldots + x_n)^{p+1}}{a_1 + a_2 + \ldots + a_n}.$$

Our task is to prove that $d_n \geq 0$, for $n \geq 2$. We are going to show more, viz., that the sequence $\{d_n\}$ monotone increasing and, since $d_1 = 0$, this will solve the entire problem of proving Radon's inequality.

To this end,

$$d_{n+1} - d_n = \sum_{k=1}^{n+1} \frac{x_k^{p+1}}{a_k^p} - \frac{(\sum_{k=1}^{n+1})^{p+1}}{(\sum_{k=1}^{n+1} a_k)^p} - \sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} + \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p}$$

$$= \left[\frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p + \frac{x_{n+1}^{p+1}}{a_{n+1}^p}}\right] - \frac{(\sum_{k=1}^{n+1} x_k)^{p+1}}{(\sum_{k=1}^{n+1} a_k)^p} \geq \frac{(\sum_{k=1}^{n+1})^{p+1}}{(\sum_{k=1}^{n+1} a_k)^p} - \frac{(\sum_{k=1}^{n+1} x_k)^{p+1}}{(\sum_{k=1}^{n+1} a_k)^p} = 0,$$

where in the penultimate step we used the earlier case of $n = 2$.

Obviously, this proof can be regarded as a proof by ***induction***. $\square$

Reverse Radon's Inequality

Daniel Sitaru has kindly alert me to the validity of what's known as the reverse Radon's inequality:

If $x_k, a_k > 0, k \in \{1, 2, ..., n\}, 0 \le p \le 1$, *then*

$$\frac{x_1^p}{a_1^{p-1}} + \frac{x_2^p}{a_2^{p-1}} + \ldots + \frac{x_n^p}{a_n^{p-1}} \le \frac{(x_1 + x_2 + \ldots + x_n)^p}{(a_1 + a_2 + \ldots + a_n)^{p-1}}.$$

Applications:
1. A Problem in Four Variables

Daniel Sitaru has posted the following problem from the Romanian Mathematical Magazine:

If $a, b, c, d \in (0, \infty)$, *and* $abcd = 1$ *then*

$$\frac{(a+b+c)^5}{(b+c+d)^4} + \frac{(b+c+d)^5}{(c+d+a)^4} + \frac{(c+d+a)^5}{(d+a+b)^4} + \frac{(d+a+b)^5}{(a+b+c)^4} \ge 12$$

Proposed by Daniel Sitaru - Romania

Proof (by Alexander Bogomolny - USA).
The inequality is solved by an application of Radon's inequality, followed by the ***AM - GM inequality***:

$$\frac{(a+b+c)^5}{(b+c+d)^4} + \frac{(b+c+d)^5}{(c+d+a)^4} + \frac{(c+d+a)^5}{(d+a+b)^4} + \frac{(d+a+b)^5}{(a+b+c)^4} \ge$$

$$\ge \frac{[3(a+b+c+d)]^5}{[3(a+b+c+d)]^4} = 3(a+b+c+d) \ge 3 \cdot 4(abcd)^{\frac{1}{4}} \ge 12.$$

$\square$

2. 42 IMO, Problem 2

Prove that, for all positive a, b, c,

$$\frac{a}{a^2 + 8bc} + \frac{b}{b^2 + 8ca} + \frac{c}{c^2 + 8ab} \ge 1$$

Proof (by Alexander Bogomolny - USA).
The left-land side can be rewritten as

$$M = \frac{a^{\frac{3}{2}}}{a^3 + 8abc} + \frac{b^{\frac{3}{2}}}{b^3 + 8abc} + \frac{c^{\frac{3}{2}}}{c^3 + 8abs}$$

which suggests using Radon's inequality with $p = \frac{1}{2}$ and $n = 3$:

$$M \ge \frac{(a+b+c)^{\frac{3}{2}}}{(a^2 + b^3 + c^3 + 24abc)^{\frac{1}{2}}} = \sqrt{\frac{(a+b+c)^3}{a^3 + b^3 + c^3 + 24abc}}$$

Thus suffice it to prove that $\frac{(a+b+c)^3}{a^3+b^3+c^3+24abc} \geq 1$. This inequality reduces to

$$ab^2 + a^2b + bc^2 + b^2c + ca^2 + c^2a \geq 6abc$$

which is an immediate consequence of the ***AM - GM inequality***. □

72. An Inequality in Triangle, IX

In an acute $\Delta ABC, A', A'' \in BC; B', B'' \in AC; C', C'' \in AB. AA', BB', CC'$ *are angle bisectors that intersect at the incenter* $I; AA'', B'', C''$ *are the altitudes that intersect at the orthocenter* H*. Prove that*

$$27 \prod_{cycl} IA' \cdot HA'' \leq \frac{1}{27} \prod_{cycl} l_a h_a,$$

where l_a, l_b, l_c*, are the lengths of the bisector and* h_a, h_b, h_c *the lengths of the altitudes in* ΔABC*.*

Proposed by Daniel Sitaru - Romania

Proof (by Daniel Sitaru - Romania).
Let M be a point in the interior of ΔABC, and AA_0, BB_0, CC_0 the ***cevians*** through M. Then by ***Gergonne's Theorem***, and, ***applying the AM - GM inequality***,

$$1 = \frac{MA_0}{AA_0} + \frac{MB_0}{BB_0} + \frac{MC_0}{CC_0} \geq \sqrt[3]{\frac{MA_0}{AA_0} \cdot \frac{MB_0}{BB_0} \cdot \frac{MC_0}{CC_0}}.$$

In particular, $1 \geq 27\frac{IA'}{l_a} \cdot \frac{IB'}{l_b} \cdot \frac{IC'}{l_c}$ and $1 \geq 27\frac{HA''}{h_a} \cdot \frac{HB''}{h_b} \cdot \frac{HC''}{h_c}$ whose product gives the required result. □

Acknowledgment (by Alexander Bogomolny - USA)
The inequality and the solution have been kindly communicated to me by Daniel Sitaru.

73. An Inequality in Triangle, X

In any ΔABC,

$$\frac{1}{r^2} \sum_{cycl} a^3 \cos B \cos C \geq 16 \left(\sum_{cycl} \sin A \right) \left(\sum_{cycl} \cos^2 A \right)$$

where r *is the inradius of* ΔABC.

Proposed by Daniel Sitaru - Romania

Proof (by Daniel Sitaru).
Since, by the ***Law of Sines***, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circumradius of ΔABC,

$$\Delta = \begin{vmatrix} a & b\cos C & c\cos B \\ b & c\cos A & a\cos C \\ c & a\cos B & b\cos A \end{vmatrix} = 8R^3 \begin{vmatrix} \sin A & \sin B\cos C & \sin C\cos B \\ \sin B & \sin C\cos A & \sin A\cos C \\ \sin C & \sin A\cos B & \sin B\cos A \end{vmatrix}$$

$$= 8R^3 \begin{vmatrix} \sin A & \sin(B+C) & \sin C\cos B \\ \sin B & \sin(A+C) & \sin A\cos C \\ \sin C & \sin(A+B) & \sin B\cos A \end{vmatrix} = 8R^3 \begin{vmatrix} \sin A & \sin(\pi - A) & \sin C\cos B \\ \sin B & \sin(\pi - B) & \sin A\cos C \\ \sin C & \sin(\pi - C) & \sin B\cos A \end{vmatrix}$$

$$= 8R^3 \begin{vmatrix} \sin A & \sin A & \sin C \cos B \\ \sin B & \sin B & \sin A \cos C \\ \sin C & \sin C & \sin B \cos A \end{vmatrix} = 0$$

On the other hand,

$$0 = \Delta = \begin{vmatrix} a & b\cos C & c\cos B \\ b & c\cos A & a\cos C \\ c & a\cos B & b\cos A \end{vmatrix}$$

$$= abc\cos^2 A + abc\cos^2 C + abc\cos^2 A -$$
$$-c^3 \cos A \cos B - a^3 \cos B \cos C - b^3 \cos A \cos C$$
$$= abc \sum_{cycl} \cos^2 A - \sum_{cycl} a^3 \cos B \cos C$$
$$= 4RS \sum_{cycl} \cos^2 A - \sum_{cycl} a^3 \cos B \cos C,$$

where $S = [\Delta ABC]$ is the area of ΔABC. (**As is well known**, abc=4RS.) Further,

$$\sum_{cycl} a^3 \cos B \cos C = 4RS \sum_{cycl} \cos^2 A$$
$$= 4Rrp \sum_{cycl} \cos^2 A$$
$$= 4Rr \cdot \frac{a+b+c}{2} \sum_{cycl} \cos^2 A$$

so that

$$\frac{\sum_{cycl} a^3 \cos B \cos C}{\sum_{cycl} \cos^2 A} = 2Rr(a+b+c) = 2Rr \cdot 2R \cdot \sum_{cycl} \sin A = 4R^2 r \sum_{cycl} \sin A$$

Now using ***Euler's inequality*** $R > 2r$,

$$\frac{\sum_{cycl} a^3 \cos B \cos C}{(\sum_{cycl} \sin A)(\sum_{cycl} \cos^2 A)} = 4R^2 r \geq 16r^3,$$

which is the same as the required inequality. □

Acknowledgment (by Alexander Bogomolny - USA)
The inequality and the solution have been kindly communicated to me by Dan Sitaru. It was published at the ***Romanian Mathematical Magazine***.

74. An Inequality with Cycling Sums

Acknowledgment (by Alexander Bogomolny - USA)
The following problem and its solution have been communicated to me by Daniel Sitaru along with Proof 1. Proof 2 has been added by Imad Zak.

Prove that, for all positive numbers $x, y, z, xyz = 1$, the following inequality holds:

$$\sum_{cycl} (x^4 + y^3 + z) \geq \sum_{cycl} \frac{x^2 + y^2}{z} + 3$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Daniel Sitaru).
We use the two special cases of ***Schur's inequality***:

$$\begin{cases} t=1: \sum_{cycl} x^3 + 3xyz \geq \sum_{cycl} xy(x+y), \\ t=2: \sum_{cycl} x^4 + xyz \sum_{cycl} x \geq \sum_{cycl} xy(x^2+y^2) \end{cases}$$

The two inequalities are simplified by nothing that $xyz = 1$. Add the two:

$$\sum_{cycl}(x^4+x^3+x)+3 \geq \sum_{cycl} \frac{x^2+y^2}{z} + \sum_{cycl}\left(\frac{x}{z}+\frac{y}{z}\right)$$
$$= \sum_{cycl} \frac{x^2+y^2}{z} + \sum_{cycl}\left(\frac{x}{z}+\frac{z}{x}\right)$$
$$\geq \sum_{cycl} \frac{x^2+y^2}{z} + 6$$

which prove the required inequality. □

Proof 2 (by Imad Zak - Lebanon).
First note that by the ***AM - GM inequality***,

$$\sum_{cycl} y^3 \geq 3xyz = 3$$

Thus (where we also use ***Schur's inequality*** with $t = 2$),

$$\sum_{cycl}(x^4+y^3+z) = \sum_{cycl} x^4 + \sum_{cycl} x^4 + \sum_{cycl} y^3 + \sum_{cycl} z$$
$$\geq \sum_{cycl x^4} + \sum_{cycl} z + 3 = \sum_{cycl} x^4 + xyz \sum_{cycl} x + 3$$
$$\geq \sum_{cycl} xy(x^2+y^2) + 3 = \sum_{cycl} \frac{1}{z}(x^2+y^2) + 3$$

Equality holds when $x = y = z = 1$. □

75. An Inequality with Determinants

Let $a, b, c, d > 0$. Then

$$\begin{vmatrix} a & -b & 0 & 0 \\ 0 & b & -c & 0 \\ 0 & 0 & c & -d \\ 1 & 1 & 1 & 1+d \end{vmatrix} \geq 3\sqrt[4]{4}(abcd)^{\frac{5}{6}}$$

Determine when the equality holds.

Proposed by Daniel Sitaru - Romania

Proof (by Ravi Prakash - India).
It could be seen that the determinant Δ in the left-hand side of the required inequality equals $\Delta = abcd + acd + abd + abc + bcd$ which is evaluated via ***AM - GM inequality***:

$$\Delta = abcd + acd + abd + abc + bcd$$

$$= \frac{1}{2}abcd + \frac{1}{2}abcd + acd + abd + abc + bcd$$
$$\geq 6\Big[\frac{1}{4}(abcd)^2(acd)(abd)(abc)(bcd)\Big]^{\frac{1}{6}} = 6\Big(\frac{1}{4}\Big)^{\frac{1}{6}}(abcd)^{\frac{5}{6}}$$
$$= 3 \cdot 4^{\frac{1}{3}}(abcd)^{\frac{5}{6}}$$

The equality holds when $\frac{1}{2}abcd = abc = bcd = \cdots$, i.e., when $a = b = c = d = 2$. □

Acknowledgment (by Alexander Bogomolny - USA)
The inequality from the book *Math Accent* has been posted at the ***CutTheKnot-Math page*** by Dan Sitaru along with a solution by Ravi Prakash.

76. An Inequality with Determinants II

Let $0 < a, b, c, d < 1$. *Then*

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} & \frac{1}{d^2} \end{vmatrix} < \frac{1}{abcd}\Big(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\Big)$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Leonard Giugiuc).
Start with subtracting the first column from the other three:

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) & (d-a)(d+a) \\ \frac{1}{a^2} & \frac{(b-a)(b+a)}{a^2b^2} & \frac{(c-a)(c+a)}{a^2c^2} & \frac{(d-a)(d+a)}{a^2d^2} \end{vmatrix}$$
$$= \frac{(b-a)(c-a)(d-a)}{a^2} \begin{vmatrix} 1 & 1 & 1 \\ b+a & c+a & d+a \\ \frac{b+a}{b^2} & \frac{c+a}{c^2} & \frac{d+a}{d^2} \end{vmatrix}$$
$$= \frac{(b-a)(c-a)(d-a)}{a^2b^2c^2d^2} \begin{vmatrix} 1 & 1 & 1 \\ b+a & c+a & d+a \\ c^2d^2(b+a) & b^2d^2(c+a) & b^2c^2(d+a) \end{vmatrix}$$
$$= \frac{(b-a)(c-a)(d-a)}{a^2b^2c^2d^2} \cdot \Delta';$$

where Δ' is being evaluated furher:

$$\Delta' = \begin{vmatrix} 1 & 0 & 0 \\ b+a & c-b & d-b \\ c^2d^2(b+a) & d^2(c-b)(ab+ac+bc) & c^2(d-b)(ab+ad+bd) \end{vmatrix}$$

It follows that

$$|\Delta| = \frac{|(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)|(abc+abd+acd+bcd)}{a^2b^2c^2d^2}$$
$$< \frac{abc+abd+acd+bcd}{a^2b^2c^2d^2} = \frac{1}{abcd}\Big(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\Big)$$

because $|(b-a)(c-a)(d-a)(c-b)(d-c)| < 1$. □

Proof 2 (by Hector Manuel Garduno Castaneda).
First of all, the required inequality is equivalent to

$$\mathbb{D} = \begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \\ 1 & 1 & 1 & 1 \end{vmatrix} < abcd\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\Big)$$

$$\text{Note that } \mathbb{D} = -\begin{vmatrix} 1 & 1 & 1 & 1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}. \text{ Set } P(x) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x^2 & b^2 & c^2 & d^2 \\ x^3 & b^3 & c^3 & d^3 \\ x^4 & b^4 & c^4 & d^4 \end{vmatrix}.$$

$P(x)$ is a polynomial of degree 4 and $P(b) = P(c) = P(d) = 0$. Thus

$$P(x) = q(b,c,d)(x-b)(x-c)(\alpha x+\beta)$$

Indeed, $P(a) = q(b,c,d)(a-b)(a-c)(a-d)(\alpha a+\beta)$. Now, since $\mathbb{D}$ is symmetric in all four variables, it is easy to show that $q(b,c,d) = (b-c)(b-d)(c-d)$ so that

$$(1) \qquad P(x) = (b-c)(b-d)(c-d)(x-b)(x-c)(x-d)(\alpha x+\beta)$$

On the other hand, in the determinant representation of $P(x)$ we get

$$P(0) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b^2 & c^2 & d^2 \\ 0 & b^3 & c^3 & d^3 \\ 0 & b^4 & c^4 & d^4 \end{vmatrix} = b^2c^2d^2 \begin{vmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{vmatrix}$$

Comparing this to (1) gives $\beta = bcd$. Thus,

$$P(x) = (b-c)(b-d)(c-d)(x-b)(x-c)(x-d)(ax+bcd)$$

Using the symmetry of $\mathbb{D}$ again, we obtain $\alpha = bc+bd+cd$, and, therefore,

$$\mathbb{D} = -P(a) = (b-c)(d-b)(d-c)(b-a)(c-a)(d-a)(abc+abd+acd+bcd)$$

and the required inequality follows. □

Acknowledgment (by Alexander Bogomolny - USA)
The inequality from the book *Math Power* has been posted at the ***CutTheKnot-Math page*** by Dan Sitaru. Solution 1 is by Leo Giugiuc; Solution 2 is by Hector Manuel Garduno Castaneda.

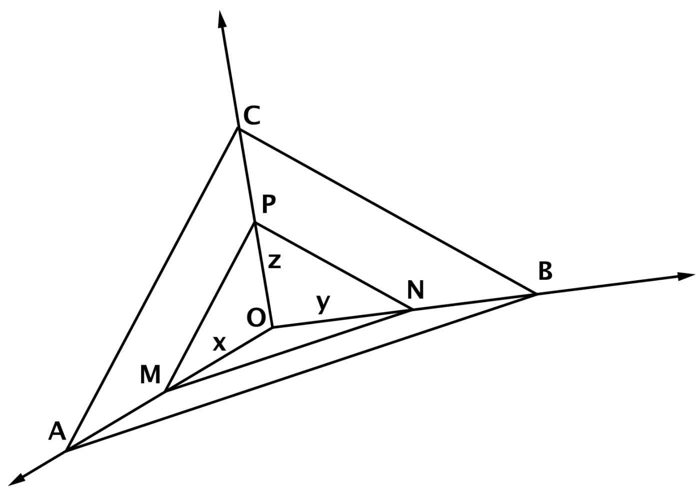
77. An Inequality with Determinants III

If $0 \le x \le a, 0 \le y \le b, 0 \le z \le c,$ *then*

$$\begin{vmatrix} 0 & x^2 & y^2 & z^2 & 1 \\ x^2 & 0 & x^2+y^2 & x^2+z^2 & 1 \\ y^2 & x^2+y^2 & 0 & y^2+z^2 & 1 \\ z^2 & x^2+z^2 & y^2+z^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \le \begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & 0 & a^2+b^2 & a^2+c^2 & 1 \\ b^2 & a^2+b^2 & 0 & b^2+c^2 & 1 \\ c^2 & a^2+c^2 & b^2+c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Daniel Sitaru - Romania).
Following the notations in the diagram below:



$$OA = a; OB = b; OC = c; AC^2 = a^2 + c^2; BC^2 = b^2 + c^2; AB^2 = a^2 + b^2 \text{ and}$$

$$OM = x; ON = y; OP = z; MP^2 = x^2 + z^2; NP^2 = y^2 + z^2; MN^2 = x^2 + y^2.$$

One may recollect that

$$V[OABC] = \frac{1}{288} \cdot \begin{vmatrix} O & OA^2 & OB^2 & OC^2 & 1 \\ OA^2 & 0 & OA^2 + OB^2 & OA^2 + OC^2 & \\ OB^2 & OA^2 + OB^2 & 0 & OB^2 + OC^2 & 1 \\ OC^2 & OC^2 + OA^2 & OC^2 + OB^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

In other words,

$$V[OABC] = \frac{1}{288} \begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & 0 & a^2 + b^2 & a^2 + c^2 & 1 \\ b^2 & a^2 + b^2 & 0 & b^2 + c^2 & 1 \\ c^2 & a^2 + c^2 & b^2 + c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

Similarly,

$$V[OMNP] = \frac{1}{288} \begin{vmatrix} 0 & x^2 & y^2 & z^2 & 1 \\ x^2 & 0 & x^2 + y^2 & x^2 + z^2 & 1 \\ y^2 & x^2 + y^2 & 0 & y^2 + z^2 & 1 \\ z^2 & x^2 + z^2 & y^2 + z^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

and the fact that, obviously, $V(OMNP) \leq V(OABC)$, proves the required inequality. $\square$

Solution 2 (by Alexander Bogomonly - USA).
We'll use column and row transformations:

$$\begin{vmatrix} 0 & x^2 & y^2 & z^2 & 1 \\ x^2 & 0 & x^2+y^2 & x^2+z^2 & 1 \\ y^2 & x^2+y^2 & 0 & y^2+z^2 & 1 \\ z^2 & x^2+z^2 & y^2+z^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & x^2 & y^2 & z^2 & 1 \\ x^2 & -x^2 & y^2 & z^2 & 1 \\ y^2 & x^2 & -y^2 & z^2 & 1 \\ z^2 & x^2 & y^2 & -z^2 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & y^2 & z^2 & 1 \\ -x^2 & y^2 & z^2 & 1 \\ x^2 & -y^2 & z^2 & 1 \\ x^2 & y^2 & -z^2 & 1 \end{vmatrix} = x^2y^2z^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{vmatrix} =$$

$$= x^2y^2z^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{vmatrix} = -x^2y^2z^2 \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 8x^2y^2z^2$$

Similarly, the determinant in the right-hand side of the required inequality equals $8a^2b^2c^2$, making the inequality obvious. □

Acknowledgment (by Alexander Bogomolny - USA)
The inequality from his book *Math Accent* has been posted at the ***CutTheKnot-Math page*** by Dan Sitaru. Dan has later communicated privately a solution (Solution 1) and placed a link to this page at the ***Romanian Mathematical Magazine***.

78. An Inequality with Integrals and Rearrangement

Acknowledgment (by Alexander Bogomolny - USA)
Leo Giugiuc has kindly communicated to me the following problem, along with a solution. The problem is from Dan Sitaru's book *Math Accent*.
If $a, b, c \in (0, \pi)$ then:

$$\sum b^2c^3 \int_0^a (\cot x)(\tan^{-1} x)dx < abc(a^3 + b^3 + c^3)$$

Proposed by Daniel Sitaru - Romania

Proof (by Leonard Giugiuc - Romania).

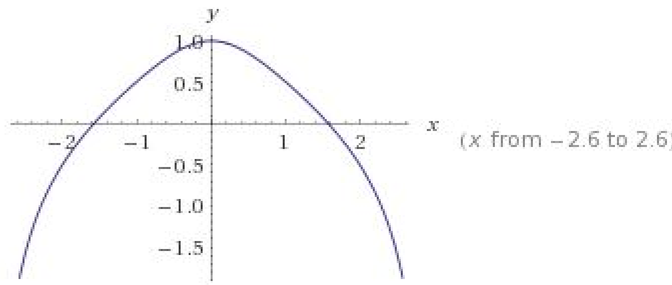
First off, $\lim\limits_{x\to 0^+}(\cot x \cdot \arctan x) = \lim\limits_{x\to 0^+}\left(\cos x \cdot \dfrac{\arctan x}{\sin x}\right) = 1$, implying that the function $f : [0, \pi) \to \mathbb{R}$, defined by

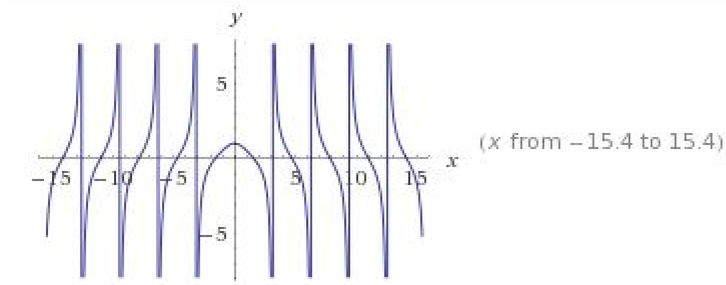
$$f(x) = \begin{cases} \cot x \cdot \arctan x, x \in (0, \pi) \\ 1, x = 0, \end{cases}$$

is continuos.

Now, for $x \in \left(0, \frac{\pi}{2}\right)$, $\arctan x < x < \tan x$, implying that $\cot x \cdot \arctan x < \cot x \cdot \tan x = 1$. It follows that on $\left[0, \frac{\pi}{2}\right]$, $f(x) < 1$ so that $\int_0^a f(x)dx < \int_0^a 1dx < a$, if $a \in \left(0, \frac{\pi}{2}\right]$. □

Illustration by Alexander Bogomolny - USA:





If $a \in \left(\frac{\pi}{2}, \pi\right)$ then

$$\int_0^a f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^a f(x)dx < \int_0^{\frac{\pi}{2}} f(x)dx < \frac{\pi}{2} < a$$

Thus, for $a \in (0, \pi)$, $\int_0^a f(x)dx < a$ and similarly $\int_0^b f(x)dx < b$ and $\int_0^c f(x)dx < c$.

$$\text{Thus, } \sum_{cycl} b^2c^3 \int_a^a f(x)dx < \sum_{cycl} ab^2c^3. \text{ But } \sum_{cycl} ab^2c^3 = abc \sum_{cycl} bc^2$$

and, by the ***rearrangement inequality,*** $\sum_{cycl} bc^2 \leq a^3 + b^3 + c^3$

79. An Inequality with Just Two Variables

Prove that, for positive a, b,

$$\left(\frac{2ab}{a+b} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab} + \sqrt{\frac{2}{a^2+b^2}}\right) \leq \frac{(a+b)^2}{ab}.$$

Proposed by Danile Sitaru - Romania

Proof 1 (by Kevin Soto Palacios - Huarmey - Peru).

The required inequality is equivalent to

$$1 + 1 + \left(\frac{a+b}{2ab}\right)\left(\sqrt{\frac{a^2+b^2}{2}}\right) + \left(\frac{2ab}{a+b}\right)\left(\sqrt{\frac{2}{a^2+b^2}}\right) \leq 2 + \frac{a}{b} + \frac{b}{a},$$

or,

$$\left(\frac{a+b}{2ab}\right)\left(\sqrt{\frac{a^2+b^2}{2}}\right)+\left(\frac{2ab}{a+b}\right)\left(\sqrt{\frac{2}{a^2+b^2}}\right)\leq \frac{a}{b}+\frac{b}{a},$$

Focusing on the left - hand side:

$$\left(\frac{a+b}{2ab}\right)\left(\sqrt{\frac{a^2+b^2}{2}}\right)+\left(\frac{2ab}{a+b}\right)\left(\sqrt{\frac{2}{a^2+b^2}}\right)$$

$$=\frac{1}{\sqrt{2(a^2+b^2)}}\left(\frac{a+b}{2ab}(a^2+b^2)+\frac{4ab}{a+b}\right)$$

$$\leq \frac{1}{\sqrt{2(a^2+b^2)}}\left(\frac{a+b}{2ab}(a^2+b^2)+(a+b)\right)\leq \frac{1}{a+b}\left(\frac{a+b}{2ab}(a^2+b^2)+(a+b)\right)$$

$$=\frac{a^2+b^2}{2ab}+1\leq \frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)=\frac{a}{b}+\frac{b}{a},$$

where we applied the ***AM - GM inequality***.

□

Proof 2 (by Soumava Chakraborty - Kolkata - India).

We know that

Harmonic mean $\leq$ Arithmetic mean $\leq$ Quadratic mean,

implying

$$\frac{2ab}{a+b}=\frac{2}{\frac{1}{a}+\frac{1}{b}}\leq \frac{a+b}{2}\leq \sqrt{\frac{a^2+b^2}{2}}$$

such that

$$\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\leq 2\sqrt{\frac{a^2+b^2}{2}}. \tag{1}$$

Also,

$$\frac{a+b}{2}\left(\frac{1}{ab}\right)\leq \sqrt{\frac{a^2+b^2}{2}}\left(\frac{1}{ab}\right),$$

such that

$$\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\leq \sqrt{\frac{a^2+b^2}{2}}\left(\frac{1}{ab}\right)+\sqrt{\frac{2}{a^2+b^2}}, \tag{2}$$

Multiplying (1) and (2) we get

$$\left(\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\right)$$

$$\leq 2\sqrt{\frac{a^2+b^2}{2}}\left(\sqrt{\frac{a^2+b^2}{2}}\left(\frac{1}{ab}\right)+\sqrt{\frac{2}{a^2+b^2}}\right)=\frac{a^2+b^2}{ab}+2=\frac{(a+b)^2}{ab}.$$

□

Proof 3 (by Daniel Sitaru - Romania).

WLOG, assume $a \le b$. As before,

$$0 < a \le \frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}} \le b.$$

We'll use *Schweitzer's inequality:*

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \le \frac{(m+M)^2 n^2}{4mM},$$

where $x_1, \ldots, x_n \in [m, M], M > 0$.

with $= 2, x_1 = \dfrac{2ab}{a+b}, x_2 = \sqrt{\dfrac{a^2+b^2}{2}}$, we directly get

$$\left(\frac{2ab}{a+b} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab} + \sqrt{\frac{2}{a^2+b^2}}\right) \le \frac{(a+b)^2}{ab}.$$

□

Acknowledgment (by Alexander Bogomolny - USA)
The problem above has been kindly posted to the ***CutTheKnotMath page*** by Dan Sitaru, along with several solutions. Solution 1 is by Kevin Soto Palacios; Solution 2 by Soumava Chakraborty; Solution 3 is by Dan Sitaru.

80. Inequality with Cubes and Cube Roots

In ΔABC,

$$\sum_{cycl} (\sqrt[3]{a} + \sqrt[3]{b} - \sqrt[3]{c})^3 \ge \sqrt[3]{3a} + \sqrt[3]{3b} + \sqrt[3]{3c} - 2$$

where, as usual, a, b, c *denote the side lengths of the triangle.*

Proposed by Daniel Sitaru - Romania

Proof 1 (by Leonard Giugiuc - Romania).
First note that $\sqrt[3]{a}$, $\sqrt[3]{b}$, $\sqrt[3]{c}$ from a triangle. Indeed,

$$(\sqrt[3]{a} + \sqrt[3]{b})^3 = a + b + 3\sqrt[3]{ab}(\sqrt[3]{a} + \sqrt[3]{b}) > a + b > c,$$

implying $\sqrt[3]{a} + \sqrt[3]{b} > \sqrt[3]{c}$. With this in mind, denote

$$-\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = 2x,$$
$$\sqrt[3]{a} - \sqrt[3]{b} + \sqrt[3]{c} = 2y,$$
$$\sqrt[3]{a} + \sqrt[3]{b} - \sqrt[3]{c} = 2z.$$

Then $x, y, z > 0$. Further $\sqrt[3]{a} = y + z$, $\sqrt[3]{b} = x + z$, and $\sqrt[3]{c} = x + y$. The required inequality becomes

$$4(x^3 + y^3 + z^3) \ge \sqrt[3]{3}(x + y + z) - 1.$$

Let $x + y + z = 3s$. By Jensen's inequality, $x^3 + y^3 + z^3 \ge 3s^3$, with equality only if $x = y = z + s$. Hence, suffice it to show that

$$12s^3 - 3\sqrt[3]{3}s + 1 \ge 0,$$

which is equivalent to $(2\sqrt[3]{3}s - 1)(\sqrt[3]{3}s + 1) \ge 0, 3$ which is obviously true.
Equality holds only if $x = y = z = \frac{1}{2\sqrt[3]{3}}$, i.e., when $a = b = c = \frac{1}{3}$. □

Proof 2 (by Daniel Sitaru - Romania).
By the AM - GM inequality,

$$(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c})^3+\frac{2}{3}=(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c})^3+\frac{1}{3}+\frac{1}{3}$$

$$\geq 3\sqrt[3]{(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c})^3\cdot\frac{1}{3}\cdot\frac{1}{3}}=\frac{3}{\sqrt[3]{9}}(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c})=\sqrt[3]{3}(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}).$$

That is,

$$(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c})^2+\frac{2}{3}\geq\sqrt[3]{3a}+\sqrt[3]{3b}-\sqrt[3]{3c}.$$

Similarly,

$$(\sqrt[3]{b}+\sqrt[3]{c}-\sqrt[3]{a})^3+\frac{2}{3}\geq\sqrt[3]{3b}+\sqrt[3]{3c}-\sqrt[3]{3a}$$

and

$$(\sqrt[3]{c}+\sqrt[3]{a}-\sqrt[3]{b})^3+\frac{2}{3}\geq\sqrt[3]{3c}+\sqrt[3]{3a}-\sqrt[3]{3b}$$

Adding the three gives the required inequality. Equality is attained for $a=b=c=\frac{1}{3}$. □

Acknowledgment (by Alexander Bogomolny - USA)
I am grateful to Dan Sitaru for communicated to me the above problem from his book *Math Accent*, with two solutions. Solution 1 is by Leo Giugiuc, Solution 2 is by Dan Sitaru.

81. A Followup on Solving A Fourth Degree Equation

Acknowledgment (by Alexander Bogomolny - USA)
I learned of a problem posted by Dan Sitaru from a solution by Kunihiko Chikaya. More than the solution I liked the question, and not even the question its being a followup on the previous one. This teaches a brilliant way to generate new problems by modifying the ones already solved. This is certainly an excellent illustration of George Polya's last step - Looking back - in problem solving.
Find

$$\sum_{i=1}^{4}|x_i|$$

where $x_i, i=1,2,3,4$ *are the roots of*

$$x^4+8x^3+23x^2+28x+10=0$$

Proposed by Daniel Sitaru - Romania

Proof (by Kunihiko Chikaya - Tokyo - Japan).
Clearly this problem is a followup on another one where a similar equation has been solved by three different methods. Any of these will be a good first step for answering the question at hand. I'll use the second solution which implies that

$$x^4+8x^3+23x^2+28x+10=(x+2)^4-(x+2)^2-2.$$

Thus we are led to four roots of the given polynomial; $-2\pm\sqrt{2}$ and $2\pm i$, whose moduli add up to $4+2\sqrt{5}$. □

82. An Inequality in Triangle III

Prove that in ΔABC, with angles A, B, C side lengths a, b, c the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\cos^2\frac{A}{2}}+\frac{b(c+a)}{ca\cdot\cos^2\frac{B}{2}}+\frac{c(a+b)}{ab\cdot\cos^2\frac{C}{2}}\geq 8.$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Daniel Sitaru, Leonard Giugiuc - Romania).
WLOG, assume $a\geq b\geq c$. Then $ab+ac\geq bc+ca\geq cb+ca$; also, $\frac{1}{bc}\geq\frac{1}{ca}\geq\frac{1}{ab}$. From these, $\frac{ab+ac}{bc}\geq\frac{ab+bc}{ac}\geq\frac{bc+ac}{ab}$. On the other hand, $\frac{1}{\cos^2 x}=1+\tan^2 x$ and the tangent function in strictly increasing and positive on $\left(0,\frac{\pi}{2}\right)$, hence
$1+\tan^2\frac{A}{2}\geq 1+\tan^2\frac{B}{2}\geq 1+\tan^2\frac{C}{2}$. We can apply now Chebyshev's inequality to get

$$\sum_{cyc}\frac{ab+ac}{bc\cdot\cos^2\frac{A}{2}}\geq$$

$$\geq\frac{1}{3}\left(\frac{ab+ac}{bc}+\frac{ab+bc}{ac}+\frac{bc+ac}{ab}\right)\left(3+\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)$$

Suffice it to prove that

$$\frac{1}{3}\left(\frac{ab+ac}{bc}+\frac{ab+bc}{ac}+\frac{bc+ac}{ab}\right)\left(3+\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)\geq 24$$

But obviously $\frac{ab+ac}{bc}+\frac{ab+bc}{ac}+\frac{bc+ac}{ab}\geq 6$. On the other hand,

$$3+\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 3+\frac{1}{3}\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2\geq 3+1=4$$

□

Proof 2 (by Kunihiko Chikaya - Japan).
From the half - angle formula and the Low of Cosine,

$$\cos^2\frac{A}{2}=\frac{1+\cos A}{2}=\frac{(b+c)^2-a^2}{4bc},$$

and similarly for the other two angles. Thus the inequality at hand is equivalent to

$$\frac{a(b+c)}{(b+c)^2-a^2}+\frac{b(c+a)}{(c+a)^2-b^2}+\frac{c(a+b)}{(a+b)^2-c^2}\geq 2,$$

or,

$$\frac{a(b+c)}{b+c-a}+\frac{b(c+a)}{c+a-b}+\frac{c(a+b)}{a+b-c}\geq 2(a+b+c),$$

or, else

$$\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c}\geq a+b+c,$$

Now, by the Cauchy - Schwarz inequality, for any $x, y, z>0$,

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\geq\frac{(a+b+c)^2}{x+y+z},$$

which with $x=b+c-a, y=c+a-b, z=a+b-c$ gives

$$\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c}\geq\frac{(a+b+c)^2}{a+b+c}=a+b+c$$

□

Proof 3 (by Alexander Bogomolny - USA).
We know that $\cos^2\frac{A}{2}=\frac{p(p-a)}{bc}$, where $p=\frac{a+b+c}{2}$ is the semiperimeter of ΔABC. Thus, the required inequality can be rewritten as

$$\frac{a(b+c)}{p(p-a)}+\frac{b(c+a)}{p(p-b)}+\frac{c(b+a)}{p(p-c)}\geq 8$$

Now observe that $\frac{a(b+c)}{p(p-a)}=\frac{2p}{p}+\frac{a^2}{p(p-a)}$, and similar for the other two fractions. Since, $\sum_{cyc}\frac{2a}{p}=4$, the required inequality reduces to

$$\frac{a^2}{p(p-a)}+\frac{b^2}{p(p-b)}+\frac{c^2}{p(p-c)}\geq 4$$

Consider the function $f(x)=\frac{x^2}{p-x}, f'(x)=\frac{2xp}{(p-x)^2}>0$ and $f''(x)=\frac{2p(p+x)}{(p-x)^3}<0$ for $x\in(0,p)$. Thus the function is convex on $(0,p)$. Keeping p fixed, we may apply Jensen's inequality:

$$\frac{a^2}{p(p-a)}+\frac{b^2}{p(p-b)}+\frac{c^2}{p(p-c)}\geq\frac{(\frac{a+b+c}{3})^2}{p(p-\frac{a+b+c}{3})}$$

$$=3\frac{(\frac{2p}{3})^2}{p(p-\frac{2p}{3})}=3\cdot\frac{4}{9}\cdot\frac{3}{1}=4.$$

The problem form the *Math Phenomenon* has been posted at the CutTheNotMath page by Dan Sitaru, along with a solution (Solution 1) by Leo Giugiuc and Dan Sitaru. Solution 2 is by Kunikiko Chikaya. □

83. An Inequality with Exponents

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted a problem form his book "Math Phenomenon" at the CutTheKnotMath page. He also posted a solution (Solution 1)

If $a,b,c\in(0,1]$, *then*

$$e^{\frac{4}{e}}\left(b\cdot a^{2\sqrt{2}}+c\cdot b^{2\sqrt{b}}+a\cdot c^{2\sqrt{c}}\right)\geq 3\sqrt[3]{abc}.$$

When does the equality hold?

Proposed by Daniel Sitaru - Romania

Proof 1 (by Daniel Sitaru - Romania).
Define $f(x):(0,1]\to\mathbb{R}$ with $f(x)=x^{2\sqrt{x}}.f'(x)=x^{2\sqrt{x}-\frac{1}{2}}(2+\ln x)$.

$$\lim_{x\to0^+}x^{2\sqrt{x}}=\lim_{x\to0^+}e^{2\sqrt{x}\ln x}=e^{2\lim_{x\to0^+}\sqrt{x}\ln x}=e^0=1.$$

$f'(x)$ has the only root that can be found from $2+\ln x=0$, giving $x=e^{-2}$. Thus $f(x)$ is monotone decreasing on $(0,e^{-2}]$ and monotone increasing on $(e^{-2},1].f(1)=1$. It follows that

$$e^{-\frac{4}{e}}\leq a^{2\sqrt{a}}<1, e^{-\frac{4}{e}}\leq b^{2\sqrt{b}}<1, e^{-\frac{4}{e}}\leq x^{2\sqrt{c}}<1.$$

And, subsequently,

$$b\cdot e^{-\frac{4}{e}}\leq ba^{2\sqrt{a}}<1, c\cdot e^{-\frac{4}{e}}\leq cb^{2\sqrt{b}}<1, a\cdot e^{-\frac{4}{e}}\leq ac^{2\sqrt{c}}<1.$$

Adding up and using the AM - GM inequality,

$$b \cdot e^{-\frac{4}{e}} + c \cdot e^{-\frac{4}{e}} + a \cdot e^{-\frac{4}{e}} \geq (a+b+c)e^{-\frac{4}{e}} \geq 3\sqrt{abc} \cdot e^{-\frac{4}{e}}.$$

In other words,

$$e^{\frac{4}{e}}\left(b \cdot e^{-\frac{4}{e}} + c \cdot e^{-\frac{4}{e}} + a \cdot e^{-\frac{4}{e}}\right) \geq 3\sqrt{abc}.$$

□

Proof 2 (by Alexander Bogomolny - USA).
This solution is much the same as the first one, with a few simplifications. First, replace $a = x^2, b = y^2, c = z^2$ to reduce the required inequality to

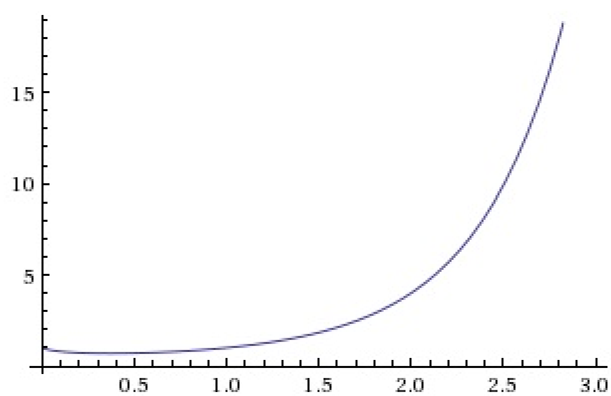
$$e^{\frac{4}{e}}(y^2x^{4x} + z^2y^{4y} + x^2z^{4z}) \geq 3\sqrt[3]{x^2y^2z^2}.$$

With the AM-GM inequality we obtain

$$\frac{1}{3}(y^2x^{4x} + z^2y^{4y} + x^2z^{4z}) \geq \sqrt[3]{x^{4x}y^{4y}z^{4z}} \cdot \sqrt[3]{x^2y^2z^2}$$

$$\geq \sqrt[3]{x^2y^2z^2}\sqrt[3]{\left[\left[\frac{1}{e}^{\frac{1}{e}}\right]\right]^{4\cdot 3}} = \sqrt[3]{x^2y^2z^2}\cdot\left(\frac{1}{e}\right)^{\frac{4}{e}}.$$

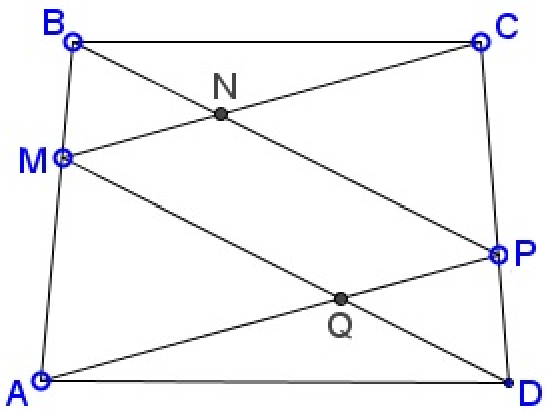
The latter inequality is the consequence of the properties of function $f(x) = x^x$, defined for $x > 0$. Its derivative $f'(x) = x^x(1 + \ln x)$ vanishes only at $x = \frac{1}{e}$, where the function attains its minimum:



Indeed, the derivative $f'(x) = x^x(1 + \ln x)$ is negative for $x < \frac{1}{e}$ and positive for $x > \frac{1}{e}$. Thus the required inequality hods for $a, b, c > 0$.
The equality is attained when $x = y = z = \frac{1}{e}$, i.e., when $a = b = c = \frac{1}{e^2}$, in which case both sides of the inequality are equal to $3e^{-2}$. □

84. Parallelogram in Trapezoid

In trapezoid $ABCD, BC \parallel AD, P \in CD$ satisfies $AP = BP; M \in AB$, with $\angle AMD = \angle BMC; N = BP \cap CM$ and $Q = AP \cap DM$.



Prove that the quadrilateral $MNPQ$ is a parallelogram.

Proposed by Miguel Ochoa Sanchez - Peru

Proof 1 (by Leonard Giugiuc - Romania).
Choose $A = (1, 0), B = (-1, 0)$, and $P = (0, a)$ with $a > 0$. Since $P \in CD$ does not cross the interior of ΔAPB, there is $m \in (-a, a)$ such that CD is defined by the equation $-mx + y = a$. Also, since $BC \parallel AD$, and neither passes through the interior of ΔAPB, there is $n \in \left(-\frac{1}{a}, \frac{1}{a}\right)$ such that BC and AD are defined by $x - ny = -1$ and $x - ny = 1$, respectively.
These gives us $C = \left(\frac{na-1}{1-mn}, \frac{a-m}{1-mn}\right)$ and $D = \left(\frac{na+1}{1-mn}, \frac{a+m}{1-mn}\right)$. Assume $M = (k, 0)$. Since $\angle BMC = \angle AMD$, the lines MC and MD have opposite *antislopes*.
Thus

$$\frac{k - \frac{na-1}{1-mn}}{\frac{a-m}{1-nm}} = \frac{-k + \frac{na+1}{1-mn}}{\frac{a+m}{1-mn}},$$

implying $k = \frac{na^2-n}{a(1-mn)}$. Using this, we can easily check that $MC \parallel AP$ and $MD \parallel BP$. Thus $MNPQ$ is indeed a parallelogram. □

Proof 2 (by Alexander Bogomolny - USA).
Find point M' on AB such that $\angle BM'C = 90^\circ - \frac{1}{2}\angle APB$. From M' draw a ray $M'D'$, with D' on line CP such that $\angle CM'D' = \angle APB$. Then, ***as we know***, $AD' \parallel BC$, so, since also $AD \parallel BC$, we see that $D' = D$. Thus M' solves ***Heron's problem*** for C and D and, as such, is unique on AB with the property that $\angle BM'C = \angle AM'D$. It follows that $M = M'$.
From the construction of M', the quadrilateral $MNPQ$ is a parallelogram. □

85. A Cyclic Inequality in Three Variables II

Let $a, b, c > 0$. Prove that

$$\frac{10a^3}{3a^2 + 7bc} + \frac{10b^3}{3b^2 + 7ca} + \frac{10c^3}{3a^2 + 7ab} \geq a + b + c$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Anas Adlany - Morroco).
Since the inequality is *homogeneous*, we may assume **WLOG**, $a^2 + b^2 + c^2 = 1$. First using the ***GM - QM inequality*** and then ***Chebyshev's inequality*** twice, we get

$$\sum_{cycl} \frac{10a^3}{3a^2 + 7bc} \geq \sum_{cycl} \left(\frac{10a^3}{3a^2 + 7(\frac{b^2+c^2}{2})} \right)$$

$$= 20 \sum_{cycl} \frac{a^3}{6a^2 + 7b^2 + 7c^2} = 20 \sum_{cycl} \Big(\frac{a^3}{7 - a^2} \Big)$$

$$\geq \frac{20}{9} \Big(\sum_{cycl} a \Big) \Big(\sum_{cycl} a^2 \Big) \Big(\sum_{cycl} \frac{1}{7 - a^2} \Big) \geq \frac{20}{9} \Big(\sum_{cycl} a \Big) \Big(\frac{9}{\sum_{cycl} (7 - a^2)} \Big) = \sum_{cycl} a,$$

as desired. □

Solution 2 (by Imad Zak - Lebanon).
Since the inequality is *homogeneous*, we may assume ***WLOG***, $abc = 1$. The inequality is then rewritten as $\sum_{cycl} f(a) \geq 0$, where

$$f(x) \frac{10x^4}{3x^3 + 7} - x = \frac{7x(x^3 - 1)}{3x^3 + 7}$$

The function is convex, so that $f(x) \geq g(x) = \frac{21}{10}(x - 1)$, which is its tangent at $x = 1$.
Thus

$$\sum_{cycl} f(a) \geq \sum_{cycl} g(a) = \frac{63}{10} - \frac{63}{10} = 0,$$

which is the required inequality. □

Proof 3 (by Soumitra Mandal - India).
By *Hölder's inequality*,

$$(a + b + c)^3 \leq \sum_{cycl} \frac{a^3}{3a^2 + 7bc} \cdot \sum_{cycl} (3a^2 + 7bc) \cdot \sum 1.$$

Thus, suffice it to prove that

$$\sum_{cycl} (3a^2 + 7bc) \leq \frac{10(a + b + c)^2}{3},$$

or, in other words, that $a^2 + b^2 + c^2 \geq a + b + c$. But then

$$\sum_{cycl} \frac{10a^3}{3a^2 + 7bc} \geq \frac{10(a + b + c)^3}{3 \sum_{cycl} (3a^2 + 7bc)} \geq \frac{10(a + b + c)^3}{10(a + b + c)^2} = a + b + c.$$

□

Proof 4 (by Soumava Chakraborty - India).
We have a series of equivalent inequalities:

$$\frac{10a^3}{3a^2 + 7bc} + \frac{10b^3}{3b^2 + 7ca} + \frac{10c^3}{3a^2 + 7ab} \geq a + b + c,$$

$$\sum_{cycl} \Big(\frac{10a^3}{3a^2 + 7bc} - a \Big) \geq 0,$$

$$\frac{7}{2}\sum_{cycl}\frac{a(a+b)(a-c)+a(a-b)(a+c)}{3a^2+7bc}\geq 0,$$

$$\frac{7}{2}\sum_{cycl}\Big[\frac{a(a-b)(a+c)}{3a^2+7bc}+\frac{a(a+b)(a-c)}{3a^2+7bc}\Big]\geq 0,$$

$$\frac{7}{2}\sum_{cycl}\Big[\frac{a(a-b)(a+c)}{3a^2+7bc}+\frac{b(b+c)(b-a)}{3b^2+7ca}\Big]\geq 0,$$

$$\frac{7}{2}\sum_{cycl}(a-b)\Big[\frac{a(a+c)}{3a^2+7bc}-\frac{b(b+c)}{3b^2+7ca}\Big]\geq 0,$$

$$\frac{7}{2}\sum_{cycl}(a-b)^2\Big[\frac{7c(a^2+b^2)+7c^2(a+b)+4abc}{(3a^2+7bc)(3b^2+7ca)}\Big]\geq 0.$$

The latter is obviously true and, so, the rest are also true. □

Acknowledgment (by Alexander Bogomolny - USA)
The problem above (from the ***Romanian Mathematical Magazine***) has been kindly communicated to me by Dan Sitaru, along with four solutions. Solution 1 is by Anas Adlany (Morroco); Solution 2 is by Imad Zak (Lebanon); Solution 3 is by Kevin Soto Palacios (Peru); Solution 3 is by Soumitra Mandal (India); Solution 4 is by Soumava Chakraborty (India).

86. A Cyclic Inequality in Three Variables IV

Let $a, b, c > 0$. Prove that

$$2\sum_{cycl}(a+b)^3+5\sum_{cycl}a^3\geq 21\sum_{cycl}a^2b$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Kevin Soto Palacios - Peru).

The required inequality is equivalent to $9\sum_{cycl}a^3+6\sum_{cycl}ab^2\geq 15\sum_{cycl}a^2b$.

Using ***AM - GM*** inequality,

$$6a^3+6b^2a\geq 12a^2b$$

$$6b^3+6c^2b\geq 12b^2c$$

$$6c^3+6a^2c\geq 12c^2a.$$

Summing the three up gives

$$6\sum_{cycl}a^3+6\sum_{cycl}ab^2\geq 12\sum_{cycl}a^2b. \tag{10}$$

On the other hand, again, by the AM - GM inequality,

$$3(a^3+b^3+c^3)=(a^3+a^3+b^3)+(b^3+b^3+c^3)+(c^3+c^3+a^3)$$
$$\geq 3a^2b+3b^2c+3c^2a.$$

Adding this to (10) proves the required inequality. □

Proof 2 (by Soumava Chakraborty - India).
The required inequality reduces to,

$$3\sum_{cycl} a^3 + 2\sum_{cycl} a^2b + 2\sum_{cycl} ab^2 \geq 7\sum_{cycl} a^2b.$$

Bu the AM - GM inequality,

$$a^3 + a^2b + ab^2 \geq 3a^2b$$
$$b^3 + b^2c + bc^2 \geq 3b^2c$$
$$c^3 + c^a + ca^2 \geq 3c^2a.$$

Summing up we get

$$\sum_{cycl} a^3 + \sum_{cycl} a^2b + \sum_{cycl} ab^2 \geq 3\sum_{cycl} a^2b. \tag{1}$$

On the other hand,

$$3(a^3 + b^3 + c^3) = (a^3 + a^3 + b^3) + (b^3 + b^3 + c^3) + (c^3 + c^3 + a^3)$$
$$\geq 3a^2b + 3b^2c + 3c^2a$$

So that $\sum_{cycl} a^3 \geq \sum_{cycl} a^2b$. adding this to twice (1) gives

$$3\sum_{cycl} a^3 + 2\sum_{cycl} a^2b + 2\sum_{cycl} ab^2 \geq 7\sum_{cycl} a^2b$$

as expected. □

Proof 3 (by Seyran Ibrahimov - Azerbaidian).
By the AM - GM inequality,

$$a^3 + a^3 + ab^2 + ab^2 \geq 4a^2b$$
$$b^3 + b^3 + bc^2 + bc^2 \geq 4b^2c$$
$$c^3 + c^3 + ca^2 + ca^2 \geq 4c^2a.$$

Summing up gives $\sum_{cycl} a^3 + \sum_{cycl} ab^2 \geq 2\sum_{cycl} a^2b$. Denotes the left - hand side X.

Further, the required inequality reduces to

$$9\sum_{cycl} a^3 + 6\sum_{cycl} ab^2 \geq 15\sum_{cycl} a^2b$$

which can be written as $\sum_{cycl} a^3 + 2X \geq 5\sum_{cycl} a^2b$. It will proved correct if

$$\sum_{cycl} a^3 \geq \sum_{cycl} a^2b$$

But this is true due to the ***Rearrangement inequality***. □

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the above problem at the ***CutTheKnotMath page***, followed by three solutions. Solution 1 is by Kevin Soto Palacios (Peru); Solution 2 is by Soumava Chakraborty (India); Solution 3 is by Seyran Ibrahimov (Azerbaijan).

87. A Cyclic Inequality in Three Variables IX

Let $x, y, z > 0$. Prove that

$$9\left(\sum_{cycl}\frac{x^2}{y^2}\right)^2 \geq 8\left(\sum_{cycl}\frac{x}{y}\right)\left(\sum_{cycl}\frac{x^3}{y^3}-3\right)$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Saptak Bhattacharya - India).
Let $x = \frac{x}{y}, b = \frac{y}{z}, a = \frac{z}{x}$. Note that $abc = 1$. The given inequality rewrites as

$$9(a^2+b^2+c^2)^2 \geq 8(a+b+c)(a^3+b^3+c^3-3abc).$$

Using $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ and rearranging, this reduces to

$$\left(\sum_{cycl}a^2\right)^2 - 8\left(\sum_{cycl}ab\right)\left(\sum_{cycl}a^2\right) + 16\left(\sum_{cycl}ab\right)^2 \geq 0,$$

or, $(a^2+b^2+c^2-4ab-4bc-4ca)^2 \geq 0$ which is obviously true. □

Proof 2 (Nassim Nicholas Taleb - USA).

Set $f = 9\sum_{cycl}\frac{x^2}{y^2} - 8\left(\sum_{cycl}\frac{x}{y}\right)\left(-3\sum_{cycl}\frac{x^3}{y^3}\right)$. We need to prove that $f \geq 0$.

Factoring we get

$$f = \frac{9(\sum_{cycl}x^2y^4)^2}{x^4y^4z^4} - \frac{8(\sum_{cycl}xy^2)^2(\sum_{cycl}x^2y^4 - \sum_{cycl}x^3y^2z)}{x^4y^4z^4}$$

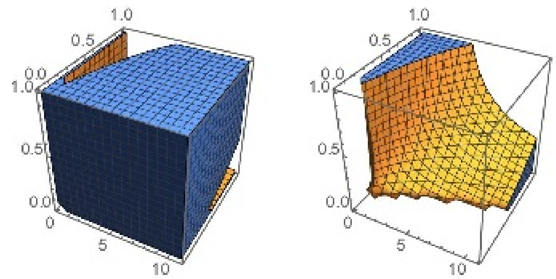
The numerator reduces to

$$\left(\sum_{cycl}x^2y^4 - 4\sum_{cycl}x^3y^2z\right)^2 \geq 0$$

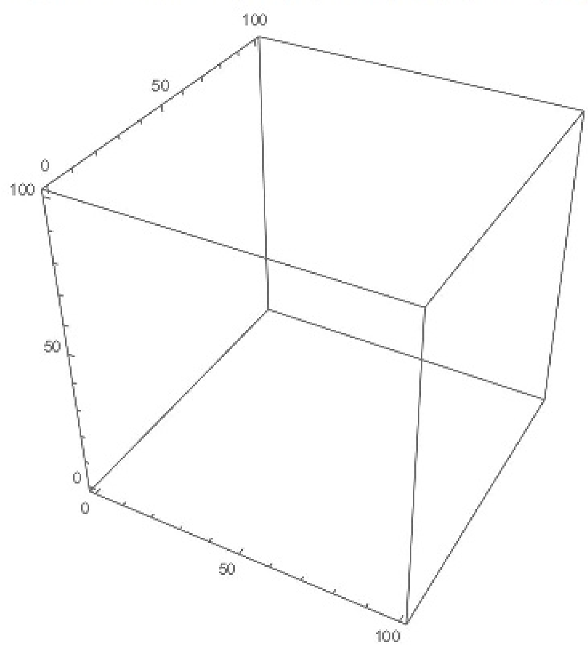
```
f1 = -4 x^3 y^2 z + x^4 z^2 - 4 x y^3 z^2 + y^2 z^4 + x^2 (y^4 - 4 y z^3);
```

We know that there is no region where f1 = 0. We can see the positive and the negatives

```
GraphicsRow[{RegionPlot3D[f1 > 0, {x, 0, 11}, {y, 0, 1}, {z, 0, 1}],
  RegionPlot3D[f1 < 0, {x, 0, 11}, {y, 0, 1}, {z, 0, 1}]}]
```

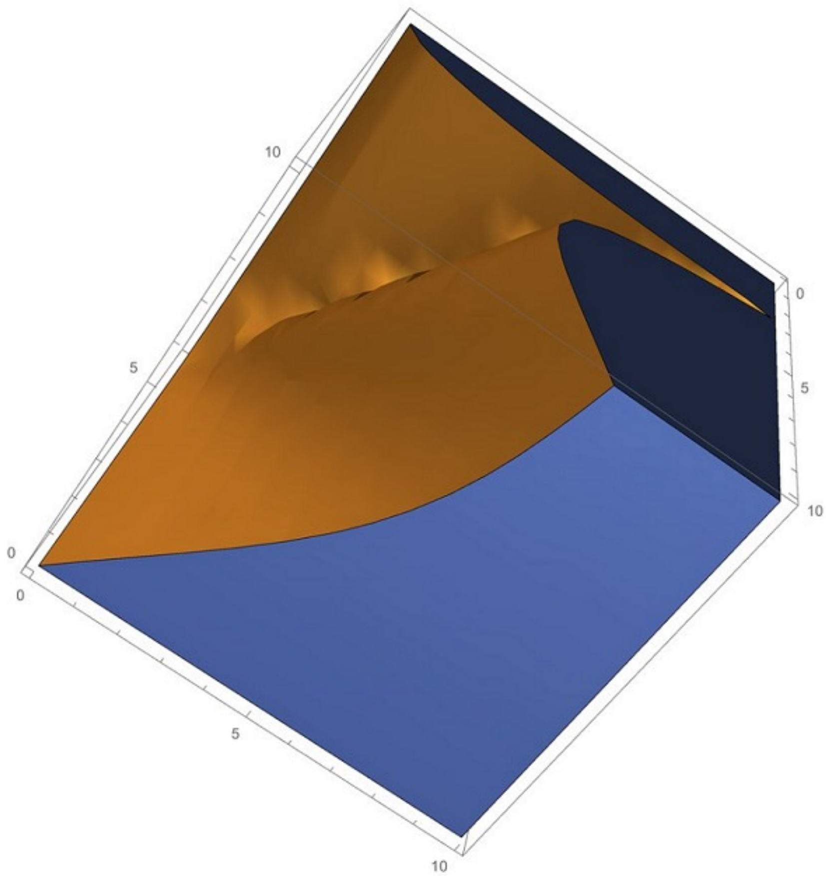


```
RegionPlot3D[f1 = 0, {x, 0, 100}, {y, 0, 100}, {z, 0, 100}]
```



□

Illustration (by Gary Davis - USA)



Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the above problem (from his book "Math Accent") at the ***CutTheKnotMath page***, along with a solution (Solution 1) by Saptak Bhattacharya. Solution 2 is by Nassim Nicholas Taleb. The illustration is by Gary Davis.

88. A Cyclic Inequality in Three Variables VI

Let $a, b, c > 0$. Prove that

$$\frac{2(a+b+c)}{abc} \geq \sum_{cycl}\left(\sqrt{\frac{a+b}{2ac}} + \sqrt{\frac{2a}{c(a+b)}}\right)$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Kevin Soto Palacios - Peru).

$$\sum_{cycl}\sqrt{\frac{a+b}{2ac}} + \sum_{cycl}\sqrt{\frac{2a}{c(a+b)}} \leq \sum_{cycl}\sqrt{\frac{(a+b)b}{2abc}} + \sum_{cycl}\sqrt{\frac{(a+b)a}{2abc}}$$

$$\leq \frac{\sum_{cycl}\sqrt{a+b}(\sqrt{a}+\sqrt{b})}{\sqrt{abc}} \leq \frac{\sum_{cycl}\sqrt{2}(\sqrt{a+b})^2}{\sqrt{2abc}}.$$

It follows that

$$\sum_{cycl}\sqrt{\frac{a+b}{2ac}} + \sum_{cycl}\sqrt{\frac{2a}{c(a+b)}} \leq \frac{\sum_{cycl}\sqrt{2}(\sqrt{a+b})^2}{\sqrt{abc}}$$

$$= \frac{2\sqrt{2}(a+b+c)}{\sqrt{2abc}} = \frac{2(a+b+c)}{\sqrt{abc}},$$

as required. □

Proof 2 (by Soumava Chakraborty - India).
Using the ***Cauchy - Schwarz inequality***,

$$\sum_{cycl}\sqrt{\frac{a+b}{2ac}} \leq \sum_{cycl}\sqrt{\frac{(a+b)b}{2abc}} \leq \frac{\sqrt{\sum_{cycl}a}\sqrt{2\sum_{cycl}a}}{\sqrt{2abc}}$$

$$= \frac{a+b+c}{\sqrt{abc}}$$

Again, using the ***Cauchy - Schwarz inequality***,

$$\sum_{cycl}\sqrt{\frac{2a}{c(a+b)}} \leq \sqrt{2\sum_{cycl}a}\sqrt{\sum_{cycl}\frac{1}{c(a+b)}} \leq \sqrt{2\sum_{cycl}a}\sqrt{\sum_{cycl}\frac{1}{c(2\sqrt{ab})}}$$

$$= \sqrt{\sum_{cycl}a}\sqrt{\frac{1}{\sqrt{abc}}\sum_{cycl}\frac{1}{\sqrt{a}}} = \sqrt{\sum_{cycl}a}\sqrt{\frac{1}{\sqrt{abc}}\sum_{cycl}\frac{bc}{\sqrt{abc}}} =$$

$$= \sqrt{\frac{\sum_{cycl}a}{abc}}\sqrt{\sum_{cycl}\sqrt{ab}} \leq \sqrt{\frac{\sum_{cycl}a}{abc}}\sqrt{\sqrt{\sum_{cycl}a}\sqrt{\sum_{cycl}b}} \leq \frac{\sum_{cycl}a}{\sqrt{abc}}$$

Now it only remains to add the two inequalities.

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the problem from his book, *Math Accent*, with two solutions, at the ***CutTheKnotMath page***. Solution 1 is by Kevin Soto Palacios; Solution 2 is by Soumava Chakraborty. □

89. A Cyclic Inequality in Three Variables VIII

Let $x, y, z > 0$. *Prove that*

$$\sum_{cycl}(x^2+y^2)z + \sum_{cycl}\frac{xy}{(x+y)^2} \geq 27xyz$$

Proposed by Daniel Sitaru - Romania

Proof (by Mihalcea Andrei Ştefan - Romania).
Dividing by xyz throughout, we get an equivalent inequality:

$$4\sum_{cycl}\left(\frac{x}{y}+\frac{y}{x}\right) + 4\sum_{cycl}\frac{1}{\frac{x}{y}+\frac{y}{x}+2} \geq 27.$$

We'll show that

$$4\Big(\frac{x}{y}+\frac{y}{x}\Big)+4\frac{1}{\frac{x}{y}+\frac{y}{x}+2}\geq 9.$$

Denote $\alpha=\frac{x}{y}+\frac{y}{x}\geq 2$ (by the ***AM - GM inequality***). In terms of α the latter inequality becomes $4\alpha+\frac{4}{\alpha+2}\geq 9$, which reduces to

$$4\alpha^2-\alpha+14=(\alpha-2)(4\alpha+7)\geq 0,$$

which is true because $\alpha\geq 2$. □

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the above problem (from the ***Romanian Mathematical Magazine***) at the ***CutTheKnotMath page***, along with a solution by Mihalcea Andrei Ştefan.

90. A Cyclic Inequality in Three Variables X

Let $a,b,c>0$ satisfy $a^2+b^2+c^2=3$. Prove that

$$\sum_{cycl}\frac{1}{(a+1)^3}+4\sum_{cycl}\frac{1}{(a+1)^4}\geq\frac{9}{8}$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Leonard Giugiuc - Romania).
First note that from ***AM - QM*** inequality, $\left(\frac{a+b+c}{3}\right)^2\leq\frac{a^2+b^2+c^2}{3}=1$ so that $a+b+c\leq 3$.
Now, both functions, $y=\frac{1}{(x+1)^3}$ and $y=\frac{1}{(x+1)^4}$ are convex on $(0,\infty)$, so that by *Jensen's inequality*

$$\sum_{cycl}\frac{1}{(a+1)^3}\geq\frac{3}{(\frac{a+b+c}{3}+1)^3}\geq\frac{3}{8}$$

and

$$\sum_{cycl}\frac{4}{(a+1)^4}\geq\frac{3\cdot 4}{(\frac{a+b+c}{3}+1)^4}\geq\frac{3}{4}.$$

□

Proof 2 (by Alexander Bogomolny - USA).
We obtain the same result by using ***Radon's inequality***

$$\sum_{cycl}\frac{1}{(a+1)^3}\geq\frac{(1+1+1)^4}{(a+b+c+3)^3}\geq\frac{3}{8}$$

and

$$\sum_{cycl}\frac{4}{(a+1)^4}\geq\frac{4(1+1+1)^5}{(a+b+c+3)^4}\geq\frac{3}{4}$$

□

Proof 3 (by Imad Zak - Lebanon).
Define $(x)=\frac{1}{(x+1)^3}+\frac{4}{(x+1)^4}$. We have

$$f(x)-\Big(\frac{17}{16}-\frac{11}{16}\Big)=\frac{(x-1)^2(11x^3+49x^2+85x+63)}{16(x+1)^4}\geq 0,$$

implying

$$\sum_{cycl} f(a) \geq \sum_{cycl} \left(\frac{17}{16} - \frac{11a}{16}\right) = 3 \cdot \frac{17}{16} - (a+b+c) \cdot \frac{11}{16}.$$

But from

$$\sum_{cycl} a^2 = 3$$

it follows that $a+b+c \leq 3$. Thus,

$$\sum_{cycl} f(a) \geq \frac{51}{16} - \frac{33}{16} = \frac{9}{8}.$$

□

Proof 4 (by Amit Itagi).
Let define $x = a^2, y = b^2, z = c^2$. The problem becomes

Let $x, y, z > 0$ satisfy $x + y + z = 3$. Prove that

$$\sum_{cycl} \frac{\sqrt{x}+5}{(\sqrt{x}+1)^4} \geq \frac{9}{8}$$

Note that the function $f(u) = \frac{\sqrt{u}+5}{(\sqrt{u}+1)^4}$ is convex on $(0, \infty)$ and thus the problem lends itself to Jensen's inequality:

$$\sum_{cycl} \frac{\sqrt{x}+5}{(\sqrt{x}+1)^4} \geq 3\frac{\sqrt{\frac{3}{3}}+5}{(\sqrt{\frac{3}{3}}+1)^4} = \frac{9}{8}.$$

□

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the above problem (from his book "Math Accent") at the ***CutTheKnotMath page*** to which Leo Giugiuc responded with Solution 1. Solution 3 is by Imad Zak; Solution 4 is by Amit Itagi.

91. A Cyclic Inequality in Three Variables XII

Prove that, for all $a, b, c > 0$

$$\left(\sum_{cycl} \frac{1}{(a^2-ab+b^2)^6}\right)^2 \leq 3\sum_{cycl}\left(\frac{a+b}{a^2+b^2}\right)^{24}$$

Proposed by Daniel Sitaru - Romania

Proof (by Leonard Giugiuc - Romania).
By *Cauchy - Schwarz inequality*, $(a^2+b^2)^2 \leq (a+b)(a^3+b^3)$ *, so that* $\frac{1}{a^2-ab+b^2} \leq \left(\frac{a+b}{a^2+b^2}\right)^2$. From here, $\frac{1}{(a^2-ab+b^2)^6} \leq \left(\frac{a+b}{a^2+b^2}\right)^{12}$.
Summing up and, subsequently, applying the ***AM - QM inequality***,

$$\left(\sum_{cycl} \frac{1}{(a^2-ab+b^2)^6}\right)^6 \leq \left(\frac{a+b}{a^2+b^2}\right)^{12} \leq 3\sum_{cycl}\left(\frac{a+b}{a^2+b^2}\right)^{24}$$

□

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the above problem (from his book "Math Accent") at the ***CutTheKnotMath page***. The solution is by Leo Giugiuc.

92. A Cyclic Inequality in Three Variables XIII

Prove that, for all $a, b, c > 0$

$$\sum_{cycl} \frac{a^2+b^2}{a+b} + 11\sum_{cycl} \frac{ab}{a+b} > 6\sum_{cycl} \sqrt{ab}$$

Proposed by Daniel Sitaru - Romania

Proof 1.
The inequality is equivalent to

$$\sum_{cycl} \frac{(a+b)^2 + 9ab - 6(a+b)\sqrt{ab}}{a+b} > 0$$

This simplifies to

$$\sum_{cycl} \frac{(a+b-3\sqrt{ab})^2}{a+b} > 0,$$

which is obvious. □

Proof 2 (by Seyran Ibrahimov - Azerbaidian).
Using the ***AM - GM inequality***,

$$\frac{a^2+b^2}{a+b} + \frac{11ab}{a+b} = \frac{(a+b)^2}{a+b} + \frac{9ab}{a+b} \geq 6\sqrt{ab}$$

and similar for the other terms. □

Proof 3.
Set $a+b=s, ab=r$. The required inequality becomes

$$s^2 - 6s\sqrt{r} + r > 0.$$

Since $s^2 - 6s\sqrt{r} + r = (s-3\sqrt{r})^2 \geq 0$. We only need to show that the equality is not possible. The equality would mean $s - 3\sqrt{r} = 0$, i.e., $a+b = 3\sqrt{ab}$, or, $\sqrt{a} = \frac{\sqrt{b}(7\pm3\sqrt{5})}{2}$. Similarly, $\sqrt{b} = \frac{\sqrt{c}(7\pm3\sqrt{5})}{2}$ and $\sqrt{c} = \frac{\sqrt{a}(7\pm3\sqrt{5})}{2}$.
The product of the three equates a rational number 1 to an irrational number $\left(\frac{7\pm3\sqrt{5}}{2}\right)^3$ which is impossible. Thus, the required inequality is indeed strict. □

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the above problem (from his book "Math Accent") at the ***CutTheKnotMath page***, with solutions by Soumitra Mandal, Ravi Prakash (India, Solution 1), Seyran Ibrahimov (Azerbaijan, Solution 2), Mihalcea Andrei Ştefan (Romania) and Abdallah El Farissi (Algeria) Solution 3.

93. A Cyclic Inequality in Three Variables XV

Prove that for positive a, b, c

$$\frac{a(a^2+b^2)}{a^3+b^3}+\frac{b(b^2+c^2)}{b^3+c^3}+\frac{c(c^2+a^2)}{c^3+a^3}\le\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}$$

Proposed by Daniel Sitaru - Romania

Proof (by Daniel Sitaru - Romania).
Consider function $f:(0,\infty)\to\mathbb{R}$; defined as

$$f(x)=\frac{x^2+x^6}{1+x^6}-x=\frac{x^2+x^6-x-x^7}{1+x^6}$$

$$=\frac{x^6(1-x)-x(1-x)}{1+x^6}=\frac{x(1-x)(x^5-1)}{x^6+1}$$

$$=\frac{-x(x-1)(x^5-1)}{x^6+1}=\frac{-x(x-1)^2(x^4+x^3+x^2+x+1)}{x^6+1}\le 0.$$

Now,

$$f\Big(\sqrt{\frac{a}{b}}\Big)=\frac{\frac{a}{b}+\frac{a^3}{b^3}}{1+\frac{a^3}{b^3}}-\sqrt{\frac{a}{b}}=\frac{a(a^2+b^2)}{a^3+b^3}-\sqrt{\frac{a}{b}}.$$

It follows that $f\Big(\sqrt{\frac{a}{b}}\Big)\le 0$ is equivalent to

$$\frac{a(a^2+b^2)}{a^3+b^3}-\sqrt{\frac{a}{b}}\le 0,$$

i.e., $\frac{a(a^2+b^2)}{a^3+b^3}\le\sqrt{\frac{a}{b}}$. The required inequality in nothing but

$$f\left(\sqrt{\frac{a}{b}}\right)+f\left(\sqrt{\frac{b}{c}}\right)+f\left(\sqrt{\frac{c}{a}}\right)\le 0.$$

□

Challenge (by Alexander Bogomolny - USA)

Prove that, for $x, y>0$,

$$\frac{1+x^2}{1+x^3}+\frac{1+y^2}{1+y^3}+\frac{xy(1+x^2y^2)}{1+x^3y^3}\le 3.$$

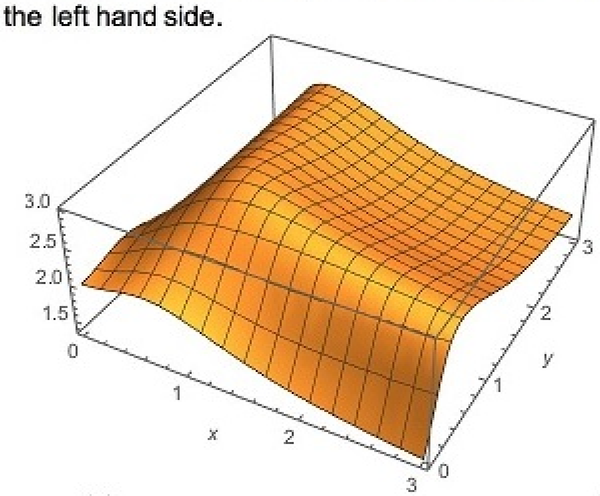
Visual support:

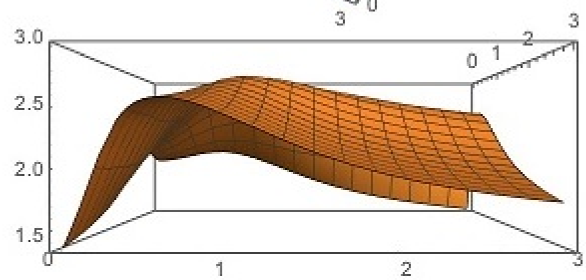
As this doesn't (easily) match standard inequalities, we can use calculus.

We get equality *lhs* = *rhs* for $x = y = 1$.

We can see visually and check analytically that this constitutes a maximum.

Let f be the left hand side.





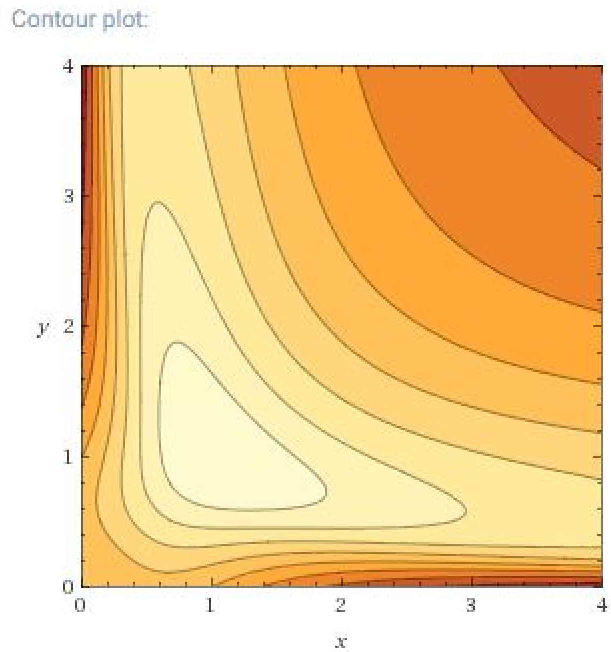
$$\Delta = \frac{\partial f}{\partial \{x,\ y\}} \text{ // FullSimplify; } \Delta \text{ // MatrixForm}$$

latrixForm=

$$\begin{pmatrix} -\frac{3x^2\left(1+x^2\right)}{\left(1+x^3\right)^2}+\frac{2x}{1+x^3}-\frac{3x^3y^4\left(1+x^2y^2\right)}{\left(1+x^3y^3\right)^2}+\frac{2x^2y^3}{1+x^3y^3}+\frac{y\left(1+x^2y^2\right)}{1+x^3y^3} \\ -\frac{3y^2\left(1+y^2\right)}{\left(1+y^3\right)^2}+\frac{2y}{1+y^3}-\frac{3x^4y^3\left(1+x^2y^2\right)}{\left(1+x^3y^3\right)^2}+\frac{2x^3y^2}{1+x^3y^3}+\frac{x\left(1+x^2y^2\right)}{1+x^3y^3} \end{pmatrix}$$

Which is satsfied for $x == 1$, with *a* matrix of local second derivatives $\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

and contour plot:



This problem with the solution has been kindly communicated to me by Dan Sitaru. Wolframalpha was instrumental in obtaining the 3d plots.

94. A Cyclic Inequality in Three Variables XVI

Prove that for $a, b, c \in \mathbb{R}$

$$\sum_{cycl} |(a+b)(1-ab)| < \frac{3}{2} + \sum_{cycl} a^2 + \frac{1}{2}\sum_{cycl} a^4$$

Proposed by Daniel Sitaru - Romania

Proof (by Daniel Sitaru - Romania).

$$[(1+a)+b(1-a)]^2 \geq 0$$

$$(1+a)^2 + 2b(1-a^2) + b^2(1-a^2) \geq 0 \Leftrightarrow$$

$$1 + 2a + a^2 + 2b - 2ba^2 + b^2 - 2b^2a + b^2a^2 \geq 0 \Leftrightarrow$$

$$1 + a^2 + b^2 + a^2b^2 + 2(a + b - a^2b - ab^2) \geq 0 \Leftrightarrow$$

$$(1+a^2)(1+b^2) \geq 2\Big(ab(a+b) - a(a+b)\Big) \Leftrightarrow$$

$$2(a+b)(ab-1) \leq (1+a^2)(1+b^2) \Leftrightarrow$$

(1)
$$2(a+b)(1-ab) \geq -1(1+a^2)(1+b^2)$$

Further,

$$[(1-a) - b(1+a)]^2 \geq 0 \Leftrightarrow$$

$$(1-a)^2 - 2b(1-a^2) + b^2(1+a)^2 \geq 0 \Leftrightarrow$$

$$1 - 2a + a^2 - 2b + 2ba^2 + b^2 + 2ab^2 + a^2b^2 \geq 0 \Leftrightarrow$$
$$1 + a^2 + b^2 + a^2b^2 - 2(a + b - ab^2 - a^2b) \geq 0 \Leftrightarrow$$
$$(1 + a^2)(1 + b^2) \geq 2\Big(a + b - ab(a + b)\Big) \Leftrightarrow$$

$$2(a + b)(1 - ab) \leq (1 + a^2)(1 + b^2) \tag{2}$$

From (1), (2) it follows that

$$2|(a + b)(1 - ab)| \leq (1 + a^2)(1 + b^2) \tag{3}$$

Similarly,

$$2|(b + c)(1 - bc)| \leq (1 + b^2)(1 + c^2) \tag{4}$$

$$2|(c + a)(1 - ca)| \leq (1 + c^2)(1 + a^2) \tag{5}$$

In (3) equality is attained for $a = 0; b = 1$ or $a = 1; b = 0$; similarly, for (4) and (5)

Thus, $\sum_{cycl} A$ of three inequality is strict:

$$2\sum_{cycl}|(a + b)(1 - ab)| < \sum_{cycl}(1 + a^2 + b^2 + a^2b^2)$$
$$3 + 2(a^2 + b^2 + c^2) + \sum_{cycl} a^2b^2 < 3 + 2(a^2 + b^2 + c^2) + \sum_{cycl} a^4$$

Dividing by 2,

$$\sum_{cycl}|(a + b)(1 - ab)| < \frac{3}{2} + a^2 + b^2 + c^2 + \frac{1}{2}\sum_{cycl} a^4$$

□

Acknowledgment (by Alexander Bogomolny - USA)
This problem with the solution has been kindly communicated to me by Dan Sitaru, all on a tex file.

95. A Cyclic Inequality with Many Sums

Let $a, b, c > 0, abc = 1$, prove that

$$\left(\sum_{cycl} a^4\right)\left(\sum_{cycl} \frac{a}{b}\right)\left(\sum_{cycl} a^3\right)\left(\sum_{cycl} \frac{a}{c}\right)\left(\sum_{cycl} a^2\right) \geq \left(\sum_{cycl} a\right)^3\left(\sum_{cycl} \frac{1}{a}\right)^2$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Daniel Sitaru - Romania).
If $n \in \mathbb{N}$,

$$a^{n+1} + b^{n+1} + c^{n+1} = a^n \cdot a + b^n \cdot b + c^n \cdot c$$
$$\underbrace{\geq}_{Chebyshev} \frac{1}{3}(a^n + b^n + c^n)(a + b + c) \underbrace{\geq}_{AM-GM} \frac{1}{3}(a^n + b^n + c^n) \cdot 3\sqrt[3]{abc}$$
$$= \frac{1}{3}(a^n + b^n + c^n) \cdot 3 = a^n + b^n + c^n.$$

It follows that

$$a^4 + b^4 + c^4 \geq a^3 + b^3 + c^3 \geq a^2 + b^2 + c^2 \geq a + b + c$$

In particular,

(1) $$\sum_{cycl} a^4 \geq \sum_{cycl} a$$

and

(2) $$\sum_{cycl} a^2 \geq \sum_{cycl} a$$

Further,

$$\sum_{cycl} a^3 = a^3 + b^3 + c^3 = a^2 \cdot a + b^2 \cdot a + c^2 \cdot c$$

$$\overbrace{\geq}^{Chebyshev} \frac{1}{3}(a^2 + b^2 + c^2)(a + b + c)$$

$$\overbrace{\geq}^{AM-GM} \frac{1}{3}(a^2 + b^2 + c^2) \cdot 3\sqrt[3]{abc} = \frac{1}{3}(a^2 + b^2 + c^2) \cdot 3 = a^2 + b^2 + c^2$$

$$\geq ab + bc + ca = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \sum_{cycl} \frac{1}{a},$$

(3) $$\text{So that } \sum_{cycl} a^3 \geq \sum_{cycl} \frac{1}{a}$$

By the AM-GM inequality,

(4) $$a = \sqrt[3]{\frac{a}{b} \cdot \frac{a}{b} \cdot \frac{b}{c}} \leq \frac{\frac{a}{b} + \frac{a}{b} + \frac{b}{c}}{3}$$

(5) $$b = \sqrt[3]{\frac{b}{c} \cdot \frac{b}{c} \cdot \frac{c}{a}} \leq \frac{\frac{b}{c} + \frac{b}{c} + \frac{c}{a}}{3}$$

(6) $$c = \sqrt[3]{\frac{c}{a} \cdot \frac{c}{a} \cdot \frac{a}{b}} \leq \frac{\frac{c}{a} + \frac{c}{a} + \frac{a}{b}}{3}$$

(7) By adding (4), (5), (6) $\sum^{cycl} \frac{a}{b} \geq \sum_{cycl} a$ and, by analogy with (4) - (6),

(8) $$\frac{1}{a} = \sqrt[3]{\frac{b}{a} \cdot \frac{b}{a} \cdot \frac{a}{c}} \leq \frac{\frac{b}{a} + \frac{b}{a} + \frac{a}{c}}{3}$$

(9) $$\frac{1}{b} = \sqrt[3]{\frac{c}{b} \cdot \frac{c}{b} \cdot \frac{b}{a}} \leq \frac{\frac{c}{b} + \frac{c}{b} + \frac{c}{b}}{3}$$

(10) $$\frac{1}{c} = \sqrt[3]{\frac{a}{c} \cdot \frac{a}{c} \cdot \frac{c}{b}} \leq \frac{\frac{a}{c} + \frac{a}{c} + \frac{c}{b}}{3}$$

(11) By adding the relationships (8), (9); (10), $\sum_{cycl} \frac{a}{c} \geq \sum_{cycl} \frac{1}{a}$

The product of (1), (2), (3), (7), (11) is exactly

$$\left(\sum_{cycl} a^4\right)\left(\sum_{cycl} \frac{a}{b}\right)\left(\sum_{cycl} a^3\right)\left(\sum_{cycl} \frac{a}{c}\right)\left(\sum_{cycl} a^2\right) \geq \left(\sum_{cycl} a\right)^3 \left(\sum_{cycl} \frac{1}{a}\right)^2$$

□

Proof 2 (by Leonard Giugiuc - Romania).
By the ***AM - GM inequality***, $\sum_{cycl} a \geq 3$, so that $(\sum_{cycl} a)^3 \geq 9(\sum_{cycl} a)$. But

$$9\left(\sum_{cycl} a^3\right) \geq \left(\sum_{cycl} a\right)^3.$$

Combining all these gives

$$\sum_{cycl} a^3 \geq \sum_{cycl} a.$$

By *Hölder's inequality*,

$$\left(\sum_{cycl} a^4\right)\left(\sum_{cycl} a^3\right)\left(\sum_{cycl} a^2\right) \geq \left(\sum_{cycl} a^3\right)^3.$$

implying

$$\left(\sum_{cycl} a^4\right)\left(\sum_{cycl} a^3\right)\left(\sum_{cycl} a^2\right) \geq \left(\sum_{cycl} a^3\right)^3$$

On the other hand,

$$\left(\sum_{cycl} \frac{1}{a}\right)^2 = \left(\sum_{cycl} ab\right)^2$$

As in (11) of Solution 1,

$$\sum_{cycl} \frac{1}{a} \geq \sum_{cycl} \frac{1}{a}$$

so that

$$\left(\sum_{cycl} \frac{a}{b}\right)\left(\sum_{cycl} \frac{a}{c}\right) \geq \left(\sum_{cycl} \frac{1}{a}\right)^2$$

which completes the proof. □

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the problem from this book *Math Accent* at the ***CutTheKnotMath page*** and supplied his solution (Solution 1) on a tex file. Solution 2 is by Leo Giugiuc.

96. A Triple Integral Inequality

Prove that, for all $a, b, c \in \left(0, \frac{\pi}{4}\right)$

$$0 \leq \int_0^a \left(\int_0^b \left(\int_0^c \left(\sum_{cycl}(\tan x - 2\tan y \tan z) + 4\prod_{cycl} \tan x\right) dx\right) dy\right) dz \leq abc$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Leonard Giugiuc - Romania).

$$T = \sum_{cycl}(\tan x - 2\tan y \tan z) + 4\prod_{cycl}\tan x = \frac{1}{2}\left(1 - \prod_{cycl}(1 - 2\tan x)\right).$$

Assuming $a, b, c \in \left(0, \frac{\pi}{4}\right), 0 \le T \le 1$. Thus,

$$0 \le \int_0^a \left(\int_0^b \left(\int_0^c \left(\sum_{cycl}(\tan x - 2\tan y \tan z) + 4\prod_{cycl}\tan x\right)dx\right)dy\right)dz$$

$$\le \int_0^a \int_0^b \int_0^c 1 dx dy dz = abc.$$

□

Proof 2 (by Soumitra Mandal - India).

Let $f : [0, c] \to \mathbb{R}^+$ *defined by*

$$f(x) = \tan x(4\tan y \tan z - 2\tan y - 2\tan z + 1) + \tan y + \tan z - 2\tan y \tan z$$

for all $x \in [0, c]$. *Now,*

$$f'(x) = \sec^2 x(4\tan y \tan z - 2\tan y - 2\tan z + 1) \ge 0 \text{ since}$$

$$x \in (0, c) \subseteq \left(0, \frac{\pi}{4}\right) \text{ and } y, z \in \left(0, \frac{\pi}{4}\right). \text{ So, } f \text{ is continuous on } [0, c] \text{ and}$$

$$f'(x) \ge 0 \text{ hence } f \text{ is increasing on } [0, c]. \text{ So, } f\left(\frac{\pi}{4}\right) \ge f(x) \ge f(x) \ge f(0)$$

$$\Rightarrow 4\tan y \tan z - 2\tan y - 2\tan z + 1 \ge f(x) \ge \tan y + \tan z - 2\tan y \tan z$$

$$\Rightarrow (2\tan y - 1)(2\tan z - 1) \ge f(x) \ge \frac{1}{2} - \frac{1}{2}(2\tan y - 1)(2\tan z - 1)$$

$$\Rightarrow (2\tan y - 1)(2\tan z - 1) \ge f(x) \ge \frac{1}{2} - \frac{1}{2}(2\tan y - 1)(2\tan z - 1)$$

$$\Rightarrow 1 \ge f(x) \ge 0 \text{ for all, } y, z \in \left(0, \frac{\pi}{4}\right)$$

$$\therefore 0 \le \int_0^a \left(\int_0^b \left(\int_0^c \sum_{cyc}(\tan x - 2\tan y \tan z) + 4\prod_{cyc}\tan x dx\right)dy\right)dz \le abc$$

(proved)

□

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the above problem (from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath page***. Solution 1 is by Leo Giugiuc. Solution 2 is by Soumitra Mandal - Chandar Nagore - India.

97. An Inequality in Triangle and in General

In any acute ΔABC,

$$\sum_{cycl}\frac{\cot A \cot^3 B}{\cot^2 B + 2\cot^2 A} + 2\sum_{cycl}\frac{\cot^2 A + \cot B}{\cot A + 2\cot B} \ge 1$$

Proposed by Daniel Sitaru - Romania

Remark (by Alexander Bogomolny - USA)
Both solutions below use ***the fact*** that in any triangle

$$\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$$

Thus, using the substitutions $a = \cot A, b = \cot B, c = \cot C$ the problem reduces to proving that

$$\textit{Prove that for positive } a, b, c \textit{such that } ab + bc + ca = 1,$$

$$\sum_{cycl} \frac{ab^3}{b^2 + 2a^2} + 2\sum_{cycl} \frac{a^2 b}{a + 2b} \geq 1.$$

Proof 1 (by Dung Thanh Tùng - Vietman).
The required inequality is equivalent to

$$\sum_{cycl} ab - 2\sum_{cycl} \frac{a^3 b}{b^2 + 2a^2} + 2\sum_{cycl} \frac{a^2 b}{a + 2b} \geq 1,$$

reducing the task to proving

(1) $$\sum_{cycl} \frac{a^2 b}{a + 2b} \geq \sum_{cycl} \frac{a^3 b}{b^2 + 2a^2}$$

We'll prove $\frac{a^2 b}{a+2b} \geq \frac{a^3 b}{b^2+2a^2}$ which is equivalent to $2a^2 + b^2 \geq a(a + 2b)$, i.e., $(a - b)^2 \geq 0$, implying (1).
Equality holds when $a = b = c = \frac{1}{\sqrt{3}}$, i.e., when $A = B = C = 60°$. □

Proof 2 (by Myagmarsuren Yadamsuren - Mongolia).

$$\frac{ab^3}{b^2 + 2a^2} + 2\frac{a^2 b}{a + 2b} = ab\Big(\frac{b^2}{b^2 + 2a^2} + \frac{(2a)(2a)}{(a + 2b)(2a)}\Big)$$

$$= ab\Big(\frac{b^2}{b^2 + 2a^2} + \frac{(2a)^2}{2a^2 + 4ab}\Big) \geq ab\Big(\frac{(b + 2a)^2}{(b + 2a)^2}\Big) = ab,$$

where, on the penultimate step, we used ***Bergström's inequality***. Summing up and using $ab + bc + ca = 1$, delivers the required inequality. □

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the above problem (from his book "Math Accent") at the ***CutTheKnotMath page***. Solution 1 is by Dung Tùng; Solution 2 is by Myagmarsuren Yadamsuren.

98. An Inequality in Triangle with Differences of Medians

$$\textit{Prove tha in a scalene } \Delta ABC:$$

$$\frac{8(m_a - m_b)(m_b - m_c)(m_c - m_a)}{(b - a)(c - b)(a - c)} > \frac{27abc}{(a + 2s)(b + 2s)(c + 2s)}$$

Proposed by Daniel Sitaru - Romania

Proof 1.

$$(m_a - m_b)(m_a + m_b) = m_a^2 - m_b^2$$
$$= \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2 - \frac{1}{2}(a^2 + c^2) + \frac{1}{4}b^2$$
$$= \frac{2b^2 + 2c^2 - a^2 - 2a^2 - 2c^2 + b^2}{4} = \frac{3(b^2 - a^2)}{4} = \frac{3(b-a)(b+a)}{4}$$
$$\frac{m_a - m_b}{b - a} = \frac{3(b+a)}{4(m_a + m_b)} > \frac{3(b+a)}{4(\frac{b+c}{2} + \frac{a+c}{2})} =$$
$$= \frac{3(b+a)}{2(a+b+2c)} = \frac{3(b+a)}{2(2s+c)} \geq \frac{3 \cdot 2\sqrt{ab}}{2(2s+c)}$$

It follows that

$$\frac{2(m_a - m_b)}{b - a} > \frac{3\sqrt{ab}}{2s + c}$$

Similarly, $\frac{2(m_b - m_c)}{c-b} > \frac{3\sqrt{bc}}{2s+a}$ and $\frac{2(m_c - m_a)}{a-c} > \frac{3\sqrt{ca}}{2s+b}$. Multiplying the three relationship yields

$$\frac{8(m_a - m_b)(m_b - m_c)(m_c - m_a)}{(b-a)(c-b)(a-c)} > \frac{27abc}{(a+2s)(b+2s)(c+2s)}$$

□

Proof 2.

First we note $a > b \Rightarrow m_a < m_b$. Indeed, ***from*** $m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4}$ and $m_b^2 = \frac{a^2+c^2}{2} - \frac{b^2}{4}$ we obtain $m_a^2 - m_b^2 = \frac{3}{4}(b^2 - a^2)$.

Now, using $m_a m_b \leq \frac{2c^2+ab}{4}$,

$$(m_a - m_b)^2 = m_a^2 + m_b^2 - 2m_a m_b$$
$$\geq \frac{a^2 + b + 2 + 4c^2}{4} - \frac{2c^2 + ab}{2}$$
$$= \frac{(b-a)^2}{4},$$

sucht that $\frac{2(m_a - m_b)}{b-a} \geq 1$. Similarly, $\frac{2(m_b - m_c)}{c-b} \geq 1$ and $\frac{2(m_c - m_a)}{a-c} \geq 1$ the product of which leads to

$$\frac{8(m_a - m_b)(m_b - m_c)(m_c - m_a)}{(b-a)(c-b)(a-c)} \geq 1.$$

Suffice it to prove that $1 > \frac{27abc}{(a+2s)(b+2s)(c+2s)}$. But, by the ***AM - GM inequality***, $a + b + c \geq 3\sqrt[3]{abc}$. Thus, we continue

$$\frac{27abc}{(a+2s)(b+2s)(c+2s)} < \frac{27abc}{(2s)(2s)(2s)} = \frac{27abc}{(a+b+c)^3} \leq \frac{27abc}{27abc} = 1.$$

This completes the proof. □

Acknowledgment (by Alexander Bogomolny - USA)

Dan Sitaru has kindly posted at the ***CutTheKnotMath page*** the above problem of his that was published in the ***Romanian Mathematical Magazine***. Dan messaged me his solution in a tex file an later added two more solutions. Solution 2 is by Soumava Chakraborty. Şerban George Florin and independently Athina Kalampolka and Chris Kyriazis gave solutions very similar to that of Dan Sitaru.

99. An Inequality in Triangle with Roots and Circumradius

Prove that in any ΔABC,

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\leq 3R\sqrt{2s},$$

where s *is the semiperimeter of* $\Delta ABC, R$ its circumradius.

Proposed by Daniel Sitaru - Romania

Proof (by Mihalcea Andrei Ştefan - Romania).
Use *Hölder's inequality followed by the* ***rearrangement inequality,***

$$(a\sqrt{b}+b\sqrt{c}+c\sqrt{a})^2\leq(a+b+c)(ab+bc+ca)$$
$$\leq 2s(a^2+b^2+c^2)$$

But we know that $a^2+b^2+c^2\leq 9R^2$. *A combination of the two gives desired result.* □

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted the above problem from his book *Math Accent*, with a solution, at the ***CutTheKnotMath page***. The solution is by Mihalcea Andrei Ştefan, a grade 9 student.

100. An Inequality in Triangle with the Sines of Half - Angles and Cube Roots

Prove that in an acute-angled triangle ΔABC:

$$2\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\sin^2\frac{C}{2}\geq\sqrt[3]{abc}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Daniel Sitaru - Romania

Proof 1(by Daniel Sitaru - Romania).

$$2\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\sin^2\frac{C}{2}=\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)2\sin^2\frac{C}{2}=$$

$$=2\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)(1-\cos C)=\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)-\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\cos C$$

(1) $$2\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\sin^2\frac{C}{2}=\sum_{cycl}\frac{a}{b}+\sum_{cycl}\frac{b}{a}-\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\cos C$$

$$\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\cos C=\sum_{cycl}\frac{a^2+b^2}{ab}\cdot\frac{a^2+b^2-c^2}{2ab}$$

$$=\frac{1}{2a^2b^2c^2}\sum_{cycl}c^2(a^2+b^2)(a^2+b^2-c^2)=\frac{1}{2a^2b^2c^2}\sum_{cycl}[c^2(a^2+b^2)^2-c^4(a^2+b^2)]$$

$$=\frac{1}{2a^2b^2c^2}\sum_{cycl}\left(c^2(a^4+b^4+2a^2b^2)-c^4a^2-c^4b^2\right)$$

$$=\frac{1}{2a^2b^2c^2}\sum_{cycl}(c^2a^4+c^2b^4+2a^2b^2c^2-c^4b^2-c^4a^2)$$

$$= \frac{1}{2a^2b^2c^2}\left(\sum_{cycl} a^4c^2 - \sum_{cycl} a^4c^2 + \sum_{cycl} b^4c^2 - \sum_{cycl} b^4c^2 + 6a^2b^2c^2\right)$$

$$= \frac{6a^2b^2c^2}{6a^2b^2c^2} = 3$$

We continue:

$$2\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\sin^2\frac{C}{2} = \sum_{cycl}\frac{a}{b} + \sum_{cycl}\frac{b}{a} - 3 \tag{2}$$

$$\frac{a}{b}+\frac{a}{b}+\frac{b}{c} \geq 3\sqrt[3]{\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{b}{c}} = 3\sqrt[3]{\frac{a^2}{bc}} = 3\frac{a}{\sqrt[3]{abc}} \tag{3}$$

$$\frac{b}{c}+\frac{b}{c}+\frac{c}{a} \geq 3\sqrt[3]{\frac{b}{c}\cdot\frac{b}{c}\cdot\frac{c}{a}} = 3\sqrt[3]{\frac{b^2}{ac}} = 3\frac{b}{\sqrt[3]{abc}} \tag{4}$$

$$\frac{c}{a}+\frac{c}{a}+\frac{a}{b} \geq 3\sqrt[3]{\frac{c}{a}\cdot\frac{c}{a}\cdot\frac{a}{b}} = 3\sqrt[3]{\frac{c^2}{ab}} = 3\frac{c}{\sqrt[3]{abc}} \tag{5}$$

Further,

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq 3\frac{a+b+c}{\sqrt[3]{abc}}$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b+c}{\sqrt{abc}} \geq \frac{3\sqrt[3]{abc}}{\sqrt[3]{abc}} = 3$$

From (2) it follows that

$$2\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\sin^2\frac{C}{2} = \sum_{cycl}\frac{a}{b} + \sum_{cycl}\frac{b}{a} - 3 \geq 3 + \sum_{cycl}\frac{b}{a} - 3 = \sum_{cycl}\frac{b}{a},$$

i.e.,

$$2\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\sin^2\frac{C}{2} \geq \sum_{cycl}\frac{b}{a} \tag{6}$$

$$\frac{a}{c}+\frac{a}{c}+\frac{b}{a} \geq 3\sqrt[3]{\frac{a}{c}\cdot\frac{a}{c}\cdot\frac{b}{a}} = \frac{3\sqrt[3]{abc}}{c}$$

$$\frac{b}{a}+\frac{b}{a}+\frac{c}{b} \geq 3\sqrt[3]{\frac{b}{a}\cdot\frac{b}{a}\cdot\frac{c}{b}} = 3\frac{\sqrt[3]{abc}}{a}$$

$$\frac{c}{b}+\frac{c}{b}+\frac{a}{c} \geq 3\sqrt[3]{\frac{c}{b}\cdot\frac{c}{b}\cdot\frac{a}{c}} = 3\frac{\sqrt[3]{abc}}{b}$$

$$\sum_{cycl}\frac{b}{a} \geq \sqrt[3]{abc}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \tag{7}$$

From (6) and (7),

$$2\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\sin^2\frac{C}{2} \geq \sqrt[3]{abc}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

□

Proof 2 (by Kevin Soto Palacios - Peru).
We'll prove instead a stronger inequality

$$\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)(1-\cos C)\geq\frac{a+b+c}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Or, equivalently,

$$A-B\geq\frac{a+b+c}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

where

$$A=\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\ \textit{and}\ B=\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\cos C$$

$$B=\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\cos C=\sum_{cycl}\left(\frac{a^2+b^2}{ab}\right)\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$=\sum_{cycl}\left(\frac{(a^2+b^2)^2}{2a^2b^2}\right)-\left(\frac{c^2(a^2+b^2)}{2a^2b^2}\right)$$

$$=\sum_{cycl}\frac{a^2}{2b^2}+\sum_{cycl}\frac{b^2}{2a^2}+\sum_{cycl}1-\sum_{cycl}\frac{c^2}{2b^2}-\sum_{cycl}\frac{c^2}{2a^2}=3.$$

$$A=\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)=\sum_{cycl}\frac{b+c}{a}.$$

We need to prove that

$$A-B=\sum_{cycl}\frac{b+c}{a}-3\geq\left(\frac{a+b+c}{3}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

This is equivalent to

$$\left(\sum_{cycl}a\right)\left(\sum_{cycl}\frac{1}{a}\right)-6\geq\left(\frac{a+b+c}{3}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

i.e.,

$$\frac{2}{3}\left(\sum_{cycl}a\right)\left(\frac{1}{a}\right)\geq 6,$$

which is true because, by the ***AM - GM inequality***,

$$\left(\sum_{cycl}a\right)\left(\frac{1}{a}\right)\geq 3\sqrt[3]{abc}\cdot 3\frac{1}{\sqrt[3]{abc}}=9.$$

□

Solution 3 (by Soumava Chakraborty - India).
First observe that

$$LHS=2\sum_{cycl}\left(\frac{a}{b}+\frac{b}{a}\right)\sin^2\frac{C}{2}=2\sum_{cycl}\left(\frac{a^2+b^2}{ab}\right)\frac{(s-a)(s-b)}{ab}$$

$$=\sum_{cycl}\frac{c^2(a^2+b^2)(b+c-a)(c+a-b)}{2a^2b^2}$$

Let's prove that

$$LHS = \sum_{cycl} \frac{c^2(a^2+b^2)(b+c-a)(c+a-b)}{2a^2b^2}$$
$$\geq \Big(\frac{a+b+c}{3}\Big)\Big(\frac{ab+bc+ca}{abc}\Big)$$

$$(1)\quad LHS = \sum_{cycl} \frac{c^2(a^2+b^2)(b+c-a)(c+a-b)}{2a^2b^2} \geq \Big(\frac{a+b+c}{3}\Big)\Big(\frac{ab+bc+ca}{abc}\Big)$$

This is equivalent to

$$3\sum_{cycl} c^2(a^2+b^2)(b+c-a)(c+a-b) \geq 2abc\sum_{cycl} a \sum_{cycl} ab \Leftrightarrow$$
$$4(a^3b^2c + a^3bc^2 + b^3c^2a + b^3ca^2 + c^3a^2b + c^3ab^2) \geq 24a^2b^2c^2 \Leftrightarrow$$
$$(a^2b + a^2c + b^2c + b^2a + c^2a + c^2b) \geq 6abc,$$

which is the same as

$$(2)\quad b(a^2+c^2) + c(a^2+b^2) + a(b^2+c^2) \geq 6abc.$$

But, by the AM - GM inequality, $b(a^2+c^2) \geq 2abc, c(a^2+b^2) \geq 2abc$, $a(b^2+c^2) \geq 2abc$, so that (2) holds and so is (1).
This is stronger that the required inequality. □

Acknowledgment (by Alexander Bogomolny - USA)
Dan Sitaru has kindly posted at the ***CutTheKnotMath page*** the above problem of his that was published in the ***Romanian Mathematical Magazine***. Dan messaged me his solution (Solution 1) in a tex file. Solution 2 is by Kevin Soto Palacios; Solution 3 is by Soumava Chakraborty.

Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru