Triangle Marathon 101 - 200

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101. **In $\Delta\ ABC$ the following relationship holds:**

$$\prod(m_a^2+m_am_b+m_b^2)\ge\left(\frac{9r_ar_br_c}{r_a+r_b+r_c}\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adil Abdullayev – Baku – Azerbaidian

Lemma. $m_a\ge\sqrt{r_br_c}$.

$$m_a=\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}\ge\frac{1}{2}\sqrt{(b+c)^2-a^2}=$$

$$=\frac{1}{2}\sqrt{(b+c+a)(b+c-a)}=\frac{1}{2}\sqrt{2p(2p-2a)}=\sqrt{p(p-a)}=\sqrt{r_br_c}.$$

By lemma: $m_a^2\ge r_br_c$.

$$LHS\ge\prod 3m_am_b=27m_a^2m_b^2m_c^2\overset{Lemma}{\ge}27r_a^2r_b^2r_c^2\overset{?}{\ge}RHS\Leftrightarrow$$

$$\Leftrightarrow\frac{r_a+r_b+r_c}{3}\ge\sqrt[3]{r_ar_br_c}.$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$\prod(m_a^2+m_am_b+m_b^2)\overset{A-G}{\ge}\prod(3m_am_b)=27m_a^2m_b^2m_c^2$$

It suffices to prove: $m_a^2m_b^2m_c^2\ge\frac{27r^3s^6}{(4R+r)^3}$. ***Now,*** $m_a^2\ge s(s-a)$, ***etc.***

$m_a^2m_b^2m_c^2\ge s^3(s-a)(s-b)(s-c)$. ***It suffices to prove:***

$r^2s^4\ge\frac{27r^4s^6}{(4R+r)^3}\Leftrightarrow s^2\le\frac{(4R+r)^3}{27r}$ ***Gerretsen*** $\Rightarrow s^2\le 4R^2+4Rr+3r^2$. ***It suffices to prove:*** $4R^2+4Rr+3r^2\le\frac{(4R+r)^3}{27r}\Leftrightarrow 16r^3-15R^2r-$

$$-24Rr^2-20r^3\ge 0\Leftrightarrow\underbrace{(t-2)(16t^2+17t+10)\ge}_{true}0\left(t=\frac{R}{r}\right)$$

$$t=\frac{R}{r}\ge 2\ \textit{(Euler)}$$

102. In $\Delta\, ABC$:

$$\frac{b^4c^7}{a^{12}}+\frac{c^4a^7}{b^{12}}+\frac{a^4b^7}{c^{12}}\ge\frac{\sqrt{3}}{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$LHS\overset{A-G}{\ge}3\sqrt[3]{\frac{1}{abc}}.\ \textit{Suffices to show: }\frac{27}{abc}\ge\frac{3\sqrt{3}}{R^3}\Rightarrow\frac{3\sqrt{3}}{4RS}\ge\frac{1}{R^3}\Leftrightarrow S\le\frac{3\sqrt{3}}{4}R^2$$

$$s\le\frac{3\sqrt{3}R}{2}\ \textit{(Mitrinovic) and } r\le\frac{R}{2};\ S\le\frac{3\sqrt{3}R^2}{4}$$

Solution 2 by Adil Abdullayev – Baku - Azerbaidjian

Lemma. $S\le\frac{p^2}{3\sqrt{3}}\le\frac{3R^2\sqrt{3}}{4}$.

$$LHS\overset{AM-GM}{\ge}3\cdot\sqrt[3]{\frac{1}{abc}}=3\cdot\sqrt[3]{\frac{1}{4SR}}\overset{?}{\ge}\frac{\sqrt{3}}{R}\Leftrightarrow S\le\frac{3R^2\sqrt{3}}{4}$$

Solution 3 by Soumava Pal – Kolkata – India

$$\sum\nolimits_{cycl}\sin(A)\le\frac{3\sqrt{3}}{2}\ \textit{and by AM-GM}$$

$$\sqrt[3]{\sin A\sin B\sin C}\le\frac{\sum_{cycl}\sin A}{3}\le\frac{\sqrt{3}}{2}\Rightarrow\sin A\sin B\sin C\le\frac{3\sqrt{3}}{8}$$

$$\sum\nolimits_{cycl}\frac{a^4b^7}{c^{12}}\ge3\sqrt[3]{\frac{1}{abc}}\ \textit{(by AM-GM)};\ 3\sqrt[3]{\frac{1}{abc}}=\frac{3}{2R}\sqrt[3]{\frac{1}{\sin A\sin B\sin C}}\ge\frac{3}{2R}\sqrt[3]{\frac{8}{3\sqrt{3}}}=\frac{\sqrt{3}}{R}$$

Solution 4 by Seyran Ibrahimov – Maasilli – Azerbaidjian

$$a\ge b\ge c;\ \frac{b^4c^7}{a^{12}}+\frac{c^4a^7}{b^{12}}+\frac{a^4b^7}{c^{12}}\ge3\sqrt[3]{\frac{1}{abc}}\Rightarrow x';\ x'\ge3\sqrt[3]{\frac{1}{abc}}\ \textit{(to prove)}$$

$$\frac{3}{\sqrt[3]{4RS}}\ge\frac{\sqrt{3}}{R};\ \frac{27}{4RS}\ge\frac{3\sqrt{3}}{R^3}.\ \textit{Equivalent inequality}\rightarrow S\le\frac{3\sqrt{3}R^2}{4}\ \textit{(to prove)}$$

$$2R^2\sin a\cdot\sin b\cdot\sin c\le\frac{3\sqrt{3}R^2}{4};\ \sin a\cdot\sin b\cdot\sin c\le\frac{3\sqrt{3}}{8}\ \textit{(proved)}$$

103. Given a triangle ABC.Prove that:

$$a\sin\frac{A}{2}+b\sin\frac{B}{2}+c\sin\frac{C}{2}\geq\frac{a+b+c}{2}.$$

Proposed by Kunihiko Chikaya – Tokyo – Japan

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $a\sin\frac{A}{2}+b\sin\frac{B}{2}+c\sin\frac{C}{2}\geq\frac{a+b+c}{2}$

Por la desigualdad de Jensen:

$$\cos B+\cos C=2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\leq 2\sin\frac{A}{2}\ldots\text{ (A)}$$

Análogamente: $\cos A+\cos C\leq 2\sin\frac{B}{2}\ldots$ *(B),*

$$\cos A+\cos B\leq 2\sin\frac{C}{2}\ldots\text{ (C)}$$

Por teorema de las proyecciones: $a=b\cos C+c\cos B$

$$b=a\cos C+c\cos A,\ c=a\cos B+b\cos A$$

La desigualdad propuesta, es equivalente:

$$\Rightarrow 2a\sin\frac{A}{2}2b\sin\frac{B}{2}+2c\sin\frac{C}{2}\geq(a\cos B+a\cos C)+(b\cos A+b\cos C)+(c\cos A+c\cos B)=a+b+c$$

104. In $\Delta\ ABC$ the following relationship holds:

$$\sqrt{m_a}+\sqrt{m_b}+\sqrt{m_c}\geq\sqrt{m_a+m_b+m_c+6\sqrt[3]{r_ar_br_c}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$m_a=\sqrt{\frac{b^2+c^2}{2}-\frac{a^2}{4}}\geq\sqrt{\frac{(b+c)^2-a^2}{4}}=\sqrt{s(s-a)},\ m_b\geq\sqrt{s(s-b)}$$

and $m_c\geq\sqrt{s(s-c)}$.

$$\sum_{cyc}\sqrt{m_a} \ge \sqrt{\sum_{cyc} m_a + 6\sqrt[3]{r_ar_br_c}} \Leftrightarrow \left(\sum_{cyc}\sqrt{m_a}\right)^2 \ge \sum_{cyc} m_a + 6\sqrt[3]{r_ar_br_c}$$

$$\Leftrightarrow \sum_{cyc}\sqrt{m_am_b} \ge 3\sqrt[3]{r_ar_br_c} \quad \dots (1)$$

Applying A.M ≥ G.M: $\frac{\sum_{cyc}\sqrt{m_am_b}}{3} \ge \sqrt[3]{m_am_bm_c} \ge$

$$\ge \sqrt[3]{s\sqrt{(s-a)(s-b)(s-c)}} = \sqrt[3]{sS} =$$

$= \sqrt[3]{r_ar_br_c}$, ***hence statement (1) is proved.***

$$\sum_{cyc}\sqrt{m_a} \ge \sqrt{\sum_{cyc} m_a + 6\sqrt[3]{r_ar_br_c}}$$

Solution 2 by Adil Abdullayev – Baku – Azerbaidjian

Lemma 1.

$$m_a \ge \sqrt{r_br_c};\ m_b \ge \sqrt{r_ar_c};\ m_c \ge \sqrt{r_ar_b}$$

Lemma 2.

$$m_am_bm_c \ge r_ar_br_c;\ LHS \ge RHS \Leftrightarrow (LHS)^2 \ge (RHS)^2 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{m_am_b} + \sqrt{m_bm_c} + \sqrt{m_cm_a} \ge 3\cdot\sqrt[3]{r_ar_br_c} \quad \dots (A)$$

$$\sqrt{m_am_b} + \sqrt{m_bm_c} + \sqrt{m_cm_a} \overset{AM-GM}{\ge} 3\cdot\sqrt[3]{m_am_bm_c} \overset{Lemma\ 2}{\ge} 3\cdot\sqrt[3]{r_ar_br_c} \Leftrightarrow (A)$$

105. In $\Delta\ ABC$ the following relationship holds:

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \le \frac{1}{4S}\sum\left(\frac{\sqrt{a}(a^2+bc)}{\sqrt{b}+\sqrt{c}} + a^2\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

$$\frac{4\Delta}{\sin A} = \frac{2bc\sin A}{\sin A} = 2bc, \frac{4\Delta}{\sin B} = \frac{2ca\sin B}{\sin B} = 2ca;$$

$$\frac{4\Delta}{\sin C} = \frac{2ab\sin C}{\sin C} = 2ab$$

given inequality $\Leftrightarrow \frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}(a^2+bc) + \frac{\sqrt{b}(b^2+ca)}{\sqrt{c}+\sqrt{a}} + \frac{\sqrt{c}(c^2+ab)}{\sqrt{a}+\sqrt{b}} + \sum a^2 \geq 2\sum ab$ ***(1)***

Now, $\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}} \leq \frac{\sqrt{b}}{\sqrt{c}+\sqrt{a}} \Leftrightarrow \sqrt{ac} + a \leq b + \sqrt{bc}$

$$\Leftrightarrow (a-b) + \sqrt{c}\left(\sqrt{a}-\sqrt{b}\right) \leq 0 \Leftrightarrow \left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \leq 0$$

$$\Leftrightarrow \sqrt{a} \leq \sqrt{b} \Leftrightarrow a \leq b \quad \textit{(i)}$$

Similarly, $\frac{\sqrt{b}}{\sqrt{c}+\sqrt{a}} \leq \frac{\sqrt{c}}{\sqrt{a}+\sqrt{b}} \Leftrightarrow b \leq c$ ***(ii)***

Also, $a^2 + bc \leq b^2 + ca \Leftrightarrow a^2 - b^2 - c(a-b) \leq 0$

$$\Leftrightarrow (a-b)(a+b-c) \leq 0 \Leftrightarrow a-b \leq 0 \quad (a+b > c)$$

$$\Leftrightarrow a \leq b \quad \textit{(iii)}$$

Similarly, $b^2 + ca \leq c^2 + ab \Leftrightarrow b \leq c$ ***(iv)***

WLOG, let's assume $a \leq b \leq c$

LHS ***of (1)*** $\overset{Chebysev}{\geq} \frac{1}{3}\left(\sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}\right)\left(\sum a^2 + \sum bc\right) + \sum a^2$ ***(using (i), (ii), (iii), (iv))***

$$\overset{Nesbitt}{\geq} \frac{1}{3}\cdot\frac{3}{2}\left(\sum a^2 + \sum ab\right) + \sum a^2 = \frac{3}{2}\sum a^2 + \frac{1}{2}\sum ab$$

it suffices to prove: $\frac{3}{2}\sum a^2 + \frac{1}{2}\sum ab \geq 2\sum ab$

$\Leftrightarrow 3\sum a^2 + \sum ab \geq 4\sum ab \Leftrightarrow \sum a^2 \geq \sum ab \rightarrow$ ***true***

(1) is true (Proved)

106. **In $\Delta\, ABC$ the following relationship holds:**

$$\frac{1}{b+c}h_a\cos A+\frac{1}{c+a}h_b\cos B+\frac{1}{a+b}h_c\cos C<\sum\frac{bc}{b^2+c^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Myagmarsuren Yadamsuren – Mongolia

$$\sum\frac{1}{b+c}\cdot h_a\cdot\cos A<\sum\frac{bc}{b^2+c^2}$$

$$\textit{1)}\ m_a\le\frac{b+c}{2}\Rightarrow\frac{1}{2m_a}\ge\frac{1}{b+c}\quad (1)$$

$$\textit{2)}\ m_a\ge\frac{b^2+c^2}{4R}\Rightarrow\frac{m_a}{b^2+c^2}\ge\frac{1}{4R}\ \Big|\cdot 2bc$$

$$\underbrace{\frac{2bc\cdot m_a}{b^2+c^2}}_{\substack{\Downarrow\\ AK_A\ (Simedian)}}\ge\frac{4bc\cdot a}{4R\cdot a}=\frac{2S}{a}=h_a$$

$$AK_A\ge h_a$$

$$\sum\frac{1}{b+c}\cdot h_a\cdot\cos A\overset{(1)}{\le}\sum\frac{h_a}{2\cdot m_a}\cdot\cos A\le$$

$$\le\sum\frac{AK_A}{2\cdot m_a}\cdot\cos A=\sum\frac{2bc\cdot m_a}{(b^2+c^2)\cdot 2m_a}\cdot\cos A$$

$$=\sum\frac{bc}{b^2+c^2}\cdot\cos A<\sum\frac{bc}{b^2+c^2};\ \cos A\le 1$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$\frac{1}{b+c}\cdot h_a\cos A<\frac{bc}{b^2+c^2}$$

$$\Leftrightarrow\frac{1}{2R(\sin B+\sin C)}\cdot\frac{bc}{2R}\cdot\cos A<\frac{bc}{4R^2(\sin^2 B+\sin^2 C)}$$

$$\Leftrightarrow\cos A<\frac{\sin B+\sin C}{\sin^2 B+\sin^2 C}\quad \textit{(1)}$$

Now, $\sin^2 B\le\sin B$ ***and,*** $\sin^2 C\le\sin C$

$\sin^2 B + \sin^2 C \le \sin B + \sin C \Rightarrow \frac{\sin B+\sin C}{\sin^2 B+\sin^2 C} \ge 1 > \cos A \Rightarrow$ ***(1) is true***

$\Rightarrow \frac{1}{b+c} \cdot h_a \cos A < \frac{bc}{b^2+c^2}$. ***Similarly,*** $\frac{1}{c+a} \cdot h_b \cos B < \frac{ca}{c^2+a^2}$ ***and***

$\frac{1}{a+b} h_c \cos C < \frac{ab}{a^2+b^2}$. ***Adding, we get the desired inequality***

107. **In $\Delta\ ABC$ the following relationship holds:**

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} > \frac{1}{2R}(m_a + m_b + m_c)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

Case 1: $\Delta\ ABC$ ***is acute***

$$m_a \le R(1 + \cos A) \Rightarrow m_a \le 2R\cos^2\frac{A}{2}$$

$$m_b \le R(1 + \cos B) \Rightarrow m_b \le 2R\cos^2\frac{B}{2}$$

$$m_c \le R(1 + \cos C) \Rightarrow m_c \le 2R\cos^2\frac{C}{2}$$

$\frac{\sum m_a}{2R} \le \sum \cos^2\frac{A}{2}$. ***It suffices to prove:*** $\sum \cos^2\frac{A}{2} < \sum \cos\frac{A}{2}$

$$0 < \cos\frac{A}{2}, \cos\frac{B}{2}, \cos\frac{C}{2} < 1,$$

$$\cos^2\frac{A}{2} < \cos\frac{A}{2}, \cos^2\frac{B}{2} < \cos\frac{B}{2}, \cos^2\frac{C}{2} < \cos\frac{C}{2} \Rightarrow \sum \cos^2\frac{A}{2} < \sum \cos\frac{A}{2}$$

hence proved

Case 2: $\Delta\ ABC$ ***is non-acute***

WLOG, let us assume $\angle A \ge 90°$. ***Now*** $\angle B, \angle C$ ***are acute angles.***

For acute angles $B, C, m_b \le R(1 + \cos B)$ ***and*** $m_c \le R(1 + \cos C)$

$$\Rightarrow \frac{m_b + m_c}{2R} \le \cos^2\frac{B}{2} + \cos^2\frac{C}{2} < \cos\frac{B}{2} + \cos\frac{C}{2}$$

$\left(0 < \cos\frac{B}{2}, \cos\frac{C}{2} < 1\right)$. ***It suffices to prove:*** $\frac{m_a}{2R} \le \cos\frac{A}{2}$, ***where*** $A \ge 90^\circ$

$$\Leftrightarrow \frac{2R\cos\frac{A}{2}\sin\frac{A}{2}}{\sin\frac{A}{2}} \ge \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2}$$

$$\Leftrightarrow \frac{2R\sin A}{2\sin\frac{A}{2}} \ge \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2} \Leftrightarrow \frac{a}{\sin\frac{A}{2}} \ge \sqrt{2b^2 + 2c^2 - a^2}$$

Now $\frac{1}{\sin\frac{A}{2}} > 1, \frac{a}{\sin\frac{A}{2}} > a$. ***It suffices to prove:*** $a \ge \sqrt{2b^2 + 2c^2 - a^2}$

$$\Leftrightarrow a^2 \ge b^2 + c^2 \Leftrightarrow \frac{b^2 + c^2 - a^2}{2bc} \le 0 \Leftrightarrow \cos A \le 0,$$

which is true: $A \ge 90^\circ$ ***(Proved)***

108. In $\Delta\, ABC$ the following relationship holds:

$$\frac{h_b^5 h_c^4}{h_a^{10}} + \frac{h_c^5 h_a^4}{h_b^{10}} + \frac{h_a^5 h_b^4}{h_c^{10}} \ge 3\sqrt[3]{\frac{R}{2S^2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\sum \frac{h_b^5 h_c^4}{h_a^{10}} \ge 3\sqrt[3]{\frac{R}{2S^2}}$

Desde que: $h_a h_b h_c = \frac{2S}{a}\cdot\frac{2S}{b}\cdot\frac{2S}{c} = \frac{8S^3}{4RS} = \frac{2S^2}{R}$. ***Por:*** $MA \ge MG$

$$\sum \frac{h_b^5 h_c^4}{h_a^{10}} \ge 3\sqrt[3]{\frac{1}{h_a h_b h_c}} = 3\sqrt[3]{\frac{R}{2S^2}}$$

Solution 2 by Adil Abdullayev – Baku – Azerbaidjian

$$LHS \overset{AM-GM}{\geq} 3\cdot\sqrt[3]{\frac{1}{h_a h_b h_c}} = RHS$$

109. **Let ABC be a triangle and x, y, z be positive real numbers. Prove that:**

$$\left(x^2+y(z-x)\right)\frac{a}{h_a}+\left(y^2+z(x-y)\right)\frac{b}{h_b}+\left(z^2+x(y-z)\right)\frac{c}{h_c}\geq 2\sqrt{x^3y+y^3z+z^3x}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sea ABC un triángulo con x, y, z números R^+. Probar que:

$$\left(x^2+y(z-x)\right)\frac{a}{h_a}+\left(y^2+z(x-y)\right)\frac{b}{h_b}+\left(z^2+x(y-z)\right)\frac{c}{h_c}\geq 2\sqrt{x^3y+y^3z+z^3x}$$

De la desigualdad Ponderada Weizenbock:

Siendo u, v, w números R^+, tales que: $uv+vw+wu\geq 0$

$$a^2u+b^2v+c^2w\geq 4\sqrt{uv+vw+wu}S \text{ ... (A)}$$

Siendo: $u=x^2+y(z-x)>0, v=y^2+z(x-y)>0,$

$w=z^2+x(y-z)>0$. ***Ahora bien:***

2) $uv+vw+wu=\left(y^2+z(x-y)\right)\left(x^2+y(z-x)\right)+$

$$+\left(x^2+y(z-x)\right)\left(z^2+x(y-z)\right)+\left(z^2+x(y-z)\right)\left(y^2+z(x-y)\right)$$

$$A=\left(y^2+z(x-y)\right)\left(x^2+y(z-x)\right)=y^2x^2+x^2z(x-y)+y^3(z-x)+yz(x-y)(z-x)$$

$$A=y^2x^2+x^3z-x^2yz+y^2z-y^3x+z^2xy-y^2z^2-x^2yz+y^2xz$$

$$B=\left(x^2+y(z-x)\right)\left(z^2+x(y-z)\right)=x^2z^2+z^2y(z-x)+x^3(y-z)+$$

$$+xy(z-x)(y-z)$$

$$B=x^2z^2+z^3y-z^2xy+x^3y-x^3z+y^2xz-y^2x^2-z^2xy+x^2yz$$

$$C = \left(z^2 + x(y-z)\right)\left(y^2 + z(x-y)\right) =$$

$$= z^2y^2 + y^2x(y-z) + z^3(x-y) + xz(y-z)(x-y)$$

$$C = z^2y^2 + y^3x - y^2xz + z^3x - z^3y + x^2yz - x^2z^2 - y^2xz + z^2xy$$

Sumando: (A) + (B) + (C) nos resulta:

$$\rightarrow uv + vw + wu = x^3y + y^3z + z^3x$$

Tenemos en ... (A) : $\Rightarrow \frac{a^2u}{2S} + \frac{b^2v}{2S} + \frac{c^2w}{2S} \geq 2\sqrt{uv + vw + wu} \Rightarrow$

$$\Rightarrow \frac{a^2u}{ah_a} + \frac{b^2v}{bh_b} + \frac{c^2w}{ch_c} \geq 2\sqrt{uv + vw + wu}$$

$$\Rightarrow \left(x^2 + y(z-x)\right)\frac{a}{h_a} + \left(y^2 + z(x-y)\right)\frac{b}{h_b} + \left(z^2 + x(y-z)\right)\frac{c}{h_c} \geq$$

$$\geq 2\sqrt{x^3y + y^2z + z^3x}$$

110. **In acute-angled $\Delta\ ABC$ the following relationship holds:**

$$\left(\sum \frac{1}{\sqrt{s-a}}\right)^2 \leq (m_a + w_b + h_c) \cdot \frac{4R + r}{3sr^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$m_a \geq w_a \geq h_a$ ***is any triangle***

$$m_a + w_b + h_c \geq h_a + h_b + h_c = \frac{\sum ab}{2R} = \frac{s^2 + 4Rr + r^2}{2R} \quad \textbf{(1)}$$

Now, $\left(\sum \frac{1}{\sqrt{s-a}}\right)^2 \leq 3\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) =$ ***(CBS)***

$$= \frac{3s\sum(s-a)(s-b)}{s(s-a)(s-b)(s-c)} = \frac{3\sum(s-a)(s-b)}{r^2s} =$$

$$= \frac{3\{3s^2 - s \cdot 2(a+b+c) + \sum ab\}}{r^2 S} =$$

$$= \frac{3(3s^2 - 4s^2 + s^2 + 4Rr + r^2)}{r^2 s} = \frac{3r(4R+r)}{r^2 S} \quad \textit{(2)}$$

It suffices to show: $\frac{3r(4R+r)}{r^2 s} \le \frac{s^2+4Rr+r^2}{2R} \cdot \frac{(4R+r)}{2sr^2}$ ***from (1), (2)***

$$\Leftrightarrow s^2 + 4Rr + r^2 \ge 18Rr \Leftrightarrow s^2 \ge 14Rr - r^2$$

Gerretsen $\Rightarrow s^2 \ge 16Rr - 5r^2$. ***It suffices to prove:***

$$16Rr - 5r^2 \ge 14Rr - r^2 \Leftrightarrow 2Rr \ge 4r^2 \Leftrightarrow Rr \ge 2r^2 \Leftrightarrow R \ge 2r$$

(true → Euler) (proved)

Solution 2 by Martin Lukarevski-Stip

Let r_a, r_b, r_c ***and*** F ***denote the exradii and the area of the triangle respectively. We use the well-known inequality*** $h_a + h_b + h_c \ge 9r$, ***and the equality*** $r_a + r_b + r_c = 4R + r$. ***Hence by the Cauchy-Schwarz inequality*** $(m_a + w_b + h_c) \cdot \frac{4R+r}{3sr^2} \ge (h_a + h_b + h_c) \cdot \frac{4R+r}{3sr^2}$

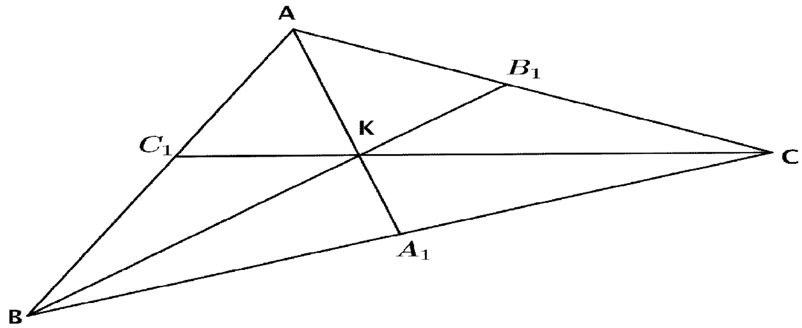
$\ge 3 \cdot \frac{4R+r}{F} = 3 \cdot \frac{r_a+r_b+r_c}{F} = 3 \cdot \sum \frac{1}{s-a} \ge \left(\sum \frac{1}{\sqrt{s-a}}\right)^2$, ***and we are done.***

111. In ΔABC, K – Lemoine's point. Prove that:

$$\frac{AK}{b^2+c^2} + \frac{BK}{c^2+a^2} + \frac{CK}{a^2+b^2} \le \frac{m_a + m_b + m_c}{a^2+b^2+c^2}$$

Proposed by Adil Abdullayev – Baku – Azerbaidian

Solution 1 by Daniel Sitaru – Romania



K – **Lemoine's point;** $\frac{A_1B}{A_1C} = \frac{c^2}{b^2}; \frac{B_1C}{B_1A} = \frac{a^2}{c^2}; \frac{C_1A}{C_1B} = \frac{b^2}{a^2}$

$$AK^2 = \frac{b^2c^2}{(\sum a^2)^2}\left(2\sum a^2 - 3a^2\right) = \frac{b^2c^2}{(a^2+b^2+c^2)^2}(2b^2+2c^2-a^2)$$

$$BK^2 = \frac{a^2c^2}{(a^2+b^2+c^2)^2}(2a^2+2c^2-b^2); CK^2 = \frac{a^2b^2}{(a^2+b^2+c^2)^2}(2a^2+2b^2-c^2)$$

$$\sum\frac{AK}{b^2+c^2} = \sum\frac{bc\sqrt{2b^2+2c^2-a^2}}{(b^2+c^2)\sum a^2} = \frac{1}{\sum a^2}\sum\frac{2bcm_a}{b^2+c^2} \le \frac{\sum m_a}{\sum a^2}$$

112. In $\Delta\ ABC$ the following equality holds:

$$\sum\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right)^3 \ge \sqrt[3]{3a}+\sqrt[3]{3b}+\sqrt[3]{3c}-2$$

Proposed by Daniel Sitaru – Romania

Solution by Daniel Sitaru-Romania:

By AM-GM: $\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right)^3+\frac{2}{3} = \left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right)^3+\frac{1}{3}+\frac{1}{3} \ge$

$$\ge 3\sqrt[3]{\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right)^3\cdot\frac{1}{3}\cdot\frac{1}{3}} = \frac{3}{\sqrt[3]{9}}\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right) = \sqrt[3]{3}\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right).$$

$$\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right)^3+\frac{2}{3} \ge \sqrt[3]{3a}+\sqrt[3]{3b}-\sqrt[3]{3c} \quad \textbf{(1)}$$

Analogous: $\left(\sqrt[3]{b}+\sqrt[3]{c}-\sqrt[3]{a}\right)^3+\frac{2}{3}\geq\sqrt[3]{3b}+\sqrt[3]{3c}-\sqrt[3]{3a}$ **(2)**

$\left(\sqrt[3]{c}+\sqrt[3]{a}-\sqrt[3]{b}\right)^3+\frac{2}{3}\geq\sqrt[3]{3c}+\sqrt[3]{3a}-\sqrt[3]{3b}$ **(3)**

By adding (1); (2); (3): $\sum\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right)^3+2\geq\sqrt[3]{3a}+\sqrt[3]{3b}+\sqrt[3]{3c}$

$$\sum\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right)^3\geq\sqrt[3]{a}+\sqrt[3]{3b}+\sqrt[3]{3c}-2$$

Equality holds for: $a=b=c=\frac{1}{3}$.

113. **In any $\Delta\ ABC$, prove that:**

$$\sum r_b\cdot r_c\cdot\tan\frac{A}{2}\leq\frac{9\sqrt{3}}{4}R^2$$

Proposed by George Apostolopoulous – Messalonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Prove in any triangle ABC: $\sum r_br_c\tan\frac{A}{2}\leq\frac{9\sqrt{3}}{4}R^2$. ***We know that:***

$$r_a=p\tan\frac{A}{2},r_b=p\tan\frac{B}{2},r_c=p\tan\frac{C}{2};\ \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\leq\frac{1}{3\sqrt{3}}$$

$$p\leq\frac{3\sqrt{3}R}{2}\Rightarrow\sum p^2\tan\frac{B}{2}\tan\frac{C}{2}\tan\frac{A}{2}=3p^2\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\leq$$

$$\leq3p^2\left(\frac{1}{3\sqrt{3}}\right)=\frac{1}{\sqrt{3}}p^2\leq\frac{1}{\sqrt{3}}\cdot\frac{27R^2}{4}=\frac{9\sqrt{3}}{4}R^2\ \textit{(Done)}$$

114. **In acute $\Delta\ ABC$:**

$$\prod(\tan A\tan B-\cot A\cot B)\geq\frac{512}{27}$$

Proposed by Daniel Sitaru – Romania

Solution by Rozeta Atanasova - Skopje

$$\prod(\tan A\tan B-\cot A\cot B)=\prod\frac{(\tan A\tan B-1)(\tan A\tan B+1)}{\tan A\tan B}=$$

$$=\prod\frac{(\tan A\tan B-1)(\tan A\tan B+1)(\tan A+\tan B)}{(\tan A+\tan B)\tan A\tan B}=$$

$$=\prod\frac{(\tan A\tan B+1)(\tan A+\tan B)}{\tan A\tan B\tan C}=\prod\frac{\left(3*\frac{\tan A\tan B}{3}+1\right)(\tan A+\tan B)}{\tan A\tan B\tan C}\geq$$

$$\overset{AM-GM}{\geq}\prod\frac{4\sqrt[4]{\left(\frac{\tan A\tan B}{3}\right)^3}\sin(A+B)}{\tan A\tan B\tan C\cos A\cos B}=\frac{64(\tan A\tan B\tan C)^{\frac{3}{2}}\sin A\sin B\sin C}{3^{\frac{9}{4}}(\tan A\tan B\tan C)^3(\cos A\cos B\cos C)^2}=$$

$$=\frac{64\sin A\sin B\sin C}{3^{\frac{9}{4}}(\tan A\tan B\tan C)^{\frac{3}{2}}(\cos A\cos B\cos C)^2}=\frac{64}{3^{\frac{9}{4}}\sqrt{\sin A\sin B\sin C\cos A\cos B\cos C}}=$$

$$=\frac{64*2\sqrt{2}}{3^{\frac{9}{4}}\sqrt{\sin 2A\sin 2B\sin 2C}}\overset{AM-GM}{\geq}\frac{64*2\sqrt{2}}{3^{\frac{9}{4}}\sqrt{\left(\frac{\sin 2A+\sin 2B+\sin 2C}{3}\right)^3}}\geq$$

$$\overset{Jensen}{\geq}\frac{64*2\sqrt{2}}{3^{\frac{9}{4}}\sqrt{\left(\sin\frac{2A+2B+2C}{3}\right)^3}}=\frac{64*2\sqrt{2}}{3^{\frac{9}{4}}\sqrt{\sin^3\frac{2\pi}{3}}}=\frac{64*2\sqrt{2}}{3^{\frac{9}{4}}\left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{2}}}=\frac{512}{27}$$

115. Prove that in any triangle ABC the following relationship holds:

$$\frac{m_a}{\sin\frac{A}{2}}+\frac{m_b}{\sin\frac{B}{2}}+\frac{m_c}{\sin\frac{C}{2}}\geq\frac{a^2+b^2+c^2}{2r}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solutin by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{m_a}{\sin\frac{A}{2}}r+\frac{m_b}{\sin\frac{B}{2}}r+\frac{m_c}{\sin\frac{C}{2}}r\geq\frac{a^2+b^2+c^2}{2}$

$$\frac{m_a}{\sin\frac{A}{2}}(p-a)\tan\frac{A}{2}+\frac{m_b}{\sin\frac{B}{2}}(p-b)\tan\frac{B}{2}+\frac{m_c}{\sin\frac{C}{2}}+\frac{m_c}{\sin\frac{C}{2}}(p-c)\tan\frac{C}{2}\geq\frac{a^2+b^2+c^2}{2}$$

De las siguientes desigualdades conocidas en un triángulo ABC:

$$m_a\geq\frac{b+c}{2}\cos\frac{A}{2},m_b\geq\frac{a+c}{2}\cos\frac{B}{2},m_c\geq\frac{a+b}{2}\cos\frac{C}{2}$$

La desigualdad es equivalente:

$$\frac{m_a}{\sin\frac{A}{2}}(p-a)\tan\frac{A}{2}+\frac{m_b}{\sin\frac{B}{2}}(p-b)\tan\frac{B}{2}+\frac{m_c}{\sin\frac{C}{2}}(p-c)\tan\frac{C}{2}\geq$$

$$\geq\sum\left(\frac{b+c}{2}\right)(p-a)$$

$$\to\sum\left(\frac{b+c}{2}\right)(p-a)=\left(\frac{b+c}{2}\right)\left(\frac{b+c-a}{2}\right)+\left(\frac{c+a}{2}\right)\left(\frac{c+a-b}{2}\right)+$$

$$+\left(\frac{a+b}{2}\right)\left(\frac{a+b-c}{2}\right)=\frac{a^2+b^2+c^2}{2}$$

116. In ΔABC the following relationship holds:

$$9\sum\frac{a^5}{(b+c-a)^2}\geq 8s(s_ar_a+s_br_b+s_cr_c)$$

s_a, s_b, s_c – simedians, s - semiperimeter

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

$s_a=\frac{2bc}{b^2+c^2}\cdot m_a\leq m_a$ $(\because 2bc\leq b^2+c^2)$**. Similarly,** $s_b\leq m_b$ **and** $s_c\leq m_c$

$$RHS=8s\left(\sum s_ar_a\right)\overset{CBS}{\leq}8s\sqrt{\sum s_a^2}\sqrt{\sum r_a^2}\leq 8s\sqrt{\sum m_a^2}\sqrt{\sum r_a^2}$$

$$\textbf{(1)}\leq 8s\sqrt{\sum r_a^2}\sqrt{\sum r_a^2}=8s\left(\sum r_a^2\right)$$

$$\left(\therefore \sum m_a^2 = \frac{3}{4}\left(\sum a^2\right) \overset{Leibniz}{\leq} \frac{27R^2}{4} \overset{Thebault}{\leq} \sum r_a^2\right)$$

$$LHS = \frac{9}{4}\sum \frac{a^5}{(s-a)^2} = \frac{9}{4}\left\{\frac{a^5}{(s-a)^2} + \frac{b^5}{(s-b)^2} + \frac{c^5}{(s-c)^2}\right\}$$

WLOG, we may assume $a \leq b \leq c$. Now, $\frac{a}{s-a} \leq \frac{b}{s-b} \Leftrightarrow as \leq bs \Leftrightarrow a \leq b$

Again, $\frac{b}{s-b} \leq \frac{c}{s-c} \Leftrightarrow bs \leq cs \Leftrightarrow b \leq c; a \leq b \leq c \Rightarrow \frac{a^2}{(s-a)^2} \leq \frac{b^2}{(s-b)^2} \leq \frac{c^2}{(s-c)^2}$

Now, $LHS = \frac{9}{4}\left\{a^3\left(\frac{a^2}{(s-a)^2}\right) + b^3\left(\frac{b^2}{(s-b)^2}\right) + c^3\left(\frac{c^2}{(s-c)^2}\right)\right\}$

$$\overset{Chebyshec}{\geq} \frac{9}{4}\cdot\frac{1}{3}\left(\sum a^3\right)\left(\sum \frac{a^2}{(s-a)^2}\right) = \frac{3}{4\Delta^2}\left(\sum a^3\right)\left(\sum a^2 \frac{\Delta^2}{(s-a)^2}\right)$$

$$= \frac{3}{4\Delta^2}\left(\sum a^3\right)\left(a^2 r_a^2 + b^2 r_b^2 + c^2 r_c^2\right)$$

Now, $r_a \leq r_b \Leftrightarrow \frac{\Delta}{s-a} \leq \frac{\Delta}{s-b} \Leftrightarrow s-b \leq s-a \Leftrightarrow a \leq b$

Similarly, $r_b \leq r_c \Leftrightarrow b \leq c \therefore a \leq b \leq c \Rightarrow r_a^2 \leq r_b^2 \leq r_c^2$

$$LHS \geq \frac{3}{4\Delta^2}\left(\sum a^3\right)\left(\sum a^2 r_a^2\right) \overset{Chebyshev}{\geq} \frac{3}{4\Delta^2}\cdot\frac{1}{3}\cdot\left(\sum a^3\right)\left(\sum a^2\right)\left(\sum r_a^2\right)$$

$$\overset{AM-GM}{\geq} \frac{1}{4\Delta^2}(3abc)\left(\sum a^2\right)\left(\sum r_a^2\right) = \frac{3R\Delta}{\Delta^2}\left(\sum a^2\right)\left(\sum r_a^2\right)$$

$$\overset{Weitzenbock}{\geq} \frac{3R\left(4\sqrt{3}\Delta\right)\left(\sum r_a^2\right)}{\Delta} = \left(3\sqrt{3}R\right)\left(4\sum r_a^2\right)$$

$\overset{Mitrinovic}{\geq}$ $(2s)(4\sum r_a^2) = 8s(\sum r_a^2) \geq RHS$ (from (1)) (proved)

117. Prove that in any ABC triangle the following relationship holds:

$$\frac{1}{r^3}\sum a^3 \cos B \cos C \geq 16\left(\sum \sin A\right)\left(\sum \cos^2 A\right)$$

Proposed by Daniel Sitaru - Romania

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Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC. Probar que:

$$\frac{1}{r^3}\sum a^3\cos B\cos C \geq 16\left(\sum \operatorname{sen} A\right)\left(\sum \cos^2 A\right) \to r \text{ (Inradio)}$$

$$\frac{R^3}{r^3}(8\operatorname{sen}^3 A\cos B\cos C + 8\operatorname{sen}^3 B\cos A\cos C + 8\operatorname{sen}^3 C\cos A\cos B) \geq$$

$$\geq 16(\operatorname{sen} A + \operatorname{sen} B + \operatorname{sen} C)(\cos^2 A + \cos^2 B + \cos^2 C)$$

Tener presente en un triángulo ABC:

1) $\operatorname{sen} 4A + \operatorname{sen} 4B + \operatorname{sen} 4C = -4\operatorname{sen} 2A\operatorname{sen} 2B\operatorname{sen} 2C$,

2) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$,

3) $\frac{r}{R} = 4\operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}$,

4) $\operatorname{sen}(B + C) = \operatorname{sen} A$ $\wedge$ ***5)*** $\operatorname{sen}(2B + 2C) = -\operatorname{sen} 2A$

$$T_1 = 8\operatorname{sen}^3 A\cos B\cos C \to T_1 = (2\operatorname{sen}^2 A)(2\operatorname{sen} A)(\cos(B + C) + \cos(B - C))$$

$$T_1 = (1 - \cos 2A)(2\operatorname{sen}(B + C)\cos(B + C) + \cos(B - C))$$

$$T_1 = (1 - \cos 2A)(\operatorname{sen}(2B + 2C) + \operatorname{sen} 2B + \operatorname{sen} 2C)$$

$$T_1 = (-\operatorname{sen} 2A + \operatorname{sen} 2B + \operatorname{sen} 2C) - \operatorname{sen} 2B\cos 2A - \operatorname{sen} 2C\cos 2A + (0,5)2\operatorname{sen} 2A\cos 2A$$

$$T_2 = 8\operatorname{sen}^3 B\cos A\cos C \to$$

$$\to T_2 = (-\operatorname{sen} 2B + \operatorname{sen} 2A + \operatorname{sen} 2C) - \operatorname{sen} 2A\cos 2B -$$

$$-\operatorname{sen} 2C\cos 2B + (0,5)2\operatorname{sen} 2B\cos 2B$$

$$T_3 = 8\operatorname{sen}^3 C\cos A\cos B \to$$

$$\to T_3 = (-\operatorname{sen} 2C + \operatorname{sen} 2A + \operatorname{sen} 2B) - \operatorname{sen} 2A\cos 2C -$$

$$-\operatorname{sen} 2B\cos 2C + (0,5)2\operatorname{sen} 2C\cos 2C$$

$$T_1 + T_2 + T_3 = 2(\operatorname{sen} 2A + \operatorname{sen} 2B + \operatorname{sen} 2C) - 2\operatorname{sen} 2A\operatorname{sen} 2B\operatorname{sen} 2C$$

$$T_1 + T_2 + T_3 = 2(4\operatorname{sen} A\operatorname{sen} B\operatorname{sen} C)(1 - 2\cos A\cos B\cos C) \geq$$

$$\geq 16(\operatorname{sen} A + \operatorname{sen} B + \operatorname{sen} C)(\cos^2 A + \cos^2 B + \cos^2 C)\left(4\operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}\right)^3$$

$$8\operatorname{sen}A\operatorname{sen}B\operatorname{sen}C \geq 16\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\left(2\operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}\right)8\,x4\left(\operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}\right)^2 \rightarrow$$

$$\rightarrow \frac{1}{64} \geq \left(\operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}\right)^2 \rightarrow \frac{1}{8} \geq \operatorname{sen}\frac{A}{2}\operatorname{sen}\frac{B}{2}\operatorname{sen}\frac{C}{2}$$

Solution 2 by Daniel Sitaru – Romania

Let be $\Delta = \begin{vmatrix} a & b\cos C & c\cos B \\ b & c\cos A & a\cos C \\ c & a\cos B & b\cos A \end{vmatrix} =$

$$= 8R^3\begin{vmatrix} \sin A & \sin B\cos C & \sin C\cos B \\ \sin B & \sin C\cos A & \sin A\cos C \\ \sin C & \sin A\cos B & \sin B\cos A \end{vmatrix}^{C_2+C_3} = 8R^3\begin{vmatrix} \sin A & \sin(B+C) & \sin C\cos B \\ \sin B & \sin(A+C) & \sin A\cos C \\ \sin C & \sin(A+B) & \sin B\cos A \end{vmatrix} =$$

$$= 8R^3\begin{vmatrix} \sin A & \sin(\pi - A) & \sin C\cos B \\ \sin B & \sin(\pi - B) & \sin A\cos C \\ \sin C & \sin(\pi - C) & \sin B\cos A \end{vmatrix} = 8R^3\begin{vmatrix} \sin A & \sin A & \sin C\cos B \\ \sin B & \sin B & \sin A\cos C \\ \sin C & \sin C & \sin B\cos A \end{vmatrix} = 0$$

On the other hand: $0 = \Delta = \begin{vmatrix} a & b\cos C & c\cos B \\ b & c\cos A & a\cos C \\ c & a\cos B & b\cos A \end{vmatrix} =$

$$= ab\cos^2 A + abc\cos^2 C + abc\cos^2 A -$$

$$-c^3\cos A\cos B - a^3\cos B\cos C - b^3\cos A\cos C =$$

$$= abc\sum\cos^2 A - \sum a^3\cos B\cos C = 4RS\sum\cos^2 A - \sum a^3\cos B\cos C.$$

It follows: $\sum a^3\cos B\cos C = 4RS\sum\cos^2 A =$

$$= 4Rrp\sum\cos^2 A = 4Rr\cdot\frac{a+b+c}{2}\sum\cos^2 A$$

$$\frac{\sum a^3\cos B\cos C}{\sum\cos^2 A} = 2Rr(a+b+c) = 2Rr\cdot 2R\cdot\sum\sin A = 8R^2r\sum\sin A$$

From Euler's inequality: $R \geq 2r$

$$\frac{\sum a^3\cos B\cos C}{(\sum\sin A)(\sum\cos^2 A)} = 8R^2r \geq 8\cdot(2r)^2\cdot r = 16r^3$$

$$\frac{1}{r^3}\sum a^3\cos B\cos C \geq 16\left(\sum\sin A\right)\left(\sum\cos^2 A\right)$$

Solution 3 by Soumava Chakraborty – Kolkata – India

In $\Delta ABC, \frac{1}{r^3}\sum a^3\cos B\cos C \ge 16(\sum \sin A)(\sum \cos^2 A); r \to$ ***inradius***

$$a^3\cos B\cos C = R^3(8\sin^3 A\cos B\cos C)$$

$$8\sin^3 A\cos B\cos C = (2\sin^2 A)(4\sin A\cos B\cos C) =$$

$$= (1-\cos 2A)(2\cos C)(2\sin A\cos B) =$$

$$= (1-\cos 2A)(2\cos C)\{\sin(A+B)+\sin(A-B)\} =$$

$$= (1-\cos 2A)\{2\cos C\sin C - 2\cos(A+B)\ \sin(A-B)\} =$$

$$= (1-\cos 2A)\{\sin 2C - (\sin 2A - \sin 2B)\} =$$

$$= (1-\cos 2A)(\sin 2B + \sin 2C - \sin 2A) =$$

$$= \sin 2B + \sin 2C - \sin 2A - \cos 2A\sin 2B - \cos 2A\sin 2C + \cos 2A\sin 2A \quad \textit{(1)}$$

Similarly, $8\sin^3 B\cos C\cos A = (1-\cos 2B)(\sin 2C + \sin 2A - \sin 2B)$

$$= \sin 2C + \sin 2A - \sin 2B - \cos 2B\sin 2C - \cos 2B\sin 2A + \sin 2B\cos 2B \quad \textit{(2)}$$

and $8\sin^3 C\cos A\cos B = (1-\cos 2C)(\sin 2A + \sin 2B - \sin 2C)$

$$= \sin 2A + \sin 2B - \sin 2C - \cos 2C\sin 2A - \cos 2C\sin 2B + \sin 2C\cos 2C \quad \textit{(3)}$$

(1) + ***(2)*** + ***(3)*** $\Rightarrow \frac{1}{r^3}\sum a^3\cos B\cos C$

$$= \frac{R^3}{r^3}\begin{pmatrix}(\sin 2A + \sin 2B + \sin 2C) - \sin 2C(\cos 2A + \cos 2B) - \\ -\sin 2B(\cos 2C + \cos 2A) - \sin 2A(\cos 2B + \cos 2C) + \\ +\cos 2A\sin 2A + \cos 2B\sin 2B + \cos 2C\sin 2C\end{pmatrix}$$

$$-\sin 2C(\cos 2A + \cos 2B) = -\sin 2C\{2\cos(A+B)\cos(A-B)\}$$

$$= -2\sin C\cos C(-2\cos C\cos(A-B))$$

$$= 4\cos^2 C\sin(A+B)\cos(A-B) = 2\cos^2 C(\sin 2A + \sin 2B)$$

$$= (1+\cos 2C)(\sin 2A + \sin 2B)$$

$$= (\sin 2A + \sin 2B) + \cos 2C(\sin 2A + \sin 2B) \quad \textit{(4)}$$

Similarly,

$$-\sin 2B(\cos 2C + \cos 2A) = (\sin 2C + \sin 2A) + \cos 2B(\sin 2C + \sin 2A) \quad \textit{(5)}$$

and

$$-\sin 2A(\cos 2B+\cos 2C)=(\sin 2B+\sin 2C)+\cos 2A(\sin 2B+\sin 2C)\quad \textbf{\textit{(6)}}$$

$$\frac{1}{r^3}\sum a^3\cos B\cos C=$$

$$=\frac{R^3}{r^3}\left\{3\sum\sin 2A+(\cos 2A+\cos 2B+\cos 2C)\left(\sum\sin 2A\right)\right\}=$$

$$=\frac{R^3}{r^3}\left(\sum\sin 2A\right)\{(1+\cos 2A)+(1+\cos 2B)+(1+\cos 2C)\}=$$

$$=\frac{R^3}{r^3}\left(\sum\sin 2A\right)(2)\left(\sum\cos^2 A\right)=\frac{2R^3}{r^3}\left(\sum\sin 2A\right)\left(\sum\cos^2 A\right)\quad \textbf{\textit{(A)}}$$

$$\sum\sin 2A=\sin 2A+\sin 2B+\sin 2C=$$

$$=2\sin(A+B)\cdot\cos(A-B)+2\sin C\cos C=$$

$$=2\sin C\{\cos(A-B)-\cos(A+B)\}=$$

$$=2\sin C\cdot 2\sin A\sin B=4\sin A\sin B\sin C=$$

$$=4\cdot 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=$$

$$=\left(8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\quad \textbf{\textit{(7)}}$$

Now, $\sin A+\sin B+\sin C=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}=$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\textbf{\textit{(7)}}\Rightarrow\sum\sin 2A=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\left(\sum\sin A\right)=$$

$$=8\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}\left(\sum\sin A\right)=$$

$$=\frac{8s(s-a)(s-b)(s-c)}{sabc}\left(\sum\sin A\right)=\left(\frac{8\Delta^2}{sabc}\right)\left(\sum\sin A\right)$$

$$\Delta = \frac{abc}{4R} \text{ and } \Delta = rs, \Delta^2 = \frac{s(abc)r}{4R}$$

$$\sum \sin 2A = \frac{8(sabc)r}{4R(sabc)}\left(\sum \sin A\right) = \frac{2r}{R}\left(\sum \sin A\right)$$

(A) $\Rightarrow \frac{1}{r^3}(\sum a^3 \cos B \cos C) = \left(\frac{2R^3}{r^3}\right)\left(\frac{2r}{R}(\sum \sin A)\right)(\sum \cos^2 A)$

$$= 4\left(\frac{R^2}{r^2}\right)\left(\sum \sin A\right)\left(\sum \cos^2 A\right) \geq 4(2^2)\left(\sum \sin A\right)\left(\sum \cos^2 A\right)$$

$(R \geq 2r) = 16(\sum \sin A)(\sum \cos^2 A)$. ***(Hence proved)***

118. Let ABC be an acute triangle. Show that

$$\sum \frac{\sec A}{\sqrt{\cos A + \cos B}} \geq 6$$

where the sum $\sum$ is over all cyclic permutations of (A, B, C).

Proposed by George Apostolopoulos – Messalonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC:

$$\frac{\sec A}{\sqrt{\cos A + \cos B}} + \frac{\sec B}{\sqrt{\cos B + \cos C}} + \frac{\sec C}{\sqrt{\cos C + \cos A}} \geq 6$$

Dado que es un triángulo acutángulo: $\cos A, \cos B, \cos C > 0$,

$\sec A \sec B \sec C \geq 8$. *Por la desigualdad de Cauchy:*

$$\frac{\sec A}{\sqrt{\cos A + \cos B}} + \frac{\sec B}{\sqrt{\cos B + \cos C}} + \frac{\sec C}{\sqrt{\cos C + \cos A}} \geq$$

$$\geq \frac{\left(\sqrt{\sec A} + \sqrt{\sec B} + \sqrt{\sec C}\right)^2}{\sqrt{\cos A + \cos B} + \sqrt{\cos B + \cos C} + \sqrt{\cos C + \cos A}} \geq 6$$

Esto es suficiente probar que:

$$3\left(\sqrt{\sec A}+\sqrt{\sec B}+\sqrt{\sec C}\right)^2 \geq 18\left(\sqrt{\cos A+\cos B}+\sqrt{\cos B+\cos C}+\sqrt{\cos C+\cos A}\right)$$

Desde que: $\sqrt{\cos A+\cos B}+\sqrt{\cos B+\cos C}+\sqrt{\cos C+\cos A} \leq$

$$\leq \sqrt{3(2)(\cos A+\cos B+\cos C)} \leq 3$$

$$18 \leq \left(3\sqrt[3]{\sqrt{\sec A\sec B\sec C}}\right)^2 \leq \left(\sqrt{\sec A}+\sqrt{\sec B}+\sqrt{\sec C}\right)^2$$

(The proof is completed)

119. In ABC the following relationship holds:

$$a^{\sin a}+b^{\sin B}+c^{\sin C} \geq 3\cdot\left(\frac{2s}{3}\right)^{\frac{s}{3R}}$$

where $a+b+c=2s$

Proposed by Mehmet Sahin – Ankara – Turkey

Solution 1 by Daniel Sitaru – Romania

$$\sum a^{\sin A} \geq 3\sqrt[3]{\prod a^{\sin A}} = 3\left(\prod a^a\right)^{\frac{1}{6R}} \geq$$

$$\geq 3\left(\frac{a+b+c}{3}\right)^{\frac{a+b+c}{6R}} = 3\left(\frac{2s}{3}\right)^{\frac{s}{3R}}$$

Solution 2 by Mehmet Sahin – Ankara – Turkey

First, let's define the function

$f(x)=x^{\frac{x}{2R}}, f''(x)>0=f$ ***is convex. If used Jensen Inequality, then***

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{1}{3}[f(a)+f(b)+f(c)]$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{a+b+c}{3\cdot 2R}} \leq \frac{1}{3}\left(a^{\frac{a}{2R}}+b^{\frac{b}{2R}}+c^{\frac{c}{2R}}\right)$$

$$\Rightarrow a^{\sin A}+b^{\sin B}+c^{\sin C}\geq 3\cdot\left(\frac{2s}{3}\right)^{\frac{s}{3R}}$$ ***as desired.***

120. In ABC the following relationship holds:

$$\sum_{a,b,c}\frac{1}{r_a-r}\geq\frac{7}{R}-\frac{2}{r}$$

Proposed by Mehmet Sahin – Ankara – Turkey

Solution 1 by Daniel Sitaru – Romania

$$\sum\frac{1}{r_a-r}=\frac{1}{S}\sum\frac{s(s-a)}{a}=\frac{s}{S}\sum\frac{s-a}{s}=\frac{s}{S}\cdot\frac{s^2+r^2-8Rr}{4Rr}\geq\frac{7r-2R}{Rr}\leftrightarrow$$

$$\leftrightarrow s^2+r^2-8Rr\geq 28r^2-8Rr\leftrightarrow s^2\geq 27r^2\leftrightarrow s\geq 3\sqrt{3}r$$

Solution 2 by George Apostolopoulos – Messalonghi – Greece

We have $\frac{1}{r_a-r}+\frac{1}{r_b-r}+\frac{1}{r_c-r}\geq\frac{(1+1+1)^2}{r_a+r_b+r_c-3r}=\frac{9}{(4R+r)-3r}=\frac{3}{4R-2r}$

It suffices to prove that $\frac{9}{4R-2r}\geq\frac{7}{R}-\frac{2}{r}$. ***We have***

$$\Leftrightarrow 9Rr\geq 7r(4R-2r)-2R(4R-2r)\Leftrightarrow 8R^2-23Rr+14r^2\geq 0$$

We have $8R^2-23Rr+14r^2=8(R-2r)\left(R-\frac{3r}{2}\right)\geq 0$

because $R\geq 2r$ ***(Euler), and*** $R-\frac{3r}{2}>2r-\frac{3r}{2}>0$

Equality holds when the triangle ABC is equilateral.

121. In $\Delta\ ABC$ the following relationship holds:

$$\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}\geq 2\sqrt{2}\left(\frac{w_a}{r_b+r_c}+\frac{w_b}{r_c+r_a}+\frac{w_c}{r_a+r_b}\right)$$

Proposed by Bogdan Fustei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{a}{s-a}} = \frac{\sqrt{a}\sqrt{s}\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c)}} = \sqrt{as}\sqrt{\frac{s-b}{\Delta}}\sqrt{\frac{s-c}{\Delta}} \overset{(1)}{=} \sqrt{as}\cdot\frac{1}{\sqrt{r_b r_c}}$$

$$\textbf{Similarly, } \sqrt{\frac{b}{s-b}} \overset{(2)}{=} \sqrt{bs}\cdot\frac{1}{\sqrt{r_c r_a}} \ \& \ \sqrt{\frac{c}{s-c}} \overset{(3)}{=} \sqrt{cs}\cdot\frac{1}{\sqrt{r_a r_b}}$$

$$\textbf{(1)+(2)+(3)} \Rightarrow LHS \overset{(4)}{=} \sqrt{s}\sum\left(\sqrt{a}\cdot\frac{1}{\sqrt{r_b r_c}}\right)$$

$$\textbf{WLOG, we may assume } a \ge b \ge c. \textbf{ Then } \sqrt{a} \ge \sqrt{b} \ge \sqrt{c} \ \& \frac{1}{\sqrt{r_b r_c}} \ge \frac{1}{\sqrt{r_c r_a}} \ge \frac{1}{\sqrt{r_a r_a}}$$

$$\therefore \textbf{(4)} \Rightarrow LHS \underset{(5)}{\overset{Chebyshev}{\ge}} \frac{\sqrt{s}}{3}\left(\sum\sqrt{a}\right)\left(\sum\frac{1}{\sqrt{r_b r_c}}\right)$$

$$\because w_a \le \sqrt{s(s-a)} \textbf{ etc } \& \ r_b + r_c \overset{A-G}{\ge} 2\sqrt{r_b r_c} \textbf{ etc } \therefore RHS \overset{(6)}{\le} \sqrt{2s}\sum\left(\sqrt{s-a}\cdot\frac{1}{\sqrt{r_b r_c}}\right)$$

$$\because a \ge b \ge c \ \textbf{ (by assumption)}$$

$$\therefore \sqrt{s-a} \le \sqrt{s-b} \le \sqrt{s-c} \ \textbf{ \& of course, } \frac{1}{\sqrt{r_b r_c}} \ge \frac{1}{\sqrt{r_c r_a}} \ge \frac{1}{\sqrt{r_a r_b}}$$

$$\therefore \textbf{(6)} \Rightarrow RHS \underset{(7)}{\overset{Chebyshev}{\ge}} \frac{\sqrt{2s}}{3}\left(\sum\sqrt{s-a}\right)\left(\sum\frac{1}{\sqrt{r_b r_c}}\right)$$

$$\textbf{(5), (7)} \Rightarrow \textbf{ it suffices to prove: } \sum\sqrt{a} \ge \sum\sqrt{2s-2a} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{a} + \sqrt{b} + \sqrt{c} \overset{(8)}{\ge} \sqrt{b+c-a} + \sqrt{c+a-b} + \sqrt{a+b-c}$$

$$\textbf{Let } b+c-a = x, c+a-b = y, a+b-c = z. \textbf{ Then } a+b+c = x+y+z$$

$$\therefore a = \frac{y+z}{2}, b = \frac{z+x}{2}, c = \frac{x+y}{2} \to x, y, z > 0$$

$$\therefore \textbf{(8)} \Leftrightarrow \sum\sqrt{\frac{y+z}{2}} \ge \sum\sqrt{x} \Leftrightarrow \sum\left(\frac{y+z}{2}\right) + 2\sum\sqrt{\frac{(y+z)(z+x)}{4}} \ge \sum x + 2\sum\sqrt{xy} \textbf{ (upon squaring)}$$

$$\Leftrightarrow \sum\sqrt{(y+z)(z+x)} \ge 2\sum\sqrt{xy} \Leftrightarrow \sum(y+z)(z+x) + 2\sqrt{\prod(x+y)}\left(\sum\sqrt{x+y}\right)$$

$$\ge 4\sum xy + 8\sqrt{xyz}\left(\sum\sqrt{x}\right) \ \textbf{ (upon squaring)}$$

$$\Leftrightarrow \sum x^2 + 3\sum xy + 2\sqrt{\prod(x+y)}\left(\sum\sqrt{x+y}\right) \ge 4\sum xy + 8\sqrt{xyz}\left(\sum\sqrt{x}\right)$$

$$\Leftrightarrow \sum x^2 + 2\sqrt{\prod(x+y)}\left(\sum\sqrt{x+y}\right) \overset{(9)}{\ge} \sum xy + 8\sqrt{xyz}\left(\sum\sqrt{x}\right)$$

Now, $\sum x^2 \geq \sum xy$***. Also,*** $\prod(x+y) \geq 8xyz \therefore$ ***in order to prove (9), it suffices to show:***

$$\sum \sqrt{2(x+y)} \overset{(10)}{\geq} 2\sum \sqrt{x}$$

Now, $\sum \sqrt{2(x+y)} \overset{Chebyshev}{\geq} \sum \sqrt{\left(\sqrt{x}+\sqrt{y}\right)^2} = \sum\left(\sqrt{x}+\sqrt{y}\right) = 2\sum\sqrt{x} \Rightarrow$ ***(10) is true***

(Hence proved)

Solution 2 by Soumitra Mandal-Chandar Nagore-India

We know $abc \geq 8\prod_{cyc}(p-a)$ ***then*** $\sum_{cyc}\sqrt{\frac{a}{p-a}} \overset{AM\geq GM}{\geq} 3\sqrt[3]{\sqrt{\prod_{cyc}\left(\frac{a}{p-a}\right)}}$

$= 3\sqrt[3]{\sqrt{8}} = 3\sqrt{2}$ ***again,*** $2\sqrt{2}\sum_{cyc}\frac{w_a}{r_b+r_c} \overset{AM\geq GM}{\leq} 2\sqrt{2}\sum_{cyc}\frac{w_a}{2\sqrt{r_br_c}} \leq$

$$\leq \sqrt{2}\sum_{cyc}\frac{\sqrt{p(p-a)}}{\sqrt{\frac{\Delta}{p-b}\cdot\frac{\Delta}{p-c}}} = 3\sqrt{2}\cdot\frac{\sqrt{p(p-a)(p-b)(p-c)}}{\Delta} = 3\sqrt{2}$$

122. In acute ΔABC the following relationship holds:

$$2\left(\sin\left(A+\frac{\pi}{4}\right)+\sin\left(B+\frac{\pi}{4}\right)+\sin\left(C+\frac{\pi}{4}\right)\right) > 3+\sqrt{\sin 2A}+\sqrt{\sin 2B}+\sqrt{\sin 2C}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC***:***

$$2\left(\sin\left(A+\frac{\pi}{4}\right)+\sin\left(B+\frac{\pi}{4}\right)+\sin\left(C+\frac{\pi}{4}\right)\right) > 3+\sqrt{\sin 2A}+\sqrt{\sin 2B}+\sqrt{\sin 2C}$$

$$\Rightarrow 2\left(\left(\frac{1}{\sqrt{2}}\right)(\sin A+\cos A)+\left(\frac{1}{\sqrt{2}}\right)(\sin B+\cos B)+\left(\frac{1}{\sqrt{2}}\right)(\sin C+\cos C)\right) >$$

$$> 3+\sqrt{\sin 2A}+\sqrt{\sin 2B}+\sqrt{\sin 2C}$$

$$\Rightarrow \sqrt{2}(\sin A+\cos A)+\sqrt{2}(\sin B+\cos B)+\sqrt{2}(\sin C+\cos C) >$$

$> 3+\sqrt{\sin 2A}+\sqrt{\sin 2B}+\sqrt{\sin 2C}$ ... ***(A)***

Dado que es un triángulo acutángulo:

$$(\sin A, \sin B, \sin C), (\cos A, \cos B, \cos C) > 0$$

Además: $\sqrt{1+\sin 2x} = \sin x + \cos x \Leftrightarrow (\sin x, \cos x) > 0$

Por la tanto tenemos en ... (A):

$$\Rightarrow \sqrt{2(1+\sin 2A)} + \sqrt{2(1+\sin 2B)} + \sqrt{2(1+\sin 2C)} >$$

$$> 3 + \sqrt{\sin 2A} + \sqrt{\sin 2B} + \sqrt{\sin 2C}$$

Es suficiente demonstrar lo siguiente:

$$\Rightarrow \sqrt{2(1+\sin 2A)} > 1 + \sqrt{\sin 2A} \to 2(1+\sin 2A) > 1 + \sin 2A +$$

$2\sqrt{\sin 2A} \to 1 + \sin 2A > 2\sqrt{\sin 2A} \Rightarrow \left(1-\sqrt{\sin 2A}\right)^2 > 0$, ***desde que:***

$0 < 2A < \pi \to 0 < \sin 2A < 1 \to 0 < \sqrt{\sin 2A} < 1 \Rightarrow$ ***Por la tanto:***

$$\sqrt{2(1+\sin 2A)} + \sqrt{2(1+\sin 2B)} + \sqrt{2(1+\sin 2C)} >$$

$> 3 + \sqrt{\sin 2A} + \sqrt{\sin 2B} + \sqrt{\sin 2C}$ ***... (LQQD)***

Solution 2 by Soumava Chakraborty – Kolkata – India

In acute $\Delta ABC, \sum 2\sin\left(A+\frac{\pi}{4}\right) > 3 + \sum\sqrt{\sin 2A}$

Given inequality $\Leftrightarrow \sum\sqrt{2}(\sin A + \cos A) > \sum\left(1+\sqrt{2\sin A\cos A}\right)$

Let's prove that $\forall x \in \left(0, \frac{\pi}{2}\right), \sqrt{2}(\sin x + \cos x) \geq 1 + \sqrt{2\sin x\cos x}$

$$\Leftrightarrow \sqrt{2}(\sin x + \cos x) - 1 - \sqrt{(\sin x + \cos x)^2 - 1} \geq 0$$

$\Leftrightarrow \sqrt{2}t - 1 - \sqrt{t^2-1} \geq 0$, ***where*** $t = \sin x + \cos x$

$$t^2 - 1 = 2\sin x\cos x > 0 \Rightarrow t^2 > 1 \Rightarrow t > 1 (\because t > 0)$$

Let $f(t) = \sqrt{2}t - 1 - \sqrt{t^2-1} \therefore f'(t) = \sqrt{2} - \frac{t}{\sqrt{t^2-1}}$ ***and*** $f''(t) = \frac{1}{(t^2-1)^{\frac{3}{2}}} > 0$

$\therefore f'(t) = 0 \Rightarrow$ ***occurs when*** $f(t)$ ***attains a minima***

$$f'(t) = 0 \Rightarrow \frac{t^2}{t^2-1} = 2 \Rightarrow t = \sqrt{2} (\because t > 1)$$

$\because f(t)$ ***never attains a maxima*** $\forall t>1 \therefore f(t)\geq f(\sqrt{2})=2-1-1=0$

$$\Rightarrow f(t)\geq 0, \text{ equality at } x=\frac{\pi}{4}$$

$$\Rightarrow \sqrt{2}(\sin x+\cos x)\geq 1+\sqrt{2\sin x\cos x}, \forall x\in\left(0,\frac{\pi}{2}\right), \text{ equality at } x=\frac{\pi}{4}$$

Case 1: 1 ***angle*** $>\frac{\pi}{4}$, 1 ***angle*** $\geq\frac{\pi}{4}$, 1 ***angle*** $<\frac{\pi}{4}$

WLOG we may assume $A>\frac{\pi}{4}, B\geq\frac{\pi}{4}, C<\frac{\pi}{4}$

Then $\sqrt{2}(\sin A+\cos A)>1+\sqrt{\sin A}, \sqrt{2}(\sin B+\cos B)\geq 1+\sqrt{\sin 2B}$,

$\sqrt{2}(\sin C+\cos C)>1+\sqrt{\sin C}$ ***... adding,***

$$\sum\sqrt{2}(\sin A+\cos A)>\sum\left(1+\sqrt{\sin 2A}\right)$$

Case 2: 2 ***angles*** $>\frac{\pi}{4}$, 1 ***angle*** $\geq\frac{\pi}{4}$***. WLOG, we may assume*** $A, B>\frac{\pi}{4}, C\geq\frac{\pi}{4}$

Then, $\sqrt{2}(\sin A+\cos A)>1+\sqrt{\sin 2A}, \sqrt{2}(\sin B+\cos B)>1+\sqrt{\sin 2B}$,

$\sqrt{2}(\sin C+\cos C)\geq 1+\sqrt{\sin C}$ ***... adding***

$$\sum\sqrt{2}(\sin A+\cos A)>\sum\left(1+\sqrt{\sin 2A}\right)$$

123. In $\Delta\, ABC$ the following relationship holds:

$$\frac{r_a h_a}{a w_a}+\frac{r_b h_b}{b w_b}+\frac{r_c h_c}{c w_c}\leq\frac{1}{2}\left(\frac{r_b+r_c}{b+c-a}+\frac{r_c+r_a}{c+a-b}+\frac{r_a+r_b}{a+b-c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC***, la siguiente desigualdad:***

$$\frac{2r_a h_a}{a w_a}+\frac{2r_b h_b}{b w_b}+\frac{2r_c h_c}{c w_c}\leq\frac{r_a+r_b}{a+b-c}+\frac{r_b+r_c}{b+c-a}+\frac{r_c+r_a}{a+c-b}$$

En un triángulo ABC***, se cumple la siguiente desigualdad:***

$$w_a \geq h_a \to \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2bc}{4R\cos\left(\frac{B-C}{2}\right)} \geq \frac{2bc}{\frac{abc}{S}\times 1} = \frac{2S}{a} = h_a \text{ ... (LQQD)}$$

Por lo tanto la desigualdad es equivalente:

$$\frac{2r_ah_a}{aw_a} + \frac{2r_bh_b}{bw_b} + \frac{2r_ch_c}{cw_c} \leq \frac{2r_a}{a} + \frac{2r_b}{b} + \frac{2r_c}{c} \leq \frac{r_a + r_b}{a+b-c} + \frac{r_b + r_c}{b+c-a} + \frac{r_c + r_a}{a+c-b}$$

Desde que: a, b, c ***son lados de un triángulo:***

$(a+b-c), (b+c-a), (a+c-b) > 0$. ***Por la desigualdad de Cauchy:***

$$r_a\left(\frac{1}{a+b-c} + \frac{1}{a+c-b}\right) \geq \frac{4r_a}{2a} = \frac{2r_a}{a},$$

$$r_b\left(\frac{1}{b+a-c} + \frac{1}{b+c-a}\right) \geq \frac{2r_b}{b}, r_c\left(\frac{1}{c+a-b} + \frac{1}{c+b-a}\right) \geq \frac{2r_c}{c}$$

$\Rightarrow$ ***Sumando se obtiene:*** $\frac{r_a+r_b}{a+b-c} + \frac{r_b+r_c}{b+c-a} + \frac{r_c+r_a}{a+c-b} \geq \frac{2r_a}{a} + \frac{2r_b}{b} + \frac{2r_c}{c} \geq$

$$\geq \frac{2r_ah_a}{aw_a} + \frac{2r_bh_b}{bw_b} + \frac{2r_ch_c}{cw_c} h_a \text{ ... (LQQD)}$$

Solution 2 by Soumava Chakraborty – Kolkata-India

$$\frac{r_b + r_c}{2(b+c-a)} = \frac{r_b + r_c}{4(s-a)} = \frac{1}{4\Delta}\left(\frac{\Delta}{s-a}\right)(r_b - r_c) = \frac{1}{4\Delta}r_a(r_b + r_c)$$

Similarly, $\frac{r_c+r_a}{2(c+a-b)} = \frac{1}{4\Delta}r_b(r_c + r_a), \frac{r_a+r_b}{2(a+b-c)} = \frac{1}{4\Delta}r_c(r_a + r_b)$

$$RHS = \frac{1}{4\Delta}\left(2\sum r_ar_b\right) = \frac{s^2}{2rs} = \frac{s}{2r}$$

$$LHS \leq \frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \ (\because h_a \leq w_a, h_b \leq w_b, h_c \leq w_c)$$

$$= \Delta\left\{\frac{1}{a(s-a)} + \frac{1}{b(s-b)} + \frac{1}{c(s-c)}\right\}$$

$$= \frac{rs\{bc(s-b)(s-c) + ca(s-c)(s-a) + ab(s-a)(s-b)\}}{abc(s-a)(s-b)(s-c)}$$

$$= \frac{rs\{s^2(\sum ab) - s\{bc(b+c) + ca(c+a) + ab(a+b)\} + \sum a^2b^2\}}{4Rrs(s-a)(s-b)(s-c)}$$

$$=\frac{rs\{s^2(\sum ab)-s\{bc(2s-a)+ca(2s-b)+ab(2s-c)\}+(\sum ab)^2-2abc(2s)\}}{4Rr(rs)^2}$$

$$=\frac{s^2(\sum ab)-2s^2(\sum ab)+3s(abc)+(\sum ab)^2-4s(abc)}{4Rr^2s}$$

$$=\frac{-s^2(s^2+4Rr+r^2)+(s^2+4Rr+r^2)^2-s(4Rrs)}{4Rr^2s}$$

$$=\frac{(s^2+4Rr+r^2)(4Rr+r^2)-4Rrs^2}{4Rr^2s}$$

$$=\frac{s^2(4Rr+r^2)-4s^2Rr+r^2(4Rr+r)^2}{4Rr^2s}=\frac{s^2r^2+r^2(4R+r)^2}{4Rr^2s}$$

$=\frac{s^2+(4R+r)^2}{4Rs}$ $\therefore$ ***it suffices to prove:*** $\frac{s^2+(4R+r)^2}{4Rs}\le\frac{s}{2r}$

$$\frac{s^2+(4R+r)^2}{4Rs}\le\frac{s}{2r}\Leftrightarrow 2Rs^2\ge r\{s^2+(4R+r)^2\}$$

$$\Leftrightarrow s^2(2R-r)\ge r(4R+r)^2 \quad \textbf{(1)}$$

Gerretsen $\Rightarrow s^2\ge(16R-5r)r$ ***and*** $\because 2R-r>0$

$$\therefore s^2(2R-r)\ge r(16R-5r)(2R-r) \quad \textbf{(2)}$$

(1) and (2) $\Rightarrow$ ***it suffices to prove:*** $(16R-5R)(2R-r)\ge(4R+r)^2$

$$\Leftrightarrow 32R^2-26Rr+5r^2\ge 16R^2+8Rr+r^2$$

$$\Leftrightarrow 16R^2-34Rr+4r^2\ge 0\Leftrightarrow 8R^2-17Rr+2r^2\ge 0$$

$\Leftrightarrow (2R-2r)(8R-r)\ge 0\to$ ***true*** $\because R\overset{Euler}{\ge}2r$ ***(Proved)***

124. Prove that in any triangle the following relationship holds:

$$\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\ge\frac{(a+b+c)^2}{ab+bc+ca}.$$

Proposed by Adil Abdullayev – Baku – Azerbaidjian

Solution by Daniel Sitaru – Romania

$$\sum \frac{m_a}{s_a} = \sum \frac{b^2 + c^2}{2bc} \geq \frac{2(a+b+c)^2}{2(ab+bc+ca)} = \frac{(a+b+c)^2}{ab+bc+ca}$$

125. Let ABC be a triangle with circumradius R and inradius r, and let w_a, w_b, w_c be the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c, respectively.Prove that:

$$\frac{3}{8} \cdot \left(\frac{a^2+b^2+c^2}{w_a^2+w_b^2+w_c^2}\right) \geq \frac{r}{R}.$$

Proposed by George Apostolopoulos – Messalonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{3}{8}\left(\frac{a^2+b^2+c^2}{w_a^2+w_b^2+w_c^2}\right) \geq \frac{r}{R} \to \frac{3}{8}\left(\frac{a^2+b^2+c^2}{w_a^2+w_b^2+w_c^2}\right)\frac{R}{r} \geq 1$

Desde que: $w_a = \frac{2\sqrt{bc}\sqrt{p(p-a)}}{b+c} \leq \sqrt{p(p-a)}$,

$$w_b \leq \sqrt{p(p-b)}, w_c \leq \sqrt{p(p-c)}$$

$$\left(\frac{a^2+b^2+c^2}{w_a^2+w_b^2+w_c^2}\right) \geq \frac{a^2+b^2+c^2}{p(p-a)+p(p-b)+p(p-c)} = \frac{a^2+b^2+c^2}{p^2}$$

$$\Rightarrow \frac{3}{8}\left(\frac{a^2+b^2+c^2}{w_a^2+w_b^2+w_c^2}\right)\frac{R}{r} \geq \frac{3}{8}\left(\frac{a^2+b^2+c^2}{p^2}\right)2 = \frac{3(a^2+b^2+c^2)}{(a+b+c)^2} \geq 1$$

126. In $\Delta\, ABC$ the following relationship holds:

$$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - \left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right) \leq \frac{3}{\sqrt[3]{(b+c-a)(c+a-b)(a+b-c)}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC, la siguiente desigualdad:

$$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)\left[\sqrt[3]{(b+c-a)(a+c-b)(b+a-c)}\right]\leq 3$$

$$\Rightarrow\left[\sqrt[3]{(b+c-a)(a+c-b)(b+a-c)}\right]\leq\frac{(b+c+a)+(a+c-b)+(b+a-c)}{3}=\frac{2p}{3}\ \dots\ \textit{(A)}$$

$$\Rightarrow 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)=\frac{2(ab+bc+ac)-(a^2+b^2+c^2)}{abc}=$$

$$=\frac{2(p^2+r^2+4Rr)-2(p^2-r^2-4Rr)}{4pRr}$$

$$\Rightarrow 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)=\frac{4r(4R+r)}{4pRr}=\frac{4R+r}{pR}\ \dots\ \textit{(B)}$$

Multiplicando (A) × (B):

$$\Rightarrow 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)\left[\sqrt[3]{(b+c-a)(a+c-b)(b+a-c)}\right]\leq$$

$$\leq\frac{2(4R+r)}{3R}\leq\frac{2\left(4R+\frac{R}{2}\right)}{3R}=3\ \dots\ \textit{(LQQD)}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\frac{3}{\sqrt[3]{(a+b-c)(b+c-a)(c+a-b)}}\geq 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)$$

$$\frac{3}{\sqrt[3]{(a+b-c)(b+c-a)(c+a-b)}}\geq\frac{9}{a+b+c}$$

so we need to prove, $\frac{9}{a+b+c}\geq 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)$

$$\Leftrightarrow\sum_{cyc}\frac{a}{bc}-\sum_{cyc}\frac{1}{a}\geq\sum_{cyc}\frac{1}{a}-\frac{9}{a+b+c}$$

$$\Leftrightarrow\frac{a^2+b^2+c^2-ab-bc-ca}{abc}\geq\frac{\sum_{cyc}a(b-c)^2}{abc(a+b+c)}$$

$$\Leftrightarrow \frac{\sum_{cyc}(x+y)^2 - \sum_{cyc}(x+y)(y+z)}{(x+y)(y+z)(z+x)} \geq \frac{\sum_{cyc}(x+y)(y-x)^2}{2(x+y+z)(x+y)(y+z)(z+x)}$$

Applying Ravi Transformation. Let $a = x+y$, $b = y+z$ ***and*** $c = z+x$

$$\Leftrightarrow \frac{x^2+y^2+z^2-xy-yz-zx}{(x+y)(y+z)(z+x)} \geq \frac{\sum_{cyc}(x+y)(y-x)^2}{2(x+y+z)(x+y)(y+z)(z+x)}$$

$$\Leftrightarrow \frac{\sum_{cyc}(x-y)^2}{(x+y)(y+z)(z+x)} \geq \frac{\sum_{cyc}(x+y)(x-y)^2}{(x+y+z)(x+y)(y+z)(z+x)}$$

$$\Leftrightarrow \frac{\sum_{cyc} z(x-y)^2}{(x+y)(y+z)(z+x)} \geq 0, \text{ which is true}$$

$$\therefore \frac{3}{\sqrt[3]{(a+b-c)(b+c-a)(c+a-b)}} \geq 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - \sum_{cyc}\frac{a}{bc}$$

127. **In any triangle ABC the following relationship holds:**

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \leq \frac{R}{2\cdot r}\sqrt{\frac{2R}{r}-1}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Prove that in any triangle ABC: $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \leq \frac{R}{2r}\sqrt{\frac{2R}{r}-1}$

We know that: $\frac{2R}{r} - 1 \geq 3 > 0$. ***Since:***

$$r_a = p\tan\frac{A}{2}, r_b = p\tan\frac{B}{2}, r_c = p\tan\frac{C}{2}, r_a + r_b + r_c = 4R + r$$

Inequality is equivalent:

$$\frac{4R+r}{p} \leq \frac{R}{2r}\sqrt{\frac{2R-r}{r}} \to (4R+r)^2 \leq p^2\left(\frac{R}{2r}\right)^2 \times \frac{2R-r}{r} \to$$

$$\rightarrow (4R+r)^2(4r)^3 \le p^2(2R-r)R^2$$

By Gerretsen's Inequality:

$$p^2(2R-r)R^2 \ge (16Rr-5r^2)(2R^3-R^2r) \ge (4R+r)^2(4r^3)$$

We need to prove that: $(16Rr-5r^2)(2R^3-R^2r) \ge (4R+r)^2(4r^3)$

$$\Rightarrow 32R^4r - 10R^3r^2 - 16R^3r^2 + 5R^2r^3 \ge 64R^2r^3 + 32Rr^4 + 4r^5 \Leftrightarrow$$

(Dividing $\div \left(r^5 > 0\right)$***)***

$$\Rightarrow 32\left(\frac{R}{r}\right)^4 - 26\left(\frac{R}{r}\right)^3 - 59\left(\frac{R}{r}\right)^2 - 32\left(\frac{R}{r}\right) - 4 \ge 0 \Leftrightarrow m = \frac{R}{r} \ge 2$$

$$\Rightarrow 32m^4 - 26m^3 - 59m^2 - 32m - 4 = (m-2)(32m^3 + 38m^2 + 17m + 2) \ge 0$$

128. Let ABC be an acute triangle area Δ. Prove that:

$$\frac{(b+c)^2}{\tan B + \tan C} + \frac{(c+a)^2}{\tan C + \tan A} + \frac{(a+b)^2}{\tan A + \tan B} \le 8\Delta$$

Proposed by Ngyuen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sea ABC un triángulo acutángulo con área Δ. Probar que:

$$\frac{(b+c)^2}{\tan B + \tan C} + \frac{(c+a)^2}{\tan A + \tan C} + \frac{(a+b)^2}{\tan A + \tan B} \le 8\Delta$$

Dado que es triángulo acutángulo: $\tan A, \tan B, \tan C > 0$

Por la desigualdad de Cauchy: $\frac{(b+c)^2}{\tan B + \tan C} \le \frac{b^2}{\tan B} + \frac{c^2}{\tan C}$ ***... (A)***

$\frac{(c+a)^2}{\tan A + \tan C} \le \frac{c^2}{\tan C} + \frac{a^2}{\tan A}$ ***... (B);*** $\frac{(a+b)^2}{\tan A + \tan B} \le \frac{a^2}{\tan A} + \frac{b^2}{\tan B}$ ***... (C)***

Sumando: (A) + (B) + (C)

$$\rightarrow \frac{(b+c)^2}{\tan B + \tan C} + \frac{(c+a)^2}{\tan A + \tan C} + \frac{(a+b)^2}{\tan A + \tan B} \le \frac{2a^2}{\tan A} + \frac{2b^2}{\tan B} + \frac{2c^2}{\tan C} \quad \text{... (D)}$$

$$\Rightarrow \frac{2a^2}{\tan A}+\frac{2b^2}{\tan B}+\frac{2c^2}{\tan C}=4R^2\frac{2\sin^2 A\cos A}{\sin A}+4R^2\frac{2\sin^2 B\cos B}{\sin B}+4R^2\frac{2\sin^2 C\cos C}{\sin C}$$

$$\Rightarrow \frac{2a^2}{\tan A}+\frac{2b^2}{\tan B}+\frac{2c^2}{\tan C}=4R^2(\sin 2A+\sin 2B+\sin 2C)=$$

$=8(2R^2\sin A\sin B\sin C)=8\Delta$. ***Por lo tanto, tenemos en (D)...***

$$\Rightarrow \frac{(b+c)^2}{\tan B+\tan C}+\frac{(c+a)^2}{\tan A+\tan C}+\frac{(a+b)^2}{\tan A+\tan B}\le \frac{2a^2}{\tan A}+\frac{2b^2}{\tan B}+\frac{2c^2}{\tan C}=8\Delta$$

129. Given a triangle ABC.Prove that:

$$\sqrt{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)\ge\sqrt{\sin\frac{A}{2}}+\sqrt{\sin\frac{B}{2}}+\sqrt{\sin\frac{C}{2}}$$

Proposed by Richdad Phuc – Hanoi – Vietnam

Solution by Daniel Sitaru – Romania

$$f(x)=\sqrt{\sin\frac{x}{2}}-\sqrt{2}\sin\frac{x}{2};\ f''(x)=\frac{1}{16}\left(\sin\frac{x}{2}\right)^{-\frac{3}{2}}+\frac{\sqrt{2}}{4}\sin\frac{x}{2}>0,f \textit{ convex}$$

$$0=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}=f\left(\frac{\pi}{3}\right)=f\left(\frac{A+B+C}{3}\right)\le\frac{1}{3}\sum f(A)=$$

$$=\frac{1}{3}\sum\left(\sqrt{\sin\frac{A}{2}}-\sqrt{2}\sin\frac{A}{2}\right)$$

130. In acute $\Delta\ ABC$ the following relationship holds:

$$\sum(A^2+\cos A)+\log|\cos A\cos B\cos C|<3$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

Let

$$f(x)=x^2+\cos x+\ln|\cos x|-1,0\le x<\frac{\pi}{2}$$

$$f'(x)=2x-\sin x-\tan x,0<x<\frac{\pi}{2},$$

$$f''(x)=2-\cos x-\sec^2 x,0<x<\frac{\pi}{2}$$

For $0<x<\frac{\pi}{2};\ \cos x+\sec^2 x>2\sqrt{\sec x}\ge 2$

$\Rightarrow 2-\cos x-\sec^2 x<0$ *for* $0<x<\frac{\pi}{2}\Rightarrow f'(x)<0$ *for* $0<x<\frac{\pi}{2}$

$\Rightarrow f(x)$ *is strictly decreasing on* $\left[0,\frac{\pi}{2}\right)\Rightarrow f(x)<f(0)\quad 0<x<\frac{\pi}{2}$

$\Rightarrow x^2+\cos x+\ln|\cos x|<1$ *for* $0<x<\frac{\pi}{2}$

Taking $x=A,B,C$ *and adding we get desired inequality.*

131. Let a,b and c be the side lengths of a triangle ABC with inradius r. Prove that:

$$\sqrt[4]{\frac{a^4}{\tan^2\frac{A}{2}}+\frac{b^4}{\tan^2\frac{B}{2}}+\frac{c^4}{\tan^2\frac{C}{2}}}\ge 6\cdot r$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC, la siguiente desigualdad:

$$\sqrt[4]{\frac{a^4}{\tan^2\frac{A}{2}}+\frac{b^4}{\tan^2\frac{B}{2}}+\frac{c^4}{\tan^2\frac{C}{2}}}\ge 6r\to\frac{a^4}{\tan^2\frac{A}{2}}+\frac{b^4}{\tan^2\frac{B}{2}}+\frac{c^4}{\tan^2\frac{C}{2}}\ge 1296r^4$$

Recordar las siguientes desigualdades: $R\ge 2r,\quad p\ge 3\sqrt{3}r\to p^2\ge 27r^2$

Tener en cuenta la siguientes identidades:

$$abc = 4RS, r_a = p\tan\frac{A}{2}, r_b = p\tan\frac{B}{2}, r_c = p\tan\frac{C}{2},$$

$$r_ar_br_c = \frac{S^2S}{(p-a)(p-b)(p-c)} = pS$$

Desde que: $(r_a, r_b, r_c), (a, b, c) > 0$***. Por: MA*** $\geq$ ***MG:***

$$\frac{a^4}{\tan^2\frac{A}{2}} + \frac{b^4}{\tan^2\frac{B}{2}} + \frac{c^4}{\tan^2\frac{C}{2}} \geq 3\sqrt[3]{\frac{(abc)^4}{\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^2}} =$$

$$= 3\sqrt[3]{\frac{(4RS)^4p^6}{(r_ar_br_c)^2}} = 3\sqrt[3]{\frac{(4RS)^4p^6}{p^2S^2}} = 3\sqrt[3]{(4R)^4p^4(pr)^2}$$

$$\Rightarrow \frac{a^4}{\tan^2\frac{A}{2}} + \frac{b^4}{\tan^2\frac{B}{2}} + \frac{c^4}{\tan^2\frac{C}{2}} \geq 3\sqrt[3]{(8r)^4(27r^2)^3r^2} = (3\times 2)^4r^4 = 1296r^4$$

132. Let ABC be an arbitrary triangle, I_a, I_b, I_c are excenters, r is inradius, and R is circumradius of ABC. Prove that:

$$12r\sqrt{3} \leq P(I_aI_bI_c) \leq 6R\sqrt{3}$$

where $P(I_aI_bI_c)$ is perimeter of ABC.

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Adil Abdullayev – Baku – Azerbaidjian

Lemma 1.

$$I_AI_B = 4R\cos\frac{C}{2} = 4R\sqrt{\frac{p(p-c)}{ab}}$$

Lemma 2.

$$\sum \cos\frac{A}{2} \le \frac{3\sqrt{3}}{2}$$

$$LHS = P(I_A I_B I_C) = 4R\sum_{cyc}\cos\frac{A}{2} \le 6R\sqrt{3} \leftrightarrow \sum_{cyc}\cos\frac{A}{2} \le \frac{3\sqrt{3}}{2}$$

$$LHS = 4R\sum_{cyc}\sqrt{\frac{p(p-a)}{bc}} \ge 12r\sqrt{3} \leftrightarrow \sum_{cyc}\sqrt{\frac{p(p-a)}{bc}} \ge \frac{3r\sqrt{3}}{R}$$

$$AM-GM \to \sum_{cyc}\sqrt{\frac{p(p-a)}{bc}} \ge 3\sqrt[6]{\frac{p^3(p-a)(p-b)(p-c)}{a^2b^2c^2}} \ge \frac{3r\sqrt{3}}{R} \leftrightarrow$$

$$\leftrightarrow p^2 \ge \frac{16\cdot 27r^6}{R^4}$$

$$\text{GERRETSEN} \to p^2 \ge 16Rr - 5r^2 \ge 27r^2 \ge \frac{16\cdot 27r^6}{R^4} \leftrightarrow$$

$$\leftrightarrow R^4 \ge 16r^4 \leftrightarrow R \ge 2r$$

133. Let m_a, m_b, m_c be the lengths of the medians of a triangle, and let w_a, w_b, w_c be the lengths of the internal bisectors of the angle opposite of the sides of lenghts a, b, c, respectively. Prove that:

$$\frac{\left(\frac{m_a^2}{a}+\frac{m_b^2}{b}+\frac{m_c^2}{c}\right)^2}{w_a^2+w_b^2+w_c^2} \ge \frac{9}{4}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto – Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$\frac{\left(\frac{m_a^2}{a}+\frac{m_b^2}{b}+\frac{m_c^2}{c}\right)^2}{w_a^2+w_b^2+w_c^2} \geq \frac{9}{4}$$. ***Desde que:***

$$w_a = \frac{2bc}{b+c}\sqrt{\frac{p(p-a)}{bc}} = \frac{2\sqrt{bc}}{b+c}\sqrt{p(p-a)} \leq \sqrt{p(p-a)},$$

$$w_b \leq \sqrt{p(p-b)},\ w_c \leq \sqrt{p(p-c)}$$. ***Ahora bien:***

$$\frac{4m_a^2}{4a} + \frac{4m_b^2}{4b} + \frac{4m_c^2}{4c} = \frac{2b^2+2c^2-a^2}{4a} + \frac{2c^2+2a^2-b^2}{4b} + \frac{2a^2+2b^2-c^2}{4c}$$

Por desigualdad de Cauchy:

$$\frac{4m_a^2}{4a} + \frac{4m_b^2}{4b} + \frac{4m_c^2}{4c} + \frac{a+b+c}{4} =$$

$$= \frac{1}{2}\left(\frac{b^2}{a}+\frac{a^2}{b}\right) + \frac{1}{2}\left(\frac{a^2}{c}+\frac{c^2}{a}\right) + \frac{1}{2}\left(\frac{c^2}{b}+\frac{b^2}{c}\right) \geq \frac{a+b}{2} + \frac{b+c}{2} + \frac{c+a}{2} =$$

$$= a+b+c \Rightarrow \frac{4m_a^2}{4a} + \frac{4m_b^2}{4b} + \frac{4m_c^2}{4c} \geq \frac{3(a+b+c)}{4} = \frac{3p}{2}$$

Por la tanto: $$\frac{\left(\frac{m_a^2}{a}+\frac{m_b^2}{b}+\frac{m_c^2}{c}\right)^2}{w_a^2+w_b^2+w_c^2} \geq \frac{\frac{9}{4}p^2}{p(p-a)+p(p-b)+p(p-c)} = \frac{9}{4}$$

134. In acute angled $\Delta\, ABC$ the following relationship holds:

$$\sum \frac{1}{\sin 2B + \sin 2C - \sin 2A} \geq \frac{3}{\sqrt[3]{\prod \sin 2A}} \geq 2\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo acutángulo ABC, probar lo siguiente:

$$\sum \frac{1}{\sin 2A + \sin 2B - \sin 2C} \geq 3\sqrt[3]{\csc 2A \csc 2B \csc 2C} \geq 2\sqrt{3}$$

Dado que es un triángulo acutángulo:

$$\sin A, \sin B, \sin C, \cos A, \cos B, \cos C > 0$$

1) $\cos A \cos B \cos C \le \frac{1}{8}$,

$$\sin A \sin B \sin C \le \frac{3\sqrt{3}}{8} \wedge \sin 2A \sin 2B \sin 2C =$$

$$= 8 \sin A \sin B \sin C \cos A \cos B \cos C \le \frac{3\sqrt{3}}{8}$$

2) Ahora bien en un triángulo ABC, por transformaciones trigonemétricas:

$$\sin 2A + \sin 2B - \sin 2C = 2\sin(A+B)\cos(A-B) - 2\sin C\cos C$$

$$\sin 2A + \sin 2B - \sin 2C = 2\sin C\left(\cos(A-B) - \cos C\right) =$$

$$= 2\sin C\left(\cos(A-B) + \cos(A+B)\right)$$

$$\sin 2A + \sin 2B - \sin 2C = 4\sin C\cos A\cos B > 0$$

Por: MA ≥ MG

$$\sum \frac{1}{\sin 2A + \sin 2B - \sin 2C} \ge 3\sqrt[3]{(\csc 2A \csc 2B \csc 2C)(8 \sec A \sec B \sec C)} \ge$$

$$\ge 3\sqrt[3]{\csc 2A \csc 2B \csc 2C} \ge 2\sqrt{3}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$\sin 2B + \sin 2C - \sin 2A = 2\sin(B+C)\cos(B-C) - 2\sin A\cos A =$$

$$= 2\sin A\cos(B-C) + 2\sin A\cos(B+C)$$

$$= 2\sin A\{\cos(B+C) + \cos(B-C)\} = 4\sin A\cos B\cos C > 0$$

(∵ ΔABC is acute – angled) $\Rightarrow \sin 2B + \sin 2C > \sin 2A$

Similarly, $\sin 2C + \sin 2A > \sin 2B$***, and*** $\sin 2A + \sin 2B > \sin 2C$

$\therefore \sin 2A, \sin 2B, \sin 2C$ ***from*** 3 ***sides of a*** Δ.

Let $x = \sin 2A, y = \sin 2B, z = \sin 2C$

Using Padoa's inequality, $xyz \ge (x+y-z)(y+z-x)(z+x-y)$

$$\Rightarrow \sqrt[3]{(x+y-z)(y+z-x)(z+x-y)} \le \sqrt[3]{xyz}$$

$$\Rightarrow \frac{1}{\sqrt[3]{\prod(x+y-z)}} \ge \frac{1}{\sqrt[3]{xyz}} \quad \text{(1)}$$

Now, $\sum \frac{1}{\sin 2B+\sin 2C-\sin 2A} \overset{A-G}{\ge} 3\sqrt[3]{\frac{1}{\prod(x+y-z)}} \ge \frac{3}{\sqrt[3]{\prod \sin 2A}}$ ***(using (1)) (Proved 1st part). Again,*** $\sqrt[3]{\prod \sin 2A} \overset{A-G}{\le} \frac{\sum \sin 2A}{3}$

(∵ ΔABC is acute, $\therefore 0 < 2A, 2B, 2C < \pi \Rightarrow \sin 2A, \sin 2B, \sin 2C > 0$***)***

$$= \frac{4\sin A \sin B \sin C}{3} \le \frac{4}{3}\cdot\frac{3\sqrt{3}}{8}\left(\because \prod \sin A \le \frac{3\sqrt{3}}{8}\right) = \frac{\sqrt{3}}{2}$$

$\Rightarrow \frac{1}{\sqrt[3]{\prod \sin 2A}} \ge \frac{2}{\sqrt{3}} \Rightarrow \frac{3}{\sqrt[3]{\prod \sin 2A}} \ge \frac{2}{\sqrt{3}}\cdot 3 = 2\sqrt{3}$ ***(Proved 2nd part) (Done)***

Solution 3 by Soumitra Mandal - Chandar Nagore – India

$$\sin 2B + \sin 2C - \sin 2A = 2\sin A\{\cos(B-C)+\cos(B+C)\} =$$

$$= 4\sin A\cos B\cos C$$

$$\sum_{cyc}\frac{1}{\sin 2B+\sin 2C-\sin 2A} = \sum_{cyc}\frac{1}{4\sin A\cos B\cos C}$$

$$\ge \frac{3}{\sqrt[3]{\prod_{cyc}(2\sin A\cos A)\cdot 8\cos A\cos B\cos C}} \ge \frac{3}{\sqrt[3]{\prod_{cyc}\sin 2A}}$$

$[\therefore 1 \ge 8\cos A\cos B\cos C]$***. Let*** $f(x) = \ln\sin 2x$ ***for all*** $x \in \left(0, \frac{\pi}{4}\right)$

$f''(x) = -4\csc^2 x < 0$ ***for all*** $x \in \left(0,\frac{\pi}{4}\right)$ $\therefore$ ***Applying Jensen's Inequality***

$$\frac{\sum_{cyc}\ln\sin 2A}{3} \le \ln\left\{\frac{2}{3}(A+B+C)\right\} = \ln\frac{\sqrt{3}}{2} \Rightarrow \sqrt[3]{\sin 2A\cdot\sin 2B\cdot\sin 2C} \le \frac{\sqrt{3}}{2}$$

$\Rightarrow \frac{3}{\sqrt[3]{\prod_{cyc}\sin 2A}} \ge 2\sqrt{3} \therefore \sum_{cyc}\frac{1}{\sin 2B+\sin 2C-\sin 2A} \ge \frac{3}{\sqrt[3]{\prod_{cyc}\sin 2A}} \ge 2\sqrt{3}$ ***(proved)***

135. **In ΔABC the following relationship holds:**

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\le 3R\sqrt{2s}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Stefan-Romania

By $C-B-S\Rightarrow LHS\le\left(\sum a\right)\left(\sum a^2\right)\le 18R^2\cdot\frac{\sum a}{2}\Leftrightarrow\sum a^2\le 9R^2\Leftrightarrow OH^2\ge 0$ ***true***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\left(\sqrt{(ab)}\right)\left(\sqrt{a}\right)+\left(\sqrt{(bc)}\right)\left(\sqrt{b}\right)+\left(\sqrt{(ca)}\right)\left(\sqrt{c}\right)$$

$$\le\sqrt{\left(\sum ab\right)}\sqrt{\left(\sum a\right)}\ \textbf{(by CBS)}=\sqrt{\left(\sum ab\right)}\sqrt{(2s)}$$

$\therefore$ ***it suffices to prove:*** $\sqrt{\left(\sum ab\right)}\le 3R\Leftrightarrow\sum ab\le 9R^2$

$$\Leftrightarrow s^2+4Rr+r^2\le 9R^2\Rightarrow s^2\le 9R^2-4Rr-r^2$$

Now, $s^2\le 4R^2+4Rr+3r^2$ ***(Gerretsen)***

$\therefore$ ***it suffices to prove*** $4R^2+4Rr+3r^2\le 9R^2-4Rr-r^2$

$\Leftrightarrow 5R^2-8Rr-4r^2\ge 0\Leftrightarrow(R-2r)(5R+2r)\ge 0$, ***which is true by Euler (Hence proved)***

Solution 3 by Rozeta Atanasova-Skopje

By Cauchy – Schwarz inequality $LHS\le\sqrt{(a^2+b^2+c^2)}\sqrt{(a+b+c)}=$

$$=2R\sqrt{(\sin^2A+\sin^2B+\sin^2C)}\sqrt{(2s)}\le 2R\sqrt{\left(\frac{9}{4}\right)}\sqrt{(2s)}=3R\sqrt{(2s)}=RHS$$

because $\sin^2A+\sin^2B+\sin^2C\le\frac{9}{4}$

136. **In acute angled $\Delta\, ABC$:**

$$\frac{a^3\cos^3 A}{\tan^{-1}\frac{1}{2}}+\frac{b^3\cos^3 B}{\tan^{-1}\frac{1}{5}}+\frac{c^3\cos^3 C}{\tan^{-1}\frac{1}{8}}\ge\frac{32r^3s^3}{3\pi R^3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Anas Adlany - El Jadida – Morroco

We have $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)=\frac{\pi}{4}$***. So by Chebyshev's inequality*** $LHS\ge\frac{36}{\pi}\left(\sum a^3\right)\left(\sum\cos(A)^3\right)\ge$

$$\ge\frac{4}{3\pi}\left(\left(\sum a\right)\left(\sum\frac{\cos(A)}{3}\right)\right)^3=\frac{32s^3}{3\pi}\left(\sum\frac{\cos(A)}{3}\right)^3=RHS$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC***:***

$\frac{a^3\cos^3 A}{\tan^{-1}\frac{1}{2}}+\frac{b^3\cos^3 B}{\tan^{-1}\frac{1}{5}}+\frac{c^3\cos^3 C}{\tan^{-1}\frac{1}{8}}\ge\frac{32r^3s^3}{3\pi R^3}$***. Por la desigualidad de Holder:***

$$\left(\frac{a^3\cos^3 A}{\tan^{-1}\frac{1}{2}}+\frac{b^3\cos^3 B}{\tan^{-1}\frac{1}{5}}+\frac{c^3\cos^3 C}{\tan^{-1}\frac{1}{8}}\right)\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right)(1+1+1)\ge$$

$$\ge(a\cos A+b\cos B+c\cos C)^3$$

A continuación, demostraremos lo siguiente:

$$\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}=\frac{\pi}{4}$$

$$\arctan\frac{1}{2}+\arctan\frac{1}{5}=\arctan\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right)=\arctan\frac{7}{9},$$

$$\arctan\frac{7}{9}+\arctan\frac{1}{8}=\arctan\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{8}\times\frac{1}{8}}\right)=\arctan 1=\frac{\pi}{4}$$

Luego:

$$\frac{a^3\cos^3 A}{\tan^{-1}\frac{1}{2}}+\frac{b^3\cos^3 B}{\tan^{-1}\frac{1}{5}}+\frac{c^3\cos^3 C}{\tan^{-1}\frac{1}{8}}\geq\frac{4\big(R(\sin 2A+\sin 2B+\sin 2C)\big)^3}{3\pi}=$$

$$=\frac{4(4R\sin A\sin B\sin C)^3}{3\pi}=\frac{3\left(4R\frac{rs}{2R^2}\right)^3}{3\pi}=\frac{32r^3s^3}{3\pi R^3}$$

Solution 3 by Myagmarsuren Yadamsuren-Ulaanbataar-Mongolia

$$\sum\frac{a^3\cdot\cos^3 A}{\left(\sqrt{\tan^{-1}\frac{1}{2}}\right)^2}\overset{Radon's}{\geq}\frac{(a\cdot\cos A+b\cdot\cos B+c\cdot\cos C)^3}{\left(\sqrt{\tan^{-1}\frac{1}{2}}+\sqrt{\tan^{-1}\frac{1}{5}}+\sqrt{\tan^{-1}\frac{1}{8}}\right)^2}\overset{(*)}{=}$$

$$=a\cdot\cos A+b\cdot\cos B+c\cdot\cos C=R\cdot(\sin 2A+\sin 2B+\sin 2C)=$$

$$=4\cdot R\cdot(\sin A\cdot\sin B\cdot\sin C)=4\cdot R\cdot\frac{abc}{8R^3}=\frac{2\Delta}{R}.$$

2. $$\left(\sqrt{\tan^{-1}\tfrac{1}{2}}+\sqrt{\tan^{-1}\tfrac{1}{5}}+\sqrt{\tan^{-1}\tfrac{1}{8}}\right)^2\leq 3\cdot\left(\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{8}\right)=$$

$$=\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}\right)=\frac{3\pi}{4}.$$

(*) $$\frac{(\sum a\cdot\cos A)^3}{\left(\sqrt{\tan^{-1}\frac{1}{2}}+\sqrt{\tan^{-1}\frac{1}{5}}+\sqrt{\tan^{-1}\frac{1}{8}}\right)^2}\geq\frac{8\Delta^2}{R^3}\cdot\frac{1}{\frac{3\pi}{4}}=\frac{32\cdot r^3\cdot s^3}{3\pi\cdot R^3}$$

137. In $\Delta\ ABC$ the following relationship holds:

$$\frac{a^4+b^4}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{b^4+c^4}{\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}+\frac{c^4+a^4}{\tan^2\frac{C}{2}+\tan^2\frac{A}{2}}\geq 48S^2$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$\frac{a^4+b^4}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{b^4+c^4}{\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}+\frac{c^4+a^4}{\tan^2\frac{C}{2}+\tan^2\frac{A}{2}}\geq 48S^2$$

Por la desigualdad de Cauchy:

$$\sum\frac{a^4+b^4}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\geq\sum\frac{(a^2+b^2)^2}{2\tan^2\frac{A}{2}+2\tan^2\frac{B}{2}}\geq\frac{(a^2+b^2+c^2)^2}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}\geq 48S^2$$

Por lo cual solo basta probar lo siguiente: $\frac{(a^2+b^2+c^2)^2}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}=$

$$=\frac{(a^2+b^2+c^2)^2S^2}{(p-b)^2(p-c)^2+(p-c)^2+(p-c)^2(p-a)^2+(p-a)^2(p-b)^2}\geq 48S^2$$

$$\Rightarrow(a^2+b^2+c^2)^2\geq 3(a+c-b)^2(b+a-c)^2+3(a+b-c)^2(b+c-a)^2+3(b+c-a)^2(a+c-b)^2$$

$$\Rightarrow\sum a^4+2\sum a^2b^2\geq 3\sum\left(a^2-(b-c)^2\right)^2=3\sum a^4-3\sum b^4-3\sum c^4+6\sum b^2c^2-6\sum a^2(b-c)^2$$

$$\Rightarrow\sum a^4+2\sum a^2b^2\geq -3\sum a^4-6\sum a^2b^2+12abc(a+b+c)$$

$$\Rightarrow 4\sum a^4+8\sum a^2b^2\geq 12abc(a+b+c)$$

Lo cual es cierto ya que, por: MA $\geq$ MG

$$4(a^4+b^4+c^4)\geq 4(a^2b^2+b^2c^2+c^2a^2)\geq 4abc(a+b+c)$$

$$8(a^2b^2+b^2c^2+c^2a^2)\geq 8abc(a+b+c)$$

Sumando obtenemos: $\Rightarrow 4\sum a^4+8\sum a^2b^2\geq 12abc(a+b+c)$ ***… (LQQD)***

Solution 2 by Ravi Prakash-New Delhi-India

$$\tan\left(\frac{A}{2}\right)=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{(s-b)(s-c)}{\Delta}$$

$$\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)=\frac{(s-c)^2}{\Delta^2}\left[(s-b)^2+(s-a)^2\right]$$

$$\therefore\frac{a^4+b^4}{\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)}=\frac{\Delta^2}{(s-c)^2}\left[\frac{a^4+b^4}{(s-a)^2+(s-b)^2}\right]$$

But $a = 2s - (b + c) = (s - b) + (s - c) \geq 2\sqrt{(s - b)(s - c)}$

$\Rightarrow a^4 \geq 16(s - b)^2(s - c)^2 \therefore a^4 + b^4 \geq 16(s - c)^2[(s - b)^2 + (s - a)^2]$

Thus, $\frac{a^4+b^4}{\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)} \geq 16\Delta^2 \Rightarrow \sum \frac{a^4+b^4}{\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)} \geq \sum 16\Delta^2 = 48\Delta^2$

Solution 3 by Soumava Pal-Kolkata-India

$s = \frac{a+b+c}{2}$. **Take** $x = s - a, y = s - b, z = s - c$, **so that** $x, y, z > 0$.

$$\tan\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}$$

$$LHS = \sum_{cyc} \frac{a^4 + b^4}{\tan^2\frac{A}{2} + \tan^2\frac{B}{2}} = \sum_{cyc} \frac{a^4 + b^4}{\frac{(s - b)(s - c)}{s(s - a)} + \frac{(s - c)(s - a)}{s(s - b)}}$$

$$= s(s - a)(s - b)(s - c)\sum_{cyc} \frac{a^4 + b^4}{z^2(x^2 + y^2)} = \Delta^2 \sum_{cyc} \frac{(z + x)^4 + (z + y)^4}{z^2(x^2 + y^2)}$$

$$\Rightarrow \begin{array}{c} z + y \geq 2\sqrt{2}y \\ (z + y)^4 \geq 16z^2y^2 \end{array} \textbf{ and} \Rightarrow \begin{array}{c} z + x \geq 2\sqrt{2}x \\ (z + x)^4 \geq 16z^2x^2 \end{array}$$

$$\Rightarrow (z + y)^4 + (z + x)^4 \geq 16z^2(x^2 + y^2) \Rightarrow \frac{(z + y)^4 + (z + x)^4}{z^2(x^2 + y^2)} \geq 16$$

$$\Rightarrow \sum_{cyc} \frac{(z + y)^4 + (z + x)^4}{z^2(x^2 + y^2)} \geq 16 \times 3 = 48$$

$\Rightarrow \Delta^2 \sum_{cyc} \frac{(z+y)^4+(z+x)^4}{z^2(x^2+y^2)} \geq 48\Delta^2 \Rightarrow LHS \geq 48\Delta^2$ ***(Proved)***

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} = \frac{(s - b)(s - c)}{s(s - a)} + \frac{(s - c)(s - a)}{s(s - b)}$$

$$=\frac{(s-c)}{s}\left(\frac{s-b}{s-a}+\frac{s-a}{s-b}\right)=\left(\frac{s-c}{s}\right)\frac{\{(s-b)^2+(s-a)^2\}}{(s-a)(s-b)}$$

$$=\frac{(s-c)^2(s-b)^2+(s-c)^2(s-a)^2}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{\left(\sqrt{(s-b)(s-c)}\right)^4+\left(\sqrt{(s-c)(s-a)}\right)^4}{s^2}\le$$

$$\overset{GM\le AM}{\le}\frac{\left(\frac{s-b+s-c}{2}\right)^4+\left(\frac{s-c+s-a}{2}\right)^4}{s^2}=\frac{a^4+b^4}{16S^2}$$

$\therefore \frac{a^4+b^4}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\ge 16S^2$. ***Similarly,*** $\frac{b^4+c^4}{\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}\ge 16S^2$ ***and*** $\frac{c^4+a^4}{\tan^2\frac{C}{2}+\tan^2\frac{A}{2}}\ge 16S^2$

$\Rightarrow LHS$ ***of (1)*** $\ge 16S^2\times 3=48S^2=RHS$ ***(Proved)***

Solution 5 by Soumitra Mandal - Chandar Nagore – India

$$\tan\frac{A}{2}=\frac{(p-b)(p-c)}{S},\ \tan\frac{B}{2}=\frac{(p-a)(p-c)}{S}\ \text{and}\ \tan\frac{C}{2}=\frac{(p-a)(p-b)}{S}$$

where p = ***semi – perimeter and*** S = ***area of triangle*** ABC

$$\sum_{cyc}\frac{a^4+b^4}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}=S^2\sum_{cyc}\frac{a^4+b^4}{\{(p-b)(p-c)\}^2+\{(p-a)(p-c)\}^2}\ge$$

$$\ge S^2\sum_{cyc}\frac{a^4+b^4}{\left(\frac{p-b+p-c}{2}\right)^4+\left(\frac{p-a+p-c}{2}\right)^4}=48S^2\ \textit{(proved)}$$

138. In ΔABC the following relationship holds:

$$\sum\frac{\sin^8 A+\sin^4 A+\cos^8 A+\cos^4 A}{\sin^6 A+\sin^4 A+\cos^4 A+\cos^6 A}\ge\frac{12r}{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sin^8 A + \cos^8 A + \sin^4 A + \cos^4 A =$$

$$= (\sin^4 A + \cos^4 A)^2 - 2\sin^4 A\cos^4 A + 1 - 2\sin^2 A\cos^2 A$$

$$= (1 - 2\sin^2 A\cos^2 A)^2 - 2\sin^4 A\cos^4 A - 2\sin^2 A\cos^2 A + 1$$

$$= 2 + 2\sin^4 A\cos^4 A - 6\sin^2 A\cos^2 A$$

$$= 2\left(t_i^2 - 3t_i + 1\right) \quad (t_i = \sin^2 A\cos^2 A)$$

Again, $\sin^6 A + \sin^4 A\cos^4 A + \cos^6 A$

$$= (\sin^2 A + \cos^2 A)(\sin^4 A + \cos^4 A - \sin^2 A\cos^2 A) + \sin^4 A\cos^4 A$$

$$= 1 - 2\sin^2 A\cos^2 A - \sin^2 A\cos^2 A + \sin^4 A\cos^4 A = t_i^2 - 3t_i + 1$$

$$\therefore LHS = \sum_{i=1}^{3}\frac{2\left(t_i^2 - 3t_i + 1\right)}{\left(t_i^2 - 3t_i + 1\right)} = \sum(2) = 6$$

$\therefore$ ***it suffices to prove:*** $6 \geq \frac{12R}{R} \Leftrightarrow R \geq 2r$ ***true (Euler) (Proved)***

Solution 2 by Saptak Bhattacharya-Kolkata-India

$$\sum\frac{\sin^8 A + \sin^4 A + \cos^8 A + \cos^4 A}{\sin^6 A + \sin^4 A\cos^4 A + \cos^6 A} =$$

$$= \sum\frac{(1 - 2\sin^2 A\cos^2 A)^2 + 1 - 2\sin^2 A\cos^2 A - 2\sin^4 A\cos^4 A}{1 - 3\sin^2 A\cos^2 A + \sin^4 A\cos^4 A} =$$

$$= \sum\frac{2\left(1 - 3\sin^2 A\cos^2 A + \sin^4 A\cos^4 A\right)}{\left(1 - 3\sin^2 A\cos^2 A + \sin^4 A\cos^4 A\right)} = \sum 2 = 6 \geq \frac{12r}{R}$$ ***since,*** $R \geq 2r$ ***(Euler)***

139. **In $\Delta\, ABC, a \neq b$ the following relationship holds:**

$$\frac{\left(2b + 2c - 3\sqrt[3]{abc}\right)\left(1 + \left(\sqrt{a} - \sqrt{b}\right)^2\right)}{\left(\sqrt{a} - \sqrt{b}\right)^2\left(1 + a + b + c - 3\sqrt[3]{abc}\right)} > 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Ștefan – Romania

$$\textbf{First, let's prove: } \left(2b+2c-3\sqrt[3]{abc}\right)\left(1+\left(\sqrt{a}-\sqrt{b}\right)^2\right)\ge$$

$$\ge\left(1+2b+2c-3\sqrt[3]{abc}\right)\left(\sqrt{a}-\sqrt{b}\right)^2$$

$$\Leftrightarrow 2b+2c-3\sqrt[3]{abc}\ge\left(\sqrt{a}-\sqrt{b}\right)^2$$

$$2b+2c-b+2\sqrt{ab}\ge a+3\sqrt[3]{abc}\Leftrightarrow b+c+c+2\sqrt{ab}\ge a+3\sqrt[3]{abc}$$

$$\textbf{but } b+c>a\textbf{. We'll prove: } c+2\sqrt{ab}\ge 3\sqrt[3]{abc}$$

$$c+\sqrt{ab}+\sqrt{ab}\ge 3\sqrt[3]{abc}\ \textbf{(true by AM-GM)}$$

$$LHS\ge\frac{\left(1+2b+2c-3\sqrt[3]{abc}\right)\left(\sqrt{a}-\sqrt{b}\right)^2}{\left(1+a+b+c-3\sqrt[3]{abc}\right)\left(\sqrt{a}-\sqrt{b}\right)^2}\Leftrightarrow$$

$$\Leftrightarrow 1+2b+2c>1+a+b+c\Leftrightarrow b+c>a\ \textbf{true}$$

Solution 2 by Myagmarsuren Yadamsuren – Mongolia

$$\sqrt{ab}+\sqrt{ab}+c\ge 3\sqrt[3]{abc}\Leftrightarrow 2\sqrt{ab}+c\ge 3\sqrt[3]{abc}\ |\cdot(-1)$$

$$-3\sqrt[3]{abc}\ge -2\sqrt{ab}-c$$

$$(a+b+c)-3\sqrt[3]{abc}\ge a+b+c-2\sqrt{ab}-c$$

$$a+b+c-3\sqrt[3]{abc}\ge a+b-2\sqrt{ab};\ a+b+c-3\sqrt[3]{abc}\ge\left(\sqrt{a}-\sqrt{b}\right)^2$$

$$a+b+c-3\sqrt[3]{abc}+\left(a+b+c-3\sqrt[3]{abc}\right)\cdot\left(\sqrt{a}-\sqrt{b}\right)^2\ge$$

$$\ge\left(\sqrt{a}-\sqrt{b}\right)^2+\left(a+b+c-3\sqrt[3]{abc}\right)\cdot\left(\sqrt{a}-\sqrt{b}\right)^2$$

$$\left(a+b+c-3\sqrt[3]{abc}\right)\cdot\left(1+\left(\sqrt{a}-\sqrt{b}\right)^2\right)\ge$$

$$\ge\left(\sqrt{a}-\sqrt{b}\right)^2\cdot\left(1+a+b+c-3\sqrt[3]{abc}\right)$$

$$LHS=\left(a+b+c-3\sqrt[3]{abc}\right)\cdot\left(1+\left(\sqrt{a}-\sqrt{b}\right)^2\right)\underset{a<b+c}{<}$$

$$< \left(2(b+c)-3\sqrt[3]{abc}\right)\cdot\left(1(\sqrt{a}-\sqrt{b})^2\right) = LHS$$

$$LHS: \left(2(b+c)-3\sqrt[3]{abc}\right)\cdot\left(1+(\sqrt{a}-\sqrt{b})^2\right) >$$

$$> (\sqrt{a}-\sqrt{b})^2\cdot\left(1+a+b+c-3\sqrt[3]{abc}\right): RHS \Rightarrow \frac{LHS}{RHS} > 1$$

Solution 3 by Soumava Chakraborty – Kolkata – India

$$3\sqrt[3]{abc} \overset{G-A}{<} a+b+c \Rightarrow -3\sqrt[3]{abc} > -a-b-c$$

$$\Rightarrow 2b+2c-3\sqrt[3]{abc} > 2b+2c-a-b-c = b+c-a > 0$$

Also, $1+a+b+c-3\sqrt[3]{abc} \overset{A-G}{>} 1+3\sqrt[3]{abc}-3\sqrt[3]{abc} = 1 > 0$

$\therefore$ ***numerator*** > 0 ***and denominator*** $> 0 \therefore$ ***given inequality*** $\Leftrightarrow$

$$\left(2b+2c-3\sqrt[3]{abc}\right)\left(1+(\sqrt{a}-\sqrt{b})^2\right) > (\sqrt{a}-\sqrt{b})^2\left(1+a+b+c-3\sqrt[3]{abc}\right)$$

$$\Leftrightarrow 2b+2b(\sqrt{a}-\sqrt{b})^2+2c+2c(\sqrt{a}-\sqrt{b})^2-3\sqrt[3]{abc}-3\sqrt[3]{abc}(\sqrt{a}-\sqrt{b})^2$$

$$> (\sqrt{a}-\sqrt{b})^2+a(\sqrt{a}-\sqrt{b})^2+b(\sqrt{a}-\sqrt{b})^2+c(\sqrt{a}-\sqrt{b})^2-3\sqrt[3]{abc}(\sqrt{a}-\sqrt{b})^2$$

$$\Leftrightarrow \underbrace{(b+c-a)(\sqrt{a}-\sqrt{b})^2}_{>0(\because b+c>a \text{ and } a\neq b)} + 2b+2c > a+b-2\sqrt{ab}+3\sqrt[3]{abc}$$

$\therefore$ ***it suffices to prove:*** $(b+c-a)+c+2\sqrt{ab} > 3\sqrt[3]{abc}$ ***(1)***

Now $c+2\sqrt{ab} = c+\sqrt{ab}+\sqrt{ab} \overset{A-G}{\geq} 3\sqrt[3]{abc}$ ***and of course,*** $b+c-a > 0$

$\Rightarrow (b+c-a)+c+2\sqrt{ab} > 0+3\sqrt[3]{abc} = 3\sqrt[3]{abc} \Rightarrow$ ***(1) is true (proved)***

140. In acute-angled ΔABC, $H-$ orthocentre, $G-$ centroid:

$$\sum\left(\frac{AH}{AG}\right)^2 \geq \frac{108r^2}{a^2+b^2+c^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: $H -$ ***ortocentro,*** $G -$ ***centro de gravedad. Probar en un triángulo acutángulo*** ABC: $\sum\left(\frac{AH}{AG}\right)^2 \geq \frac{108r^2}{a^2+b^2+c^2}$. ***Recordar las siguientes identidades:*** $AH = 2R\cos A$, $BH = 2R\cos B$, $CH = 2R\cos C$

$$GA = \frac{2}{3}m_a, GB = \frac{2}{3}m_b, GC = \frac{2}{3}m_c$$

$$4m_a^2 + 4m_b^2 + 4m_c^2 = 3(a^2 + b^2 + c^2), \cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

Dado que es un triángulo acutángulo: $\cos A, \cos B, \cos C > 0$

La desigualdad es equivalente: $4R^2\sum\frac{9\cos^2 A}{4m_a^2} \geq \frac{108r^2}{a^2+b^2+c^2}$

Por la desigualdad de Cauchy:

$$4R^2\sum\frac{9\cos^2 A}{4m_a^2} \geq 4R^2\frac{(3\cos A+3\cos B+3\cos C)^2}{3(a^2+b^2+c^2)} = 4\times\frac{3(R+r)^2}{a^2+b^2+c^2} \geq \frac{108r^2}{a^2+b^2+c^2} \ldots$$

(LQQD)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{(AH)^2}{(AG)^2} \geq \frac{(\sum AH)^2}{\sum AG^2} \text{ (Bergstrom's Inequality)}$$

$$= \frac{(\sum AH)^2}{\sum\left(\frac{2}{3}m_a\right)^2} = \frac{(\sum AH)^2}{\frac{4}{9}\sum m_a^2} = \frac{(\sum AH)^2}{\frac{4}{9}\cdot\frac{3}{4}\sum a^2} = \frac{3(\sum AH)^2}{\sum a^2}$$

$\therefore$ ***given inequality will be proved if it can be proved***

$$\frac{3(\sum AH)^2}{\sum a^2} \geq \frac{108r^2}{\sum a^2} \Leftrightarrow \sum AH \geq 6r$$

$$\Leftrightarrow 2R\left(\sum\cos A\right) \geq 6R \;\; (\because \Delta ABC \text{ is acute – angled})$$

$$\Leftrightarrow R\left(1+\frac{r}{R}\right) \geq 3r \Leftrightarrow R + r \geq 3r \Leftrightarrow R \geq 2r, \text{ which is true (Euler)}$$

141. In $\Delta\ ABC$ the following relationship holds:

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\le\frac{1}{6S}(\sin A+\sin B+\sin C)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

$$\sum\frac{1}{a^2+b^2+ab}\le\sum\frac{1}{3ab}=\frac{a+b+c}{3abc}=\frac{2p}{12RS}$$

$$=\frac{1}{6S}(\sin A+\sin B+\sin C)$$

Solution 2 by Mihalcea Andrei Stefan-Romania

$$a^2+ab+b^2\overset{AM-GM}{\ge}3ab\Rightarrow LHS\le\frac{1}{3}\sum\frac{1}{ab}$$

We'll prove: $\sum\frac{1}{ab}=\frac{1}{2S}\sum\sin A$

But $\sum\sin A=\frac{\sum a}{2R}\Leftrightarrow\frac{a+b+c}{abc}=\frac{\sum a}{4RS}\Leftrightarrow abc=4RS$ ***true***

Solution 3 by Pham Quy-Vietnam

We have: $2S=ab\sin C=bc\sin A=ca\sin B$

$$\Rightarrow\frac{\sin A}{6S}=\frac{1}{3bc};\frac{\sin B}{6S}=\frac{1}{3ca};\frac{\sin C}{6S}=\frac{1}{3ab}\Rightarrow RHS=\sum\frac{1}{3ab}$$

$$LHS=\sum\frac{1}{a^2+ab+b^2}\overset{AM-GM}{\le}\sum\frac{1}{3ab}=RHS\ \textbf{(q.e.d.)}$$

Solution 4 by Seyran Ibrahimov-Maasilli-Azerbaidjian

$$\frac{a+b+c}{12SR}\ge\sum\frac{1}{a^2+ab+b^2}$$

$$\sum\frac{1}{a^2+ab+b^2}\le\sum\frac{1}{3ab}=\frac{a+b+c}{3abc}=\frac{a+b+c}{12SR}$$

Solution 5 by Soumitra Mandal - Chandar Nagore – India

$$\sum_{cyc}\frac{1}{a^2+ab+b^2}\le\frac{1}{3}\sum_{cyc}\frac{1}{ab}=\frac{1}{3}\left(\sum_{cyc}\frac{\sin C}{2S}\right)$$

since $S=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B$

$$\therefore\sum_{cyc}\frac{1}{a^2+ab+b^2}\le\frac{1}{6S}(\sin A+\sin B+\sin C)$$

142. In acute angled $\Delta\, ABC$ the following relationship holds:

$$\sum a^4\cos^3 A\sin B\sin C\ge\frac{8r^4s^4}{9R^4}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo: $\sum a^4\cos^3 A\sin B\sin C\ge\frac{8r^4s^4}{9R^4}$

Por la desigualdad de Holder:

$$\left(\sum a^4\cos^3 A\sin B\sin C\right)\left(\sum\frac{\cos A}{\sin B\sin C}\right)(1+1+1)(1+1+1)\ge$$

$$\ge(a\cos A+b\cos B+c\cos C)^4$$

Ahora bien, tener en cuenta lo siguiente:

$$\sum\frac{\cos A}{\sin B\sin C}=\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin C\sin A}+\frac{\cos C}{\sin A\sin B}=$$

$$=(1-\cot B\cot C)+(1-\cot C\cot A)+(1-\cot A\cot B)$$

$$\sum\frac{\cos A}{\sin B\sin C}=3-1=2\ \wedge a\cos A+b\cos B+c\cos C=2Rrs$$

Por lo tanto, la desigualdad equivalente es:

$$\sum a^4\cos^3 A\sin B\sin C \geq \frac{(a\cos A + b\cos B + c\cos C)^4}{18} = \frac{(2Rrs)^4}{18} = \frac{8r^4s^4}{9R^4}$$

Solution 2 by Soumitra Mandal - Chandar Nagore – India

$$\sum_{cyc} a^4\cos^3 A\sin B\sin C = (\sin A\sin B\sin C)\sum_{cyc}\frac{a^4\cos^3 A}{\sin A} =$$

$$= \frac{1}{4}\left(\sum_{cyc}\sin 2A\right)\left(\sum_{cyc}\frac{a^4\cos^3 A}{\sin A}\right)$$

[since, $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$***]***

$$= \frac{1}{2}\left(\sum_{cyc}\sin A\cos A\right)\left(\sum_{cyc}\frac{a^4\cos^3 A}{\sin A}\right)$$

$$= \frac{1}{18}\left(\sum_{cyc}\sin A\cos A\right)\left(\sum_{cyc}\frac{a^4\cos^3 A}{\sin A}\right)(1+1+1)(1+1+1)$$

$$\geq \frac{1}{18}(a\sin A + b\sin B + c\sin C)^4$$ ***[Holder's Inequality]***

$$= \frac{1}{18}\left(\frac{abc}{2R^2}\right)^4 = \frac{16S^4}{18R^4} = \frac{8r^4s^4}{9R^4}$$ ***[since*** $abc = 4SR$ ***and*** $S = rs$***]***

$$\therefore \sum_{cyc} a^4\cos^3 A\sin B\sin C \geq \frac{8s^4r^4}{9R^4}$$

Solution 3 by Soumava Chakraborty – Kolkata – India

$$LHS = \sum a^4\cos^3 A\sin B\sin C = 16R^4\sum \sin^4 A\cos^3 A\sin B\sin C$$

$$(\because c = 2R\sin A) = 16R^4\sin A\sin B\sin C\left(\sum \sin^3 A\cos^3 A\right)$$

$$= 2R^4\sin A\sin B\sin C\left(\sum 8\sin^3 A\cos^3 A\right)$$

$$= 2R^4(\pi \sin A)\left(\sum (2\sin A\cos A)^3\right) = 2R^4(\pi \sin A)\left(\sum \sin^3 2A\right)$$

$$RHS = \frac{8}{9}\left(\frac{rs}{R}\right)^4 = \frac{8}{9}\left(\frac{\Delta}{R}\right)^4 = \frac{8}{9}\left(2R(\pi \sin A)\right)^4 = \frac{128}{9}R^4(\pi \sin A)^4$$

$\therefore$ ***given inequality*** $\Leftrightarrow \sum \sin^3 2A \geq \frac{64}{9}(\pi \sin A)^3$ ***(1)***

$\because$ ***ΔABC is acute-angled,*** $\therefore 0 < 2A, 2B, 2C < \pi$

$\Rightarrow 0 < \sin 2A, \sin 2B, \sin 2C < 1 \therefore$ ***if*** $\sin 2A \geq \sin 2B \geq \sin 2C$***, then,***

$\sin^2 2A \geq \sin^2 2B \geq \sin^2 2C$***. Now,*** $\sin^3 2A + \sin^3 2B + \sin^3 2C$

$\geq \frac{1}{3}(\sum \sin 2A)(\sum \sin^2 2A)$ ***(Chebyshev Inequality, because WLOG, we may assume*** $\sin 2A \geq \sin 2B \geq \sin 2C$ ***which*** $\Rightarrow \sin^2 2A \geq \sin^2 2B \geq$

$\geq \sin^2 2C$***)*** $\geq \frac{1}{3}(\sum \sin 2A)\left(\frac{1}{3}(\sum \sin 2A)^2\right)$

$$\left(\because 3\left(\sum x^2\right) \geq \left(\sum x\right)^2\right)$$

$= \frac{1}{9}(\sum \sin 2A)^3 = \frac{1}{9}(4\pi \sin A)^3 = \frac{64}{9}(\pi \sin A)^3 \Rightarrow$ ***(1) is true (proved)***

143. In acute-angled $\Delta\ ABC$ the following relationship holds:

$$2\sum \frac{a^3 \cos A}{b\cos B + c\cos C} \geq \sum (2bc - a^2)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal - Chandar Nagore – India

$$2\sum_{cyc} \frac{a^3 \cos A}{b\cos B + c\cos C} = 2\sum_{cyc} \frac{a^4(b^2 + c^2 - a^2)}{b^2(a^2 + c^2 - b^2) + c^2(a^2 + b^2 - c^2)}$$

$$\geq 2\sum_{cyc} \frac{a^4(b^2 + c^2 - a^2)}{a^2(b^2 + c^2) - (b^2 - c^2)^2} \geq 2\sum_{cyc} \frac{a^2(b^2 + c^2 - a^2)}{b^2 + c^2}$$

$\therefore$ ***we need to prove,*** $2\sum_{cyc}\left(a^2-\frac{a^4}{b^2+c^2}\right)\geq\sum_{cyc}(2bc-a^2)$

$$\Leftrightarrow \sum_{cyc} a^2+\sum_{cyc}(a-b)^2-2\sum_{cyc}\frac{a^4}{b^2+c^2}\geq 0\ \dots\ \textbf{(1)}$$

now,

$$\sum_{cyc} a^2-2\sum_{cyc}\frac{a^4}{b^2+c^2}\geq\sum_{cyc} ab-2\sum_{cyc}\frac{a^4}{b^2+c^2}\geq\sum_{cyc}\frac{2b^2c^2-a^4}{b^2+c^2}$$

$$=\sum_{cyc}\frac{(2b^2c^2-a^4)^2}{(2b^2c^2-a^4)(b^2+c^2)}\geq\frac{\left(\sum_{cyc}2b^2c^2-\sum_{cyc}a^4\right)^2}{\sum_{cyc}(2b^2c^2-a^4)\,(b^2+c^2)}>0$$

$\therefore$ ***statement (1) is true***

$$\therefore \sum_{cyc}\frac{a^3\cos A}{b\cos B+c\cos C}\geq\sum_{cyc}(2bc-a^2)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum(2bc-a^2)=\sum(2bc+a^2-2a^2)=\left(\sum a\right)^2-2\sum a^2=4s^2-2\sum a^2$$

$\therefore$ ***given inequality*** $\Leftrightarrow 2\sum\left\{\frac{a^3}{(b\cos B+c\cos C)}+a^2\right\}\geq 4s^2$

$$\Leftrightarrow \sum(a\cos A)\left\{\sum\frac{a^2}{(a\cos(B-C))}\right\}\geq 2s^2(\because b\cos B+c\cos C=a\cos(B-C))$$

Now $\sum a\cos A=\frac{abc}{(2R^2)}$ ***and*** $\sum\frac{a^2}{\{a\cos(B-C)\}}\geq\frac{4s^2}{\{\sum a\cos(B-C)\}}$ ***(Bergstrom)***

$\therefore$ ***it suffices to prove*** $\left(\frac{abc}{(2R^2)}\right)\left\{\frac{4s^2}{(\sum a\cos(B-C))}\right\}\geq 2s^2$

$$\Leftrightarrow abc\geq R^2\left(\sum a\cos(B-C)\right)\quad \textbf{(1)}$$

Now, $\sum a\cos(B-C)=\sum 2R(\sin A)(\cos(B-C))$

$$=\sum R(\sin 2B+\sin 2C)=2R\left(\sum\sin 2B\right)=8R\sin A\sin B\sin C$$

$\therefore$ ***(1)*** $\Leftrightarrow 4RS\geq 4R(2R^2\sin A\sin B\sin C)\Leftrightarrow 4RS\geq 4RS$ ***(true) (Proved)***

144. In $\Delta\, ABC, a \neq b \neq c \neq a$:

$$\sum \frac{a+b}{(c-b)(c-b)\sin C} > \frac{2}{R}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{a+b}{(c-a)(c-b)\sin C} > \frac{2}{R} \Leftrightarrow \frac{2R(a+b)}{c(c-a)(c-b)} > \frac{2}{R}$$

$$\Leftrightarrow \sum \frac{a+b}{c(c-a)(c-b)} > \frac{1}{R^2} \Leftrightarrow \sum \frac{2s-c}{c(c-a)(c-b)} > \frac{1}{R^2}$$

$$\Leftrightarrow (2s)\sum \frac{1}{c(c-a)(c-b)} - \sum \frac{1}{(c-a)(c-b)} > \frac{1}{R^2} \quad \textbf{(1)}$$

$$\sum \frac{1}{(c-a)(c-b)} = \frac{1}{(c-a)(c-b)} + \frac{1}{(a-b)(a-c)} + \frac{1}{(b-c)(b-a)}$$

$$= \frac{-(a-b)-(b-c)-(c-a)}{(a-b)(b-c)(c-a)} = 0 \quad \textbf{(2)}$$

$\therefore$ ***given inequality*** $\Leftrightarrow \sum \frac{1}{c(a-c)(b-c)} > \frac{1}{2sR^2}$ ***(from (1), (2))***

$$\Leftrightarrow \sum \frac{ab}{abc(a-c)(b-c)} > \frac{1}{2sR^2} \Leftrightarrow \sum \frac{ab}{(a-c)(b-c)} > \frac{4Rrs}{2sR^2} = \frac{2r}{R}$$

$$\Leftrightarrow \frac{ab}{(a-c)(b-c)} + \frac{bc}{(b-a)(c-a)} + \frac{ca}{(c-b)(a-b)} > \frac{2r}{R}$$

$$\Leftrightarrow \frac{-ab(a-b)-bc(b-c)-ca(c-a)}{(a-b)(b-c)(c-a)} > \frac{2r}{R}$$

$$\Leftrightarrow -\left\{\frac{a^2b-ab^2+b^2c-bc^2+c^2a-ca^2}{(a-b)(b-c)(c-a)}\right\} > \frac{2r}{R}$$

$$\Leftrightarrow \frac{-\{a^2(b-c)+bc(b-c)-a(b-c)(b+c)\}}{(a-b)(b-c)(c-a)} > \frac{2r}{R}$$

$$\Leftrightarrow \frac{-(b-c)(a^2+bc-ab-ca)}{(a-b)(b-c)(c-a)} > \frac{2r}{R}$$

$$\Leftrightarrow \frac{-(b-c)\{a(a-b)-c(a-b)\}}{(a-b)(b-c)(c-a)} > \frac{2r}{R} \Leftrightarrow \frac{\pi(a-b)}{\pi(a-b)} > \frac{2r}{R}$$

$\Leftrightarrow R > 2r \rightarrow$ ***true (Euler) (Proved)***

145. **In $\Delta\ ABC$ the following relationship holds:**

$$2\sum a^4 < 2\sum b^2c^2\cos 2A + \sum a^2(b+c)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$2b^2c^2\cos 2A = 2b^2c^2(2\cos^2 A - 1) = 4b^2c^2\left(\frac{b^2+c^2-a^2}{2bc}\right)^2 - 2b^2c^2$$

$= (b^2+c^2-a^2)^2 - 2b^2c^2$. **Let** $S = 2\sum b^2c^2\cos 2A + \sum a^2(b+c)^2$

$$= (b^2+c^2-a^2)^2 + (c^2+a^2-b^2)^2 + (a^2+b^2-c^2)^2 - 2b^2c^2 - 2c^2a^2 -$$

$$-2a^2b^2 + \sum a^2(b^2+c^2+2bc) >$$

$$> \sum(b^2+c^2-a^2)^2 + \sum a^2(2bc\cos A) =$$

$= \sum(b^2+c^2-a^2)^2 + \sum a^2(b^2+c^2-a^2)$. ***Now,***

$$(b^2+c^2-a^2)^2 + (c^2+a^2-b^2)^2 + (a^2+b^2-c^2)^2 + (a^2+b^2+c^2)^2$$

$= 2c^4 + 2(b^2-a^2)^2 + 2(a^2+b^2)^2 + 2c^4 = 4c^4 + 4b^4 + 4a^4$. ***Thus,***

$$S > 4(a^4+b^4+c^4) - (a^2+b^2+c^2)^2 + 2\sum a^2b^2 - \sum a^4 = 2(a^4+b^4+c^4)$$

146. **In acute-angled $\Delta\ ABC$ the following relationship holds:**

$$\prod (a^{\cos A})(\cos A)^a \le \frac{(\sum\cos A)^{2s}\cdot(2s)^{\sum\cos A}}{3^{2s+\sum\cos A}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal - Chandar Nagore – India

Applying Weighted A.M ≥ G.M

$$\left(\frac{a\cos A+b\cos B+c\cos C}{a+b+c}\right)^{a+b+c}\geq\prod_{cyc}(\cos A)^a$$

$$\left(\frac{a\cos A+b\cos B+c\cos C}{\cos A+\cos B+\cos C}\right)^{\cos A+\cos B+\cos C}\geq\prod_{cyc}a^{\cos A}$$

$$\frac{(a\cos A+b\cos B+c\cos C)^{2s+\sum_{cyc}\cos A}}{(2s)^{2s}\left(\sum_{cyc}\cos A\right)^{\sum_{cyc}\cos A}}\geq\prod_{cyc}a^{\cos A}(\cos A)^a$$

Let $a\geq b\geq c$ then $\cos A\leq\cos B\leq\cos C$ and applying Chebyshev

$$\frac{1}{3}\left(\sum_{cyc}\cos A\right)(a+b+c)\geq a\cos A+b\cos B+c\cos C$$

$$\therefore\ \frac{(2s)^{\sum\cos A}\cdot\left(\sum\cos A\right)^{2s}}{3^{2s+\sum\cos A}}\geq\prod_{cyc}a^{\cos A}(\cos A)^a$$

Solution 2 by Myagmarsuren Yadamsuren – Ulaanbataar – Mongolia

$$\left(a^{\cos A}\cdot b^{\cos B}\cdot c^{\cos C}\right)\cdot\left((\cos A)^a\cdot(\cos B)^b\cdot(\cos C)^c\right)\cdot$$

$$\cdot\,3^{a+b+c}\cdot3^{\cos A+\cos B+\cos C}\leq$$

$$\leq(\cos A+\cos B+\cos C)^{a+b+c}\cdot(a+b+c)^{\cos A+\cos B+\cos C}$$

$$\left(\frac{3a}{a+b+c}\right)^{\cos A}\cdot\left(\frac{3b}{a+b+c}\right)^{\cos B}\cdot\left(\frac{3c}{a+b+c}\right)^{\cos C}\cdot$$

$$\cdot\left(\frac{3\cos A}{\cos A+\cos B+\cos C}\right)^{a}\cdot\left(\frac{3\cos B}{\cos A+\cos B+\cos C}\right)^{b}\cdot\left(\frac{3\cos C}{\cos A+\cos B+\cos C}\right)^{c}\leq1$$

(ASSURE)

$$\prod\left(\frac{3a}{a+b+c}\right)^{\cos A}\cdot\left(\frac{3\cos A}{\cos A+\cos B+\cos C}\right)^{a}\underset{Cauchy}{\leq}$$

$$\leq\left(\frac{3(a\cdot\cos A+b\cdot\cos B+c\cos C)}{(a+b+c)\cdot(\cos A+\cos B+\cos C)}\right)^{\cos A+\cos B+\cos C}\cdot$$

$$\cdot\left(\frac{3\cdot(a\cdot\cos A+b\cdot\cos B+c\cdot\cos C)}{(a+b+c)(\cos A+\cos B+\cos C)}\right)^{a+b+c}=$$

$$=\left(\frac{3\cdot(a\cdot\cos A+b\cdot\cos B+c\cdot\cos C)}{(a+b+c)\cdot(\cos A+\cos B+\cos C)}\right)^{\sum\cos A+\sum a}=$$

$$a\cdot\cos A+b\cdot\cos B+c\cdot\cos C=\frac{2\Delta}{R};\ a+b+c=2p$$

$$\cos A+\cos B+\cos C=1+\frac{r}{R}$$

$$=\left(\frac{3\cdot\frac{2\Delta}{R}}{2p\cdot\left(1+\frac{r}{R}\right)}\right)^{\sum\cos A+\sum a}=\left(\frac{3r}{R+r}\right)^{\sum\cos A+\sum a}\underset{Euler}{\leq}$$

$$\leq\left(\frac{R+r}{R+r}\right)^{\sum\cos A+\sum a}=1^{\sum\cos A+\sum a}=1$$

147. In $\Delta\ ABC$ the following relationship holds:

$$\frac{ab}{c(a+b)^2}+\frac{bc}{a(b+c)^2}+\frac{ca}{b(a+c)^2}\geq\frac{\sqrt{3}}{4R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo:** x,y,z **números** R^+. **Probar que:

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}$$

Previamente demostraremos la siguiente desigualdad:

1) Siendo:** $a,b,c,x,y,z\quad R^+$**, se cumple que:

$$\frac{2}{(a+b)(x+y)}+\frac{2}{(b+c)(y+z)}+\frac{2}{(c+a)(y+z)}\geq\frac{9}{(b+c)x+(c+a)y+(a+b)z}$$

$$\sum\frac{2\big(c(x+y)+(a+b)z+(ax+by)\big)}{(a+b)(x+y)}\geq 9$$

$$\Rightarrow\sum\frac{2c}{a+b}+\sum\frac{2z}{x+y}\geq 9-2\sum\frac{(ax+by)}{(a+b)(x+y)}=3-2\sum\frac{ax+by}{(a+b)(x+y)}+6$$

Ahora bien: $3-2\sum\frac{ax+by}{(a+b)(x+y)}=\left(1-\frac{2(ax+by)}{(a+b)(x+y)}\right)+\left(1-\frac{2(by+cz)}{(b+c)(y+z)}\right)+\left(1-\frac{2(cz+ax)}{(c+a)(z+x)}\right)$

$$\Rightarrow 3-2\sum\frac{ax+by}{(a+b)(x+y)}=\frac{(a-b)(x-y)}{(a+b)(x+y)}+\frac{(b-c)(y-z)}{(b+c)(y+z)}+\frac{(c-a)(z-x)}{(c+a)(z+x)}$$

Luego: $\Rightarrow 9-2\sum\frac{ax+by}{(a+b)(x+y)}=6+\sum\frac{(a-b)(x-y)}{(a+b)(x+y)}\leq$

$$\leq 6+\frac{1}{2}\sum\frac{(a-b)^2}{(a+b)^2}+\frac{1}{2}\sum\frac{(x-y)^2}{(x+y)^2}$$

Lo cual es suficiente probar: $\sum\frac{2c}{a+b}+\sum\frac{2z}{x+y}\geq 6+\frac{1}{2}\sum\frac{(a-b)^2}{(a+b)^2}+\frac{1}{2}\sum\frac{(x-y)^2}{(x+y)^2}$

$$\Rightarrow\sum\frac{4c}{a+b}+\sum\frac{4z}{x+y}\geq 12+\sum\frac{(a-b)^2}{(a+b)^2}+\sum\frac{(x-y)^2}{(x+y)^2}$$

$$\Rightarrow\sum\frac{4c}{a+b}+\sum\frac{4z}{x+y}\geq\left(\sum\frac{(a-b)^2}{(a+b)^2}-3\right)+\left(\sum\frac{(x-y)^2}{(x+y)^2}-3\right)+18$$

$$\Rightarrow\sum\frac{4c(a+b)}{(a+b)^2}+\sum\frac{4z(x+y)}{(x+y)^2}\geq-\sum\frac{4ab}{(a+b)^2}-\sum\frac{4xy}{(x+y)^2}+18$$

$$\Rightarrow(ab+bc+ca)\left(\sum\frac{1}{(a+b)^2}\right)+(xy+yz+zx)\left(\sum\frac{1}{(x+y)^2}\right)\geq\frac{9}{2}$$

Siendo: $x=a, y=b, z=c$***. Se obtiene lo siguiente:***

$$\Rightarrow(xy+yz+xz)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}$$

(LQQD). Probar en un triángulo ABC***:*** $\frac{ab}{c(a+b)^2}+\frac{bc}{a(b+c)^2}+\frac{ca}{b(c+a)^2}\geq\frac{\sqrt{3}}{4R}$

$$\Rightarrow \frac{ab}{c(a+b)^2}+\frac{bc}{a(b+c)^2}+\frac{ca}{b(c+a)^2}\geq \frac{\sqrt{3}S}{abc}$$

$$\Rightarrow \frac{(ab)^2}{(a+b)^2}+\frac{(bc)^2}{(b+c)^2}+\frac{(ca)^2}{(c+a)^2}\geq \sqrt{3}S$$

De la siguiente desigualdad $\forall\, x,y,z\in R^+$**:**

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq \frac{9}{4} \dots \textit{(A)}$$

Sea: $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$***. La desigualdad equivalente en (A) es …***

$$\Rightarrow \frac{(ab)^2}{(a+b)^2}+\frac{(bc)^2}{(b+c)^2}+\frac{(ca)^2}{(c+a)^2}\geq \frac{9}{4}\frac{abc}{a+b+c}\geq \sqrt{3}S \dots \textit{(LQQD)}$$

Lo cual es cierto ya que:

$$\frac{9abc}{a+b+c}\geq 4\sqrt{3}S \to \frac{9\times 4RS}{2s}\geq 4\sqrt{3}S \to 9R\geq 2\sqrt{3}s \to \frac{3\sqrt{3}R}{2}\geq s$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow \frac{1}{abc}\sum\frac{a^2b^2}{(a+b)^2}\geq\frac{\sqrt{3}}{4R}$

$$\Leftrightarrow \frac{1}{4R\Delta}\sum\frac{a^2b^2}{(a+b)^2}\geq\frac{\sqrt{3}}{4R}\Leftrightarrow \sum\frac{a^2b^2}{(a+b)^2}\geq\sqrt{3}\Delta \quad \textit{(1)}$$

Now, $\sum\frac{a^2b^2}{(a+b)^2}\geq\frac{(\sum ab)^2}{2\sum a^2+2\sum ab}$ ***(Bergstrom) (2)***

Let's first prove that: $\sum ab\geq 4\sqrt{3}\Delta$

$$\Leftrightarrow s^2+r(4R+r)\geq 4\sqrt{3}\Delta \quad \textit{(3)}$$

Now, $s^2+r(4R+r)\geq s(3\sqrt{3}r)+r(4R+r)$

$$\geq 3\sqrt{3}rs+r(s\sqrt{3}) \textit{ (Trucht)} = 4\sqrt{3}rs = 4\sqrt{3}\Delta$$

$\Rightarrow$ ***(3) is true*** $\Rightarrow \sum ab\geq 4\sqrt{3}\Delta$ ***(A)***

Now, $\sum a^2\geq\sum(a-b)^2+4\sqrt{3}\Delta$ ***(Hadwiger – Finsler)***

$$\Rightarrow 2\sum ab \geq \sum a^2 + 4\sqrt{3}\Delta \Rightarrow \sum a^2 \leq 2\sum ab - 4\sqrt{3}\Delta$$

$$\Rightarrow 2\sum a^2 \leq 4\sum ab - 8\sqrt{3}\Delta \Rightarrow 2\sum a^2 + 2\sum ab \leq 6\sum ab - 8\sqrt{3}\Delta$$

$$\Rightarrow \frac{1}{2\sum a^2 + 2\sum ab} \geq \frac{1}{6\sum ab - 8\sqrt{3}\Delta} \left(\because 6\sum ab - 8\sqrt{3}\Delta \geq 24\sqrt{3}\Delta - 8\sqrt{3}\Delta > 0\right)$$

$$\Rightarrow \frac{(\sum ab)^2}{2\sum a^2+2\sum ab} \geq \frac{(\sum ab)^2}{6\sum ab-8\sqrt{3}\Delta} \quad (4)$$

(1), (2), (4) $\Rightarrow$ ***it suffices to prove that*** $\frac{(\sum ab)^2}{6\sum ab-8\sqrt{3}\Delta} \geq \sqrt{3}\Delta$

$$\Leftrightarrow \left(\sum ab\right)^2 \geq 6\sqrt{3}\left(\sum ab\right)\Delta - 24\Delta^2$$

$$\Leftrightarrow \left(\sum ab\right)^2 - 6\sqrt{3}\left(\sum ab\right)\Delta + 24\Delta^2 \geq 0$$

$\Leftrightarrow \left(\sum ab - 4\sqrt{3}\Delta\right)\left(\sum ab - 2\sqrt{3}\Delta\right) \geq 0$, ***which is true,***

$\because \sum ab \geq 4\sqrt{3}\Delta$ ***(from (A)) (Proved)***

148. If $\alpha + \beta + \gamma = \pi$ then:

1. $\frac{\cos\alpha}{1+\sin\alpha} + \frac{\cos\beta}{1+\sin\beta} + \frac{\cos\gamma}{1+\sin\gamma} = \frac{p+2R-r}{p+2R+r}$

2. $\frac{\cos\alpha}{1+\sin\alpha} + \frac{\cos\beta}{1+\sin\beta} + \frac{\cos\gamma}{1+\sin\gamma} \geq 6 - 3\sqrt{3}$

Proposed by Adil Abdullayev – Baku – Azerbaidjian

Solution by George Apostolopoulos – Messolonghi – Greece

We have

1. $\frac{\cos\alpha}{1+\sin\alpha} + \frac{\cos\beta}{1+\sin\beta} + \frac{\cos\gamma}{1+\sin\gamma} = \frac{1-\sin\alpha}{\cos\alpha} + \frac{1-\sin\beta}{\cos\beta} + \frac{1-\sin\gamma}{\cos\gamma} =$

$$\frac{1}{\cos\alpha} + \frac{1}{\cos\beta} + \frac{1}{\cos\gamma} - (\tan\alpha + \tan\beta + \tan\gamma) =$$

$$\frac{\cos\alpha\cdot\cos\beta+\cos\beta\cdot\cos\gamma+\cos\gamma\cdot\cos\alpha}{\cos\alpha\cdot\cos\beta\cdot\cos\gamma}-(\tan\alpha+\tan\beta+\tan\gamma)$$

It is well-known that

$$\cos\alpha\cdot\cos\beta+\cos\beta\cdot\cos\gamma+\cos\gamma\cdot\cos\alpha=\frac{r^2+p^2-4R^2}{4R^2}$$

$$\cos\alpha\cdot\cos\beta\cdot\cos\gamma=\frac{p^2-(2R+r)^2}{4R^2}$$

$$\tan\alpha+\tan\beta+\tan\gamma=\frac{2pr}{p^2-(2R+r)^2}$$

So $\frac{\cos\alpha}{1+\sin\alpha}+\frac{\cos\beta}{1+\sin\beta}+\frac{\cos\gamma}{1+\sin\gamma}=$

$$\frac{\frac{r^2+p^2-4R^2}{4R^2}}{\frac{p^2-(2R+r)^2}{4R^2}}-\frac{2pr}{p^2-(2R+r)^2}=\frac{r^2+p^2-4R^2-2pr}{p^2-(2R+r)^2}=$$

$$\frac{(p-r)^2-(2R)^2}{p^2-(2R+r)^2}=\frac{(p+2R-r)(p-2R-r)}{(p+2R+r)(p-2R-r)}=\frac{p+2R-r}{p+2R+r}$$

2. $\frac{p+2R-r}{p+2R+r}=\frac{p+2R+r-2r}{p+2R+r}=1-\frac{2r}{p+2R+r}\geq 1-\frac{R}{p+2R+r}\geq$

$$\geq 1-\frac{R}{\left(\frac{3\sqrt{3}}{2}+2+\frac{1}{2}\right)R}=1-\frac{1}{\frac{3\sqrt{3}}{2}+\frac{5}{2}}=\frac{3\sqrt{3}+3}{3\sqrt{3}+5}=6-3\sqrt{3}.$$

149. In acute – angled $\Delta\ ABC$:

$$2\left(\sum\cot A\cot B\right)\left(\sum\frac{\cot A\tan B\tan C}{\tan B+\tan C}\right)\geq\left(\sum\cot A\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo acutángulo ABC. Probar que:

$$2\left(\sum \cot A \cot B\right)\left(\sum \frac{\cot A \tan B \tan C}{\tan B + \tan C}\right) \geq \left(\sum \cot A\right)^2$$

Dado que es un triángulo acutángulo: $0 < A, B, C < \frac{\pi}{2} \Leftrightarrow$

$$\Leftrightarrow (\tan A, \tan B, \tan C), (\cot A, \cot B, \cot C) > 0$$

La desigualdad es equivalente, y por la desigualdad de Cauchy:

$$2\left(\sum \cot A \cot B\right)\left(\sum \frac{\cot A}{\cot B + \cot C}\right) \geq$$

$$\geq 2\left(\sum \cot A \cot B\right)\left(\sum \frac{\cot^2 A}{\cot A \cot B + \cot A \cot C}\right) \geq \left(\sum \cot A\right)^2$$

Solution 2 by Soumitra Mandal – Chandar Nagore – India

$$2\left(\sum_{cyc} \cot A \cot B\right)\left(\sum_{cyc} \frac{\cot A \tan B \tan C}{\tan C + \tan B}\right) =$$

$$= 2\frac{(\tan A + \tan B + \tan C)}{\tan A \tan B \tan C}\left(\sum_{cyc} \frac{\cot A \tan B \tan C}{\tan B + \tan C}\right) =$$

$$= \left\{\sum_{cyc}(\tan B + \tan C)\right\}\left(\sum_{cyc} \frac{\cot^2 A}{\tan B + \tan C}\right) \geq \left(\sum_{cyc} \cot A\right)^2$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{(\cot A \tan B \tan B)}{(\tan B + \tan C)} = \frac{(\cot A \tan B \tan C)(\tan^2 A)}{\{(\tan B + \tan C)(\tan^2 A)\}}$$

$$= \frac{(\prod \tan A)(\cot^2 A)}{(\tan B + \tan C)}$$

$$\therefore \frac{\sum(\cot A \tan B \tan C)}{(\tan B + \tan C)} = \left(\prod \tan A\right)\sum\left[\frac{(\cot^2 A)}{(\tan B + \tan C)}\right] =$$

$$= \left(\prod \tan A\right)\left[\frac{(\sum \cot A)^2}{(\sum(\tan B + \tan C))}\right] \text{ (Bergstrom=)}$$

$$= \left(\sum \tan A\right)\left[\frac{(\sum \cot A)^2}{(2\sum \tan A)}\right] = \frac{(\sum \cot A)^2}{2}$$

Again $2(\sum \cot A \cot B) = 2$

$$\therefore LHS \geq 2\left[\frac{(\sum \cot A)^2}{2}\right] = \left(\sum \cot A\right)^2 = RHS$$

150. In ΔABC the following relationship holds:

$$\frac{1}{R}\sum \frac{m_c}{a+b} < \sum \frac{m_c}{aR+br} < \frac{1}{2R}\left(3 + \sum \frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Stefan-Romania

$$\frac{1}{R}\cdot\frac{m_c}{a+b} < \frac{m_c}{aR+br} \Leftrightarrow \frac{1}{aR+bR} < \frac{1}{aR+br} \Leftrightarrow r < R \text{ (true)}$$

$$\Rightarrow \frac{1}{R}\sum \frac{m_c}{a+b} < \sum \frac{m_c}{aR+br}. \text{ We'll prove: } \frac{m_c}{aR+br} < \frac{1}{2R} + \frac{b}{2Ra}$$

$$\text{But } m_c \leq \frac{a+b}{2}\left(m_c^2 = \frac{2a^2+2b^2-a^2-b^2+2ab\cos c}{4} \leq \left(\frac{a+b}{2}\right)^2\right)$$

$$\Leftrightarrow \frac{a+b}{aR+br} < \frac{1}{r} + \frac{b}{Ra} \Leftrightarrow \frac{a+b}{aR+br} < \frac{a+b}{Ra} \Leftrightarrow Ra < Ra + br$$

$$\Leftrightarrow 0 < br \text{ true} \Rightarrow \sum \frac{m_c}{aR+br} < \frac{1}{2R}\left(3 + \sum \frac{b}{a}\right)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{m_c}{aR+br} > \frac{m_c}{Ra+Rb} \Leftrightarrow R > r \to \text{true}, \because R \geq 2r$$

$$\text{Similarly, } \frac{m_a}{bR+cr} > \frac{m_a}{R_b+R_c} \text{ and, } \frac{m_b}{cR+ar} > \frac{m_b}{Rc+Ra}$$

$$\text{Adding, } \sum \frac{m_c}{aR+br} > \frac{1}{R}\sum \frac{m_c}{a+b} \text{ (proved 1st inequality)}$$

$$m_c=\frac{\sqrt{2a^2+2b^2-c^2}}{2}=\frac{\sqrt{(a+b)^2+(a-b)^2-c^2}}{2}=$$

$$=\frac{\sqrt{(a+b)^2+(a-b+c)(a-b-c)}}{2}<$$

$$<\frac{a+b}{2}\ (\because c+a>b; a-c-c<0\ as\ a<b+c)$$

$\therefore \frac{m_c}{aR+br}<\frac{a+b}{2(aR+br)}$. ***Let's prove:*** $\frac{a+b}{2(aR+br)}<\frac{1}{2R}\left(1+\frac{b}{a}\right)=\frac{a+b}{a(2R)}$

$\Leftrightarrow aR<aR+br$, ***which is true*** $\therefore \frac{m_c}{aR+br}<\frac{a+b}{2(aR+br)}\underset{(1)}{<}\frac{1}{2R}\left(1+\frac{b}{a}\right)$,

Similarly, $\frac{m_a}{bR+cr}<\frac{b+c}{2(bR+cr)}\underset{(2)}{<}\frac{1}{2R}\left(1+\frac{c}{b}\right)$, ***and*** $\frac{m_b}{cR+ar}<\frac{c+a}{2(cR+ar)}\underset{(3)}{<}\frac{1}{2R}\left(1+\frac{a}{c}\right)$

(1) + (2) + (3) $\Rightarrow \sum\frac{m_c}{aR+br}<\frac{1}{2R}\left(3+\sum\frac{b}{a}\right)$

151. In $\Delta\ A_1A_2A_3$ the following relationship holds:

$$4\cos\frac{A_1}{2}\cos\frac{A_2}{2}\cos\frac{A_3}{2}-3\sqrt[3]{\sin A_1\sin A_2\sin A_3}\geq\max_{1\leq i<j\leq 3}\left(\sqrt{\sin A_i}-\sqrt{\sin A_j}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un $\Delta\ A_1A_2A_3$:

$$4\cos\frac{A_1}{2}\cos\frac{A_2}{2}\cos\frac{A_3}{2}-3\sqrt[3]{\sin A_1\sin A_2\sin A_3}\geq\max_{1\leq i<j\leq 3}\left(\sqrt{\sin A_i}-\sqrt{\sin A_j}\right)^2$$

Si: $i=1, j=2$. ***Además en un $\Delta\ A_1A_2A_3$:***

$$4\cos\frac{A_1}{2}\cos\frac{A_2}{2}\cos\frac{A_3}{2}=\sin A_1+\sin A_2+\sin A_3$$

$$\sin A_1+\sin A_2+\sin A_3-3\sqrt[3]{\sin A_1\sin A_2\sin A_3}\geq\geq\max\left(\sqrt{\sin A_1}-\sqrt{\sin A_2}\right)^2$$

$$\Rightarrow \sin A_1+\sin A_2+\sin A_3-3\sqrt[3]{\sin A_1\sin A_2\sin A_3}\geq$$

$$\geq \sin A_1 + \sin A_2 - 2\sqrt{\sin A_1 \sin A_2}$$

$$\Rightarrow \sin A_3 + 2\sqrt{\sin A_1 \sin A_2} \geq 3\sqrt[3]{\sin A_1 \sin A_2 \sin A_3}$$

$$\Rightarrow \sin A_3 + \sqrt{\sin A_1 \sin A_2} + \sqrt{\sin A_1 \sin A_2} \geq 3\sqrt[3]{\sin A_1 \sin A_2 \sin A_3}$$

(LQQD). Análogamente para los demás términos ...

Solution 2 by Ravi Prakash - New Delhi – India

We may assume $\sin A_1 \geq \sin A_2 \geq \sin A_3$

$$\therefore \max_{1\leq i<j\leq 3}\left(\sqrt{\sin A_i} - \sqrt{\sin A_j}\right)^2 = \left(\sqrt{\sin A_1} - \sqrt{\sin A_3}\right)^2 =$$

$= \sin A_1 + \sin A_3 - 2\sqrt{\sin A_1 \sin A_3}$. ***Also, in a triangle:***

$$4\cos\frac{A_1}{2}\cos\frac{A_2}{2}\cos\frac{A_3}{3} = \sin A_1 + \sin A_2 + \sin A_3$$

Now, consider :

$$4\cos\frac{A_1}{2}\cos\frac{A_2}{2}\cos\frac{A_3}{3} - 3(\sin A_1 \sin A_2 \sin A_3)^{\frac{1}{3}} - \max_{1\leq i<j\leq 3}\left(\sqrt{\sin A_i} - \sqrt{\sin A_j}\right)^2 =$$

$$= \sin A_2 + 2\sqrt{\sin A_1 \sin A_3} - 3(\sin A_1 \sin A_2 \sin A_3)^{\frac{1}{3}} \geq$$

$$\geq 3\left[\sin A_2\left(\sqrt{\sin A_1 \sin A_3}\right)^2\right]^{\frac{1}{3}} - 3(\sin A_1 \sin A_2 \sin A_3)^{\frac{1}{3}} = 0$$

152. Prove that in any triangle ΔABC the following relationship holds:

$$\frac{m_a}{s_a} + \frac{m_b}{s_b} + \frac{m_c}{s_c} \geq \sqrt{(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}$$

Proposed by Adil Abdullayev-Baku-Azerbaidjian

Solution 1 by Daniel Sitaru – Romania

$$X = \sum\left(\frac{b}{c}+\frac{c}{b}\right) \geq 6$$

$$\sum\frac{m_a}{s_a} = \frac{1}{2}\sum\left(\frac{b}{c}+\frac{c}{b}\right) \geq \sqrt{\sum a\sum\frac{1}{a}} \leftrightarrow X \geq 2\sqrt{3+X} \leftrightarrow X \geq 6$$

153. In ΔABC the following relationship holds:

$$\sum\sqrt{a^2+(2s-a)^2+2a(2s-a)\cos A} < 6s$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Lets first prove that: $\sqrt{a^2+(2s-a)^2+2a(2s-a)\cos A} < 2s$

$$\Leftrightarrow a^2+4s^2-4sa+a^2+2a(2s-a)\cos A < 4s^2$$

$$\Leftrightarrow a^2+a(b+c)\cos A < 2sa = a(a+b+c)$$

$\Leftrightarrow a(b+c)\cos A < a(b+c) \Leftrightarrow \cos A < 1$, ***which is true.***

Similarly, $\sqrt{b^2+(2s-b)^2+2b(2s-b)\cos B} < 2s$

And, $\sqrt{c^2+(2s-c)^2+2c(2s-c)\cos C} < 2s$

$\therefore LHS$ ***of (1)*** $< 2s+2s+2s = 6s$ ***(Proved)***

Solution 2 by Soumitra Mandal - Chandar Nagore – India

In $\Delta ABC, \cos A, \cos B, \cos C \leq 1$

$$\therefore \sum_{cyc}\sqrt{a^2+(2s-a)^2+2a(2s-a)\cos A}$$

$$\leq \sum_{cyc}\sqrt{a^2+(2s-a)^2+2a(2s-a)} = 6s$$

154. In ΔABC the following relationship holds:

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{m_a+m_b+m_c}\le\frac{R}{2r}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Adil Abdullayev – Baku – Azerbaidjian

$$LHS\le RHS\leftrightarrow m_a+m_b+m_c\ge\frac{4R+r}{\frac{R}{2r}}\to$$

$$\to A=m_a+m_b+m_c\ge\frac{8Rr+2r^2}{R}$$

$$TERESHIN\to A\ge\frac{p^2-r^2-4Rr}{R}\ge\frac{8Rr+2r^2}{R}\leftrightarrow$$

$$p^2\ge 12Rr+3r^2=16Rr-5r^2-4r(R-2r)$$

GERRETSEN + EULER inequality.

155. If $x,y>0$ then in $\Delta\ ABC$ the following relationship holds:

$$(s-b)(s-c)x^2+(s-c)(s-a)y^2+(s-a)(s-b)(x^2+y^2)>2xyS$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS=x^2(s-b)(s-c+s-a)+y^2(s-a)(s-c+s-b)$$

$$=x^2b(s-b)+y^2a(s-a)\overset{A-G}{\ge}2xy\sqrt{ab(s-a)(s-b)}$$

$\therefore$ *it suffices to prove:*

$$2xy\sqrt{ab(s-a)(s-b)}>2xy\sqrt{s(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow ab>s(s-c)=\frac{(a+b+c)(a+b-c)}{4}$$

$$\Leftrightarrow 4ab>(a+b)^2-c^2\Leftrightarrow c^2>(a-b)^2\Leftrightarrow c^2-(a-b)^2>0$$

$\Leftrightarrow (c+a-b)(c-a+b)>0$, ***which is true,*** $\because c+a>b$ ***and*** $b+c>a$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\Delta\ ABC:\ x;y;a;b;c\in(0,+\infty)$$

$$(p-b)(p-c)x^2+(p-a)(p-c)y^2+(p-a)(p-b)(x^2+y^2)>2xyS$$

$$\Delta ABC\Rightarrow b+c>a\Rightarrow a-(b+c)<0$$

$$(a+c-b)\big(a-(b+c)\big)<0\Rightarrow[(a-b)+c]\cdot[(a-b)-c]<0$$

$$(a-b)^2-c^2<0\Rightarrow a^2+b^2+2ab-c^2<4ab$$

$$(a+b)^2-c^2<4ab\Rightarrow(a+b+c)\cdot(a+b-c)<4ab$$

$$p\cdot(p-c)<ab\Rightarrow\sqrt{p(p-c)}<\sqrt{ab}|\cdot\sqrt{(p-a)(p-b)}$$

$$\sqrt{p(p-a)(p-b)(p-c)}<\sqrt{ab(p-a)(p-b)}|\cdot 2xy$$

$$2xyS<2xy\cdot\sqrt{ab(p-a)(p-b)}\ \text{(*)}$$

$$2xy\cdot\sqrt{ab(p-a)(p-b)}\underset{Cauchy}{\le}8(p-b)x^2+a\cdot(p-a)y^2=$$

$$=(p-b)(p-a+p-c)x^2+(p-a)(p-b+p-c)y^2=$$

$$=(p-b)(p-c)x^2+(p-a)(p-c)y^2+(p-a)(p-b)(x^2+y^2)\ \text{(**)}$$

$$\text{(*); (**)}\Rightarrow 2xyS<(p-b)(p-c)x^2+(p-a)(p-c)y^2+(p-a)(p-b)(x^2+y^2)$$

156. **Prove that in any triangle ABC:**

$$\frac{m_a}{l_a}+\frac{m_b}{l_b}+\frac{m_c}{l_c}\ge\frac{\sqrt{b}+\sqrt{c}}{2\sqrt{a}}+\frac{\sqrt{c}+\sqrt{a}}{2\sqrt{b}}+\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{c}}.$$

where l_a,l_b,l_c are internal angle bisectors from A,B,C respectively.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{m_a}{l_a}+\frac{m_b}{l_b}+\frac{m_c}{l_c}\ge\frac{\sqrt{b}+\sqrt{c}}{2\sqrt{a}}+\frac{\sqrt{c}+\sqrt{a}}{2\sqrt{b}}+\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{c}}$.

Teniendo en cuenta las siguientes identidades y desigualdades:

$$m_a \geq \frac{b+c}{2}\cos\frac{A}{2}, m_b \geq \frac{c+a}{2}\cos\frac{B}{2}, m_c \geq \frac{a+b}{2}\cos\frac{C}{2}$$

$$l_a = \frac{2bc}{b+c}\cos\frac{A}{2}, l_b = \frac{2ca}{c+a}\cos\frac{B}{2}, l_c = \frac{2ab}{a+b}\cos\frac{C}{2}$$

Reemplazando en la desigualdad inicial es equivalente:

$$\frac{m_a}{l_b} + \frac{m_b}{l_b} + \frac{m_c}{l_c} \geq \frac{(b+c)^2}{4bc} + \frac{(c+a)^2}{4ca} + \frac{(a+b)^2}{4ab} \geq$$

$$\geq \frac{(b+c)\left(2\sqrt{bc}\right)}{4bc} + \frac{(c+a)\left(2\sqrt{ca}\right)}{4ca} + \frac{(a+b)\left(2\sqrt{ab}\right)}{4ab}$$

$$\Rightarrow \frac{m_a}{l_a} + \frac{m_b}{l_b} + \frac{m_c}{l_c} \geq \frac{(b+c)^2}{4bc} + \frac{(c+a)^2}{4ca} + \frac{(a+b)^2}{4ab} \geq$$

$$\geq \frac{b+c}{2\sqrt{bc}} + \frac{c+a}{2\sqrt{bc}} + \frac{a+b}{2\sqrt{ab}}$$

$$\Rightarrow \frac{m_a}{l_a} + \frac{m_b}{l_b} + \frac{m_c}{l_c} \geq \frac{(b+c)^2}{4bc} + \frac{(c+a)^2}{4ca} + \frac{(a+b)^2}{4ab} \geq$$

$$\geq \frac{\sqrt{b}}{2\sqrt{c}} + \frac{\sqrt{c}}{2\sqrt{b}} + \frac{\sqrt{c}}{2\sqrt{a}} + \frac{\sqrt{a}}{2\sqrt{c}} + \frac{\sqrt{a}}{2\sqrt{b}} + \frac{\sqrt{b}}{2\sqrt{a}}$$

$$\Rightarrow \frac{m_a}{l_a} + \frac{m_b}{l_b} + \frac{m_c}{l_c} \geq \frac{(b+c)^2}{4bc} + \frac{(c+a)^2}{4ca} + \frac{(a+b)^2}{4ab} \geq$$

$$\geq \frac{\sqrt{b}+\sqrt{c}}{2\sqrt{a}} + \frac{\sqrt{c}+\sqrt{a}}{2\sqrt{b}} + \frac{\sqrt{a}+\sqrt{b}}{2\sqrt{c}}$$ ***... (LQQD)***

157. In $\Delta\, ABC$ the following relationship holds:

$$a^6 + b^6 + c^6 \geq 8r^2 s \sum \frac{a^5}{b^2 - bc + c^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal - Chandar Nagore – India

$$8r^2s\sum_{cyc}\frac{a^5}{b^2-bc+c^2}\le 8r^2s\sum_{cyc}\frac{a^5}{bc}$$

applying A.M ≥ G.M

$$=\frac{8r^2s}{abc}\left(\sum_{cyc}a^6\right)=\frac{8r^2s}{4SR}\left(\sum_{cyc}a^6\right)\le\sum_{cyc}a^6$$

since, $R\ge 2r$ ***and*** $S=rs$

$$8r^2s\sum_{cyc}\frac{a^5}{b^2-bc+c^2}\le\sum_{cyc}a^6$$

Solution 2 by Seyran Ibrahimov-Baku-Azerbaidjian

$$RHS\le 8r^2s\left(\frac{a^5}{bc}+\frac{b^5}{ac}+\frac{c^5}{ab}\right)=8r^2s\left(\frac{a^6+b^6+c^6}{abc}\right)=$$

$$=4RS\cdot\frac{1}{abc}\cdot(a^6+b^6+c^6)=a^6+b^6+c^6$$

158. In ΔABC the following relationship holds:

$$\sum\left(\frac{am_a}{m_b}+b\right)^2+\sum\left(\frac{am_b}{m_a}+b\right)^2\ge 4(3s^2-r^2-4Rr)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS=\sum a^2\left[\left(\frac{m_a}{m_b}\right)^2+\left(\frac{m_b}{m_a}\right)^2\right]+2\sum a^2+\sum 2ab\left[\left(\frac{m_a}{m_b}\right)+\left(\frac{m_b}{m_a}\right)\right]\ge$$

$$\ge 4\sum a^2+4\sum ab\overset{(by\ AM-GM)}{=}4(3s^2-4Rr-r^2)=RHS$$

159. In $\Delta\, ABC$ the following relationship holds:

$$\frac{w_a}{\sqrt{bc}}+\frac{w_b}{\sqrt{ca}}+\frac{w_c}{\sqrt{ab}}\le\frac{\sqrt[4]{27}}{2}\sqrt{\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$\frac{w_a}{\sqrt{bc}}+\frac{w_b}{\sqrt{ca}}+\frac{w_c}{\sqrt{ab}}\le\frac{\sqrt[4]{27}}{2}\sqrt{\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}}=\cdots\ (A)$$

Teniendo cuenta las siguientes identidades y desigualdades en un triangulo ABC $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr},\ \frac{p}{4R}=\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\le\frac{3\sqrt{3}}{8},$

$$\frac{p}{r}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2};\ w_a\sqrt{p(p-a)},w_b\le\sqrt{p(p-b)},w_c\le\sqrt{p(p-c)}$$

Por la desigualdad de Cauchy en ... (A):

$$\frac{w_a}{\sqrt{bc}}+\frac{w_b}{\sqrt{ca}}+\frac{w_c}{\sqrt{ab}}\le\sqrt{\left(w_a^2+w_b^2+w_c^2\right)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)}\le\sqrt{\left(\frac{p}{2R}\right)\left(\frac{p}{r}\right)}\le$$

$$\le\sqrt{\frac{3\sqrt{3}}{4}\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}=\frac{\sqrt[4]{27}}{2}\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}$$

160. Let ABC be an arbitrary triangle and Ω is the first Brocard Point of ABC. Let R_a, R_b, R_c then circumradius of triangles $\Omega BC, \Omega CA, \Omega AB$ respectively.

Prove that $R_a\cdot R_b\cdot R_c=R^3$

where R is the circumradius of ABC.

Proposed by Mehmet Șahin – Ankara – Turkey

Solution by Daniel Sitaru – Romania

$$R_a = \frac{c}{2\sin\big(\pi - \omega - (B-\omega)\big)} = \frac{2R\sin C}{2\sin B}$$

$$\prod R_a = \prod \frac{R\sin C}{\sin B} = R^3$$

161. In $\Delta\ ABC$ the following relationship holds:

$$\sum \left(\frac{2s-a}{2s-b}\right)^2 \geq \sum \frac{\cos\frac{A}{2}\cos\frac{B-C}{2}}{\cos\frac{C}{2}\cos\frac{A-B}{2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

$$\sum \frac{(2s-a)^2}{(2s-b)} \geq \sum \frac{\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)}{\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right)}$$

La desigualdad es equivalente: $\sum\left(\frac{b+c}{c+a}\right)^2 \geq \sum \frac{b+c}{a+b}$

$$\Rightarrow \left(\frac{b+c}{c+a}\right)^2 + \left(\frac{c+a}{a+b}\right)^2 + \left(\frac{a+b}{b+c}\right)^2 \geq \frac{b+c}{a+b} + \frac{a+b}{c+a} + \frac{c+a}{b+c}$$

Siendo: $x, y, z \in \mathbb{R}$***, se cumple la siguiente desigualdad:***

$x^2 + y^2 + z^2 \geq xy + yz + zx$***. Desde que:*** $x = \frac{a+b}{b+c}, y = \frac{b+c}{c+a}, z = \frac{c+a}{a+b}$***, nos***

resulta: $\Rightarrow \left(\frac{b+c}{c+a}\right)^2 + \left(\frac{c+a}{a+b}\right)^2 + \left(\frac{a+b}{b+c}\right)^2 \geq \frac{b+c}{a+b} + \frac{a+b}{c+a} + \frac{c+a}{b+c}$ ***... (LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS = \left(\frac{b+c}{c+a}\right)^2 + \left(\frac{c+a}{a+b}\right)^2 + \left(\frac{a+b}{b+c}\right)^2$$

$$RHS=\sum\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}=\sum\frac{\sin B+\sin C}{\sin A+\sin B}$$

$=\sum\frac{b+c}{a+b}=\frac{b+c}{a+b}+\frac{a+b}{c+a}+\frac{c+a}{b+c}$. ***Let*** $x=\frac{b+c}{c+a}, y=\frac{c+a}{a+b}, z=\frac{a+b}{b+c}$

Given inequality $\Leftrightarrow\sum x^2\geq\sum\frac{1}{x}=\frac{\sum xy}{xyz}=\sum xy\ (\because xyz=1)$

which is true (Proved)

162. In $\Delta\ ABC$ the following relationship holds:

$$\left(\sum\sin A\right)^2\geq 6\sqrt[3]{\prod\sin^2 A}+\frac{27\prod\sin^2 A}{(\sum\sin A\sin B)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $(\sum\sin A)^2-6\sqrt[3]{\prod\sin^2 A}\geq\frac{27\prod\sin^2 A}{(\sum\sin A\sin B)^2}$

En un triángulo ABC***, se cumple lo siguiente:*** $\sin A,\sin B,\sin C>0$

La desigualdad es equivalente:

$$\left[\left(\sum\sin A\right)^2-6\sqrt[3]{\prod\sin^2 A}\right]\left(\sum\sin A\sin B\right)^2\geq 27\prod\sin^2 A$$

Lo cual es cierto ya que, por: MA $\geq$ ***MG***

$$\left(\sum\sin A\right)^2-6\sqrt[3]{\prod\sin^2 A}\geq 3\sqrt[3]{\prod\sin^2 A}\wedge\left(\sum\sin A\sin B\right)^2\geq 9\sqrt[3]{\prod\sin^4 A}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left.\begin{matrix}\sin A=x\\ \sin B=y\\ \sin C=z\end{matrix}\right\}$$

$$(x+y+z)^2 \geq 6\cdot\sqrt[3]{x^2y^2z^2}+\frac{27\prod x^2}{(\sum xy)^2}=6\cdot\sqrt[3]{x^2y^2z^2}+3\left(\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right)^2$$

$$6\cdot\sqrt[3]{x^2y^2z^2}+3\left(\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right)^2 \underset{GM\geq HM}{\leq} 6\cdot\sqrt[3]{x^2y^2z^2}+3\cdot\sqrt[3]{x^2y^2z^2}=$$

$$=3\cdot 2\cdot\sqrt[3]{(xy)\cdot(yz)\cdot(zx)}+3\sqrt[3]{x^2y^2z^2}\leq 2\cdot(xy+yz+zx)+$$

$$+(x^2+y^2+z^2)=(x+y+z)^2=(\sin A+\sin B+\sin C)^2$$

Solution 3 by Anas Adlany-El Jadida-Morroco

First, set

$$X=\sum\sin(A)\,;\ \bar{X}=\sum\frac{1}{\sin(A)}\,;Y=\prod\sin(A)$$

So the inequality to prove is $X^2\geq 6\sqrt[3]{Y}+\frac{27Y}{Y^2X^2}\Leftrightarrow X^2\geq 6\sqrt[3]{Y}+\frac{27}{X^2}$

But, by Cauchy's inequality we have $\frac{27}{X^2}\leq\frac{27}{81}X^2=\frac{X^2}{3}$

And the fact that (AM-GM) $X\geq 3\sqrt[3]{Y}$

Combining all these inequality, we shall have obtained the desired inequality.

Solution 4 by Nirapada Pal-India

$$\left(\sum\sin A\right)^2\geq\left[3(\sin A\sin B\sin C)^{\frac{1}{3}}\right]^2 \text{ by } AM\geq GM$$

$$=9(\sin A\sin B\sin C)^{\frac{2}{3}}=6(\sin A\sin B\sin C)^{\frac{2}{3}}+3(\sin A\sin B\sin C)^{\frac{2}{3}}=$$

$$=6\sqrt[3]{\prod\sin^2 A}+\frac{27\prod\sin^2 A}{\left[3(\sin^2 A\sin^2 B\sin^2 C)^{\frac{1}{3}}\right]^2}=$$

$$= 6\sqrt[3]{\prod \sin^2 A} + \frac{27\prod \sin^2 A}{[\sum \sin A \sin B]^2} \text{ by } GM \le AM$$

Solution 5 by Soumava Chakraborty – Kolkata – India

Given inequality $\Leftrightarrow$

$$\sum \sin^2 A + 2\sum \sin A \sin B \ge 6\sqrt[3]{\prod \sin^2 A} + 27\frac{\prod \sin^2 A}{(\sum \sin A \sin B)^2}$$

Now, $\sum \sin^2 A \overset{A-G}{\ge} 3\sqrt[3]{\prod \sin^2 A}$ $\therefore$ ***it suffices to prove:*** $2x \ge 3y + \frac{27y^3}{x^2}$

(where $x = \sum \sin A \sin B$ ***and*** $y = \sqrt[3]{\prod \sin^2 A}$***)***

$$\Leftrightarrow 2x^3 - 3x^2y - 27y^3 \ge 0 \Leftrightarrow 2t^3 - 3t^2 - 27 \ge 0 \left(t = \frac{x}{y}\right)$$

$$\Leftrightarrow (t-3)(2t^2 + 3t + 9) \ge 0 \Leftrightarrow t \ge 3$$

$$\Delta < 0 \Rightarrow 2t^2 + 3t + 9 > 0$$

But $\sum \sin A \sin B \ge 3\sqrt[3]{\prod \sin^2 A} \Rightarrow x \ge 3y \Rightarrow t \ge 3$ ***(Proved)***

163. In $\Delta\ ABC$ the following relationship holds:

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \ge \sqrt{4 + \frac{2\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{S}}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

In $\Delta\ ABC, \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \ge \sqrt{4 + \frac{2\sqrt{\sum a^2b^2}}{\Delta}}$

Given inequality $\Leftrightarrow 2R\left(\sum \frac{1}{a}\right) \ge \sqrt{4 + \frac{2\sqrt{\sum a^2b^2}}{\Delta}}$

$$\Leftrightarrow 2R\left(\frac{\sum ab}{abc}\right) \ge \sqrt{4 + \frac{2\sqrt{\sum a^2b^2}}{\Delta}} \Leftrightarrow \frac{2R(\sum ab)}{4R\Delta} \ge \sqrt{4 + \frac{2 + \sqrt{\sum a^2b^2}}{\Delta}}$$

$$\Leftrightarrow \frac{(\sum ab)^2}{4\Delta^2} \geq 4 + \frac{2\sqrt{\sum a^2b^2}}{\Delta} \Leftrightarrow \frac{\sum a^2b^2}{4\Delta^2} + \frac{2abc(2s)}{4\Delta^2} \geq 4 + \frac{2\sqrt{\sum a^2b^2}}{\Delta}$$

$$\Leftrightarrow \frac{\sum a^2b^2}{4\Delta^2} + \frac{2(4Rrs)(2s)}{4r^2s^2} \geq 4 + \frac{2\sqrt{\sum a^2b^2}}{\Delta}$$

$$\Leftrightarrow \frac{\sum a^2b^2}{4\Delta^2} - \frac{2\sqrt{\sum a^2b^2}}{\Delta} + 4 + \frac{4R}{r} \geq 8 \quad (i)$$

Now, $\frac{4R}{r} \geq 8$ ***(1) (Euler) and***

$$\frac{\sum a^2b^2}{4\Delta^2} - \frac{2\sqrt{\sum a^2b^2}}{\Delta} + 4 = \left(\frac{\sqrt{\sum a^2b^2}}{2\Delta}\right)^2 - 2\left(\frac{\sqrt{\sum a^2b^2}}{2\Delta}\right) \cdot 2 + 2^2$$

$$= \left(\frac{\sqrt{\sum a^2b^2}}{2\Delta} - 2\right)^2 \geq 0 \quad (2)$$

(1) + (2) ⇒ (i) is true (Proved)

164. In $\Delta\, ABC, \omega -$ Brocard angle, $\Omega -$ first Brocard point:

$$a^2 + b^2 + c^2 \leq 4S\cot\omega \sum \frac{\Omega A \cdot \Omega B}{ab}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo $ABC, \omega -$ Brocard angle, $\Omega -$ first Brocard point:

$\frac{a^2+b^2+c^2}{4S} \leq \cot\omega \sum \frac{\Omega A \Omega B}{ab}$***. De la desigualdad Hayashi:***

$$\frac{PAPB}{ab} + \frac{PBPC}{bc} + \frac{PCPA}{ca} \geq 1$$

$\cot\omega = \cot A + \cot B + \cot C$ ***(Punto Brocard)***

La desigualdad es equivalente: Siendo $P = \Omega -$ first Brocard point

$$\cot\omega \sum \frac{\Omega A \Omega B}{ab} \geq (\cot A + \cot B + \cot C)(1) = \frac{a^2+b^2+c^2}{4S}$$ ***... (LQQD)***

Solution 2 by Soumava Chakraborty – Kolkata – India

In $\Delta ABC, \sum a^2 \leq 4\Delta \cot\omega \sum \frac{\Omega A\cdot \Omega B}{ab}$, ***where*** $\omega \to$ ***Brocard angle and***

$\Omega \to$ ***first Brocard point. For any point*** P ***in the plane of*** ΔABC,

$$\sum \frac{PA\cdot PB}{ab} \geq 1 \text{ (Hayashi's inequality)}$$

$$\therefore \sum \frac{\Omega A\cdot \Omega B}{ab} \geq 1 \quad (1)$$

$$\text{Again, } \cot\omega = \frac{a^2+b^2+c^2}{4\Delta} \Rightarrow 4\Delta\cot\omega = \sum a^2 \quad (2)$$

$$(1)\times(2) \Rightarrow 4\Delta\cot\omega \sum \frac{\Omega A\cdot \Omega B}{ab} \geq \sum a^2 \text{ (Proved)}$$

165. In ΔABC the following relationship holds:

$$\sum \frac{a^2+a+1}{\sqrt{a}} \geq 9\left(r\sqrt{3}+\frac{1}{2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Myagmarsuren Yadamsuren – Darkhan – Mongolia

$$(a-1)^2 \geq 0 \text{ (True)} \Rightarrow 4a^2+4a+4 \geq 3a^2+6a+3$$

$$2a^2+2a+2 \geq \frac{3(a+1)^2}{2} = 3\cdot\left(\frac{1+a}{2}\right)\cdot(1+a) \overset{Cauchy}{\geq} 3\sqrt{a}\cdot(1+a)$$

$$2a^2+2a+2 \geq 3\sqrt{a}\cdot(1+a) \Rightarrow \frac{a^2+a+1}{\sqrt{a}} \geq \frac{3\cdot a+3}{2}$$

$$\sum \frac{a^2+a+1}{\sqrt{a}} \geq \sum \frac{3a+3}{2} = 3\cdot\left(\frac{a+b+c}{2}+\frac{3}{2}\right) \underset{p\geq 3\sqrt{3}2}{\geq} 9\cdot\left(2\sqrt{3}+\frac{1}{2}\right)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$s \geq 3\sqrt{3}r \Rightarrow 3s \geq 9\sqrt{3}r \Rightarrow 3s+\frac{9}{2} \geq RHS$$

$\therefore$ *it suffices to show:* $\sum\frac{a^2+a+1}{\sqrt{a}}\geq 3s+\frac{9}{2}$ *(i)*

Let us show: $\frac{a^2+a+1}{\sqrt{a}}\geq\frac{3a}{2}+\frac{3}{2}$ *(1)*

$$\Leftrightarrow 2a^2+2a+2\geq 3a\sqrt{a}+3\sqrt{a}$$

$\Leftrightarrow 2t^2+2t^2+2\geq 3t^3+3t$ *(where* $t=\sqrt{a}$*)*

$$\Leftrightarrow 2t^2-3t^3+2t^2-3t+2\geq 0$$

$\Leftrightarrow (t-1)^2\underbrace{(2t^2+t+2)}_{>0}\geq 0$, *which is true. Similarly,* $\frac{b^2+b+1}{\sqrt{b}}\geq\frac{3b}{2}+\frac{3}{2}$

and $\frac{c^2+c+1}{\sqrt{c}}\geq\frac{3c}{2}+\frac{3}{2}$ *(3)*

(1) + (2) + (3) $\Rightarrow\sum\frac{a^2+a+1}{2}\geq 3s+\frac{9}{2}\Rightarrow$ *(i) is true. (Proved)*

166. In $\Delta\ ABC,\omega-$ Brocard angle:

$$\frac{s}{\sin A+\sin B+\sin C}\geq\frac{r}{\sqrt[3]{\sin(A-\omega)\sin(B-\omega)\sin(C-\omega)}}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

In $\Delta\ ABC,\frac{s}{\sum\sin A}\geq\frac{r}{\sqrt[3]{\sin(A-\omega)\sin(B-\omega)\sin(C-\omega)}}$

Now, $\sqrt[3]{\sin(A-\omega)\sin(B-\omega)\sin(C-\omega)}=\sqrt[3]{\sin^3\omega}=\sin\omega$

$\therefore$ *given inequality* $\Leftrightarrow\frac{s}{\frac{a+b+c}{2R}}\geq\frac{r}{\sin\omega}$

$$\Leftrightarrow\frac{sR}{S}\geq\frac{r}{\sin\omega}\Leftrightarrow\csc\omega\leq\frac{R}{r}\Leftrightarrow\frac{R^2}{r^2}\geq\csc^2\omega$$

$$\Leftrightarrow\frac{R^2}{r^2}\geq\sum\csc^2 A=\sum(1+\cot^2 A)=3+\sum\cot^2 A$$

$$= 1 + \left(2\sum \cot A \cot B + \sum \cot^2 A\right) = 1 + \left(\sum \cot A\right)^2 = 1 + \left(\frac{\sum a^2}{4\Delta}\right)^2$$

$$\Leftrightarrow \frac{R^2-r^2}{r^2} \geq \frac{\left(\sum a^2\right)^2}{16r^2s^2} \Leftrightarrow R^2 - r^2 \geq \frac{\left(\sum a^2\right)^2}{16s^2} \quad \text{(1)}$$

We shall now prove that: $R^2 - r^2 \geq \frac{\sum a^3}{8s}$ ***(2)***

$$\text{(2)} \Leftrightarrow R^2 - r^2 \geq \frac{\sum a^3 - 3abc + 3abc}{8s}$$

$$= \frac{(2s)(\sum a^2 - \sum ab)}{8s} + \frac{3(4Rrs)}{8s}$$

$$= \frac{s^2 - 12Rr - 3r^2}{4} + \frac{3Rr}{2} = \frac{s^2 - 6Rr - 3r^2}{4}$$

$$\Leftrightarrow 4R^2 - 4r^2 \geq s^2 - 6Rr - 3r^2 \Leftrightarrow s^2 \leq 4R^2 + 6Rr - r^2 \quad \text{(3)}$$

Gerretsen $\Rightarrow s^2 \leq 4R^2 + 4Rr + 3r^2$ ***(4)***

∴ to prove (2), it suffices to prove that:

$4R^2 + 4Rr + 3r^2 \leq 4R^2 + 6Rr - r^2$ ***(from (3), (4))***

$\Leftrightarrow 2Rr \geq 4r^2 \Leftrightarrow R \geq 2r$, → ***true*** ⇒ ***(2) is true.***

∴ to prove (1), it suffices to prove that:

$$\frac{\sum a^3}{8s} \geq \frac{(\sum a^2)^2}{16s^2} \Leftrightarrow (2s)\left(\sum a^3\right) \geq \left(\sum a^2\right)^2$$

$$\Leftrightarrow \left(\sum a\right)\left(\sum a^3\right) \geq \left(\sum a^2\right)^2$$

$$\Leftrightarrow (a^3b + ab^3 - 2a^2b^2) + (b^3c + bc^3 - 2b^2c^2) + (c^3a + ca^3 - 2c^2a^2) \geq 0$$

$$\Leftrightarrow ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2 \geq 0,$$

which is true, equality at $a = b = c$ ⇒ ***(1) is true (Proved)***

167. **In $\Delta\ ABC$ the following relationship holds:**

$$\sum a^2b^2\cos^2 A\cos^2 B > 16S^2\cos^2 A\cos^2 B\cos^2 C$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Myagmarsuren Yadamsuren – Darkhan – Mongolia

$$\sin 2A + \sin 2B + \sin 2C \le \frac{3\sqrt{3}}{2} < 3\sqrt{3};\ (\sin 2A + \sin 2B + \sin 2C)^2 < 27 \mid \cdot 16S^2$$

$$16S^2 < \frac{27}{(\sin 2A+\sin 2B+\sin 2C)^2}\cdot 16S^2 \underset{RADON}{\le} \left(\frac{1}{\sin^2 2A}+\frac{1}{\sin^2 2B}+\frac{1}{\sin^2 2C}\right)\cdot 16S^2$$

$$16S^2 < \frac{16S^2}{\sin^2 2A}+\frac{16S^2}{\sin^2 2B}+\frac{16S^2}{\sin^2 2C} = \sum\frac{16\cdot\left(\frac{bc\cdot\sin A}{2}\right)^2}{\sin^2 2A} =$$

$$= \sum\frac{4\cdot(bc)^2\cdot\sin^2 A}{4\cdot\sin^2 A\cdot\cos^2 A} = \sum\frac{(bc)^2}{\cos^2 A};\ \sum\frac{(bc)^2}{\cos^2 A} > 16S^2 \mid \cdot\cos^2 A\cdot\cos^2 B\cdot\cos^2 C$$

$$\sum(bc)^2\cdot\cos^2 B\cdot\cos^2 C > 16\cdot S^2\cdot\cos^2 A\cdot\cos^2 B\cdot\cos^2 C$$

Solution 2 by Seyran Ibrahimov – Maasilli – Azerbaidjian

$$\sum a^2b^2\cos^2 A\cos^2 B > 16S^2\cos^2 A\cos^2 B\cos^2 C$$

$$\sum a^2b^2\cos^2 A\cos^2 B \overset{AM-GM}{\ge} 3abc\cdot\cos A\cos B\cos C\sqrt[3]{abc\cdot\cos A\cos B\cos C}$$

$$12RS\cdot\cos A\cos B\cos C\sqrt[3]{4RS\cdot\cos A\cdot\cos B\cdot\cos C} > 16S^2\cos^2 A\cdot\cos^2 B\cdot\cos^2 C$$

$$3R\sqrt[3]{4RS\cdot\cos A\cdot\cos B\cdot\cos C} > 4S\cdot\cos A\cos B\cos C$$

$$27R^4\cdot S\cdot\cos A\cos B\cos C > 16S^3\cdot\cos^3 A\cdot\cos^3 B\cos^3 C$$

$$\frac{3\sqrt{3}}{4}\cdot\frac{R^2}{S} > \cos A\cos B\cos C$$

$$\cos A\cos B\cos C \le \frac{1}{8} < \frac{3\sqrt{3}}{4}\cdot\frac{R^2}{S} \Rightarrow S \le \frac{3\sqrt{3}R^2}{4} \Rightarrow \frac{1}{8} < 1\ \textit{(proved)}$$

168. **In $\Delta\, ABC$ the following relationship holds:**

$$R(\sin A \sin 5A + \sin B \sin 5B + \sin C \sin 5C) < 10(2R - r)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Pal – Kolkata – India

$$\sin A \sin 5A \le 1;\ \sin B \sin 5B \le 1;\ \sin C \sin 5C \le 1$$

$$\Rightarrow R\left(\sum \sin A \sin 5A\right) \le 3R \text{ and } R \ge 2r \ge \frac{10}{17}r \Rightarrow 17R > 10r$$

$$\Rightarrow 20R - 10r > 3r \ge R\left(\sum \sin A \sin 5A\right) \Rightarrow 10(2R - r) > R\left(\sum \sin A \sin 5A\right)$$

169. **In acute-angled $\Delta\, ABC$ the following relationship holds:**

$$\sum \frac{\cot A \cot^3 B}{\cot^2 B + 2\cot^2 A} + 2\sum \frac{\cot^2 A \cot B}{\cot A + 2\cot B} \ge 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Dang Thanh Tung-Vietnam

Set $\cot A = a, \cot B = b; \cot C = c \Rightarrow ab + bc + ca = 1$ **with** $a, b, c > 0$

We prove that: $\sum \frac{ab^3}{2a^2+b^2} + 2\sum \frac{a^2b}{a+2b} \ge 1$

$$\Leftrightarrow \sum ab - 2\sum \frac{a^3b}{2a^2+b^2} + 2\sum \frac{a^2b}{a+2b} \ge 1 \Leftrightarrow 2\sum \frac{a^2b}{a+2b} \ge 2\sum \frac{a^3b}{2a^2+b^2} \quad (1)$$

We have: $\frac{a^2b}{a+2b} \ge \frac{a^3b}{2a^2+b^2} \Leftrightarrow 2a^2 + b^2 \ge a(a + 2b)$

$\Leftrightarrow (a - b)^2 \ge 0$ ***(true)$\Rightarrow$ (1) true***

Equality when $a = b = c = \frac{1}{\sqrt{3}} \Leftrightarrow A = B = C = 60^\circ$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left.\begin{array}{l}\cot A=x\\ \cot B=y\\ \cot C=z\end{array}\right\};I=\sum\frac{x\cdot y^3}{y^2+2x^2}+2\cdot\sum\frac{x^2y}{x+2y}\geq 1$$

$$\frac{x\cdot y^3}{y^2+2x^2}+\frac{2x^2y}{x+2y}=xy\cdot\left(\frac{y^2}{y^2+2x^2}+\frac{(2x)\cdot 2x}{(x+2y)\cdot 2x}\right)=$$

$$=xy\cdot\left(\frac{y^2}{y^2+2x^2}+\frac{(2x)^2}{2x^2+4xy}\right)\geq xy\cdot\left(\frac{(y+2x)^2}{(y+2x)^2}\right)=xy$$

$$I\geq\sum xy=\sum\cot A\cdot\cot B=1;\sum\cot A\cdot\cot B=1\ \textit{(ASSURE)}$$

$$\cot A\cdot(\cot B+\cot C)+\cot B\cdot\cot C=\cot A\cdot\frac{\sin(B+C)}{\sin B\sin C}+\cot B\cdot\cot C=$$

$$=\frac{\cos A}{\sin B\cdot\sin C}+\frac{\cos B\cdot\cos C}{\sin B\cdot\sin C}=\frac{\cos A+\cos B\cdot\cos C}{\sin B\cdot\sin C}=$$

$$=\frac{\cos B\cdot\cos C-\cos(B-C)}{\sin B\cdot\sin C}=\frac{\sin B\cdot\sin C}{\sin B\cdot\sin C}=1$$

170. **In acute-angled $\Delta\ ABC$ the following relationship holds:**

$$a^2b^2c^2\left(\sum a\cos A\right)\geq 8S^2\prod(a\cos A+b\cos B)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$a^2b^2c^2\left(\sum a\cos A\right)\geq 8S^2\prod(a\cos A+b\cos B)$$

Tener en cuenta la siguiente identidad "S" en un triángulo ABC:

$4RS=abc$ ***... la desigualdad es equivalente:***

$$\Rightarrow 2S^2R^3(\sin 2A+\sin 2B+\sin 2C)\geq$$

$$\geq R^3(\sin 2A+\sin 2B)(\sin 2B+\sin 2C)(\sin 2C+\sin 2A)$$

Dado qie es un triángulo acutángulo:

$$A, B, C \in < 0, \frac{\pi}{2} > \rightarrow \sin A, \sin B, \sin C > 0.$$

$$2(\sin 2A + \sin 2B + \sin 2C) = 8 \sin A \sin B \sin C \leq$$

$$\leq 8 \sin A \sin B \sin C \cos(A - B) \cos(B - C) \cos(C - A)$$

Lo cual es cierto ya que: $\cos(A - B) \cos(B - C) \cos(C - A) \leq 1$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$a \cos A + b \cos B = 2R \sin A \cos B + 2R \sin B \cos B$$

$$= R(\sin 2A + \sin 2B) = 2R \sin(A + B) \cos(A - B) =$$

$$= 2R \sin C \cos(A - B) = c \cos(A - B)$$

Similarly, $b \cos B + c \cos C = a \cos(B - C)$ ***and*** $c \cos C + a \cos A = b \cos(C - A)$

$$\therefore \prod(a \cos A + b \cos B) = (abc) \cos(A - B) \cos(B - C) \cos(C - A) \quad \textbf{(1)}$$

$$\because 0 < A, B, C < \frac{\pi}{2}, \therefore -\frac{\pi}{2} < A - B, B - C, C - A < \frac{\pi}{2}$$

$$\therefore 0 < \cos(A - B), \cos(B - C), \cos(C - A) \leq 1$$

$$\Rightarrow abc \geq abc \cos(A - B) \cos(B - C) \cos(C - A)$$

$$\Rightarrow abc \geq \prod(a \cos A + b \cos B) \quad \textbf{(2) (from (1))}$$

$\therefore$ ***it suffices to prove:*** $a^2b^2c^2(\sum a \cos A) \geq 8S^2abc$ ***(from (2))***

$$\Leftrightarrow 16R^2S^2\left(\frac{abc}{2R^2}\right) \geq 8S^2abc \Leftrightarrow 8S^2abc \geq 8S^2abc \quad \textbf{(true)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$RHS \geq LHS| \cdot 8S^2$$

$$2R^2 \cdot (\sum a \cdot \cos A) \geq \prod_{\Delta} \cdot (a \cos A + b \cos B) \quad \textbf{(*)}$$

$$\prod_{\Delta}(a \cos A + b \cos B) \underset{Cauchy}{\leq} \left(\frac{2 \cdot (a \cos A + b \cos B + c \cos C)}{3}\right)^3 =$$

$$= \frac{8}{27} \cdot (\sum a \cdot \cos A)^3 \quad \textbf{(**)}$$

(*); (**) $\Rightarrow 2k^2\cdot\left(\sum a\cos A\right)\ge\frac{8}{27}\left(\sum a\cos A\right)^3$, $\frac{27R^2}{4}\ge\left(\sum a\cdot\cos A\right)^2$

$\frac{3\sqrt{3}R}{2}\ge\sum a\cdot\cos A$ ***(ASSURE)***

$$\sum a\cdot\cos A=R\cdot\sum 2\cdot\frac{a}{2R}\cdot\cos A=R\cdot\sum\sin 2A=$$

$$=R\cdot(\sin 2A+\sin 2B+\sin 2C)\le\frac{3\sqrt{3}}{2}\cdot R$$

171. **Let a,b,c be positive real numbers such that $ab+bc+ca=3$. Prove that**

$$\frac{a^3+b^3}{b^2+bc+c^2}+\frac{b^3+c^3}{c^2+ca+a^2}+\frac{c^2+a^3}{a^2+ab+b^2}\ge 2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a,b,c ***números*** R^+ ***de tal manera que*** $ab+bc+ca=3$***. Probar que:*** $\frac{a^3+b^3}{a^2+ab+b^2}+\frac{b^3+c^3}{b^2+bc+c^2}+\frac{c^3+a^3}{c^2+ac+a^2}\ge 2$ ***… (A)***

Ahora bien se puede observar claramente que $\forall: x,y\in\mathbb{R}^+$***:***

1. $\frac{x^2+y^2-xy}{x^2+y^2+xy}\ge\frac{1}{3}\Leftrightarrow 2(x-y)^2\ge 0$***; 2.*** $x+y+z\ge\sqrt{3(xy+yz+zx)}$,

3. $(x+y)(y+z)(z+x)\ge\frac{8}{9}(x+y+z)(xy+yz+zx)\ \forall: x,y,z>0$

Desde que: $a,b,c>0$. ***Aplicando en … (A) → (MA ≥ MG)***

$$\sum\frac{a^3+b^3}{a^2+ab+b^2}\ge 3\sqrt[3]{\prod\left(\frac{a^2-ab+b^2}{a^2+ab+b^2}\right)\prod(a+b)}\ge$$

$$\ge 3\sqrt[3]{\frac{1}{27}\times\frac{8}{9}(a+b+c)(ab+bc+ca)}\ge 3\sqrt[3]{\frac{1}{27}\times\frac{8}{9}\times 3\times 3}=2$$

172. In ΔABC the following relationship holds:

$$\sum |2(\sin C + a\cos C)(\sin C + b\cos C) - 1 - ab| < 3 + a^2 + b^2 + c^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash - New Delhi – India

$$|2(\sin C + a\cos C)(\sin C + b\cos C) - 1 - ab| =$$

$$= \left|2\sin^2 C + 2ab\cos^2 C + 2(a+b)\cos C\sin C - 1 - ab\right| =$$

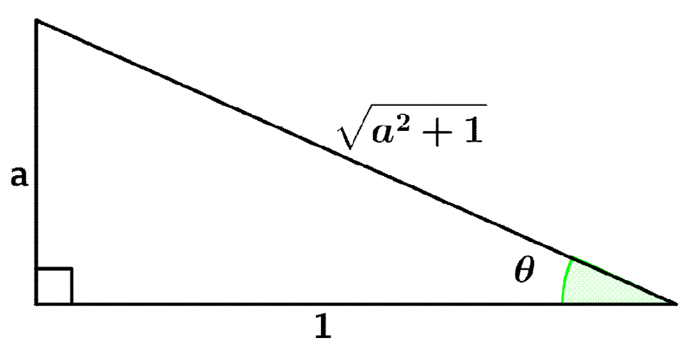
$$= |(ab-1)\cos 2C + (a+b)\sin 2C| \le \sqrt{(ab-1)^2 + (a+b)^2} =$$

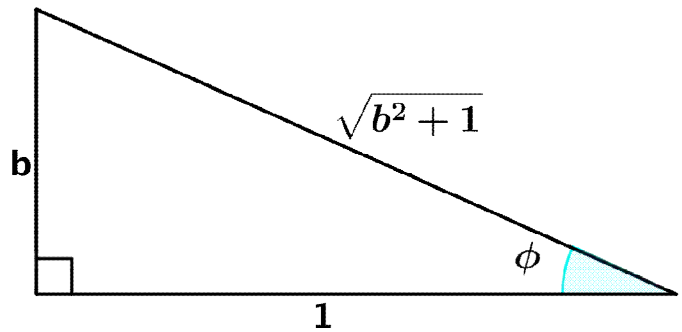
$$= \sqrt{(a^2+1)(b^2+1)} \le \frac{1}{2}[(a^2+1) + (b^2+1)] = \frac{1}{2}(a^2+b^2) + 1$$

Similarly for other two expressions.

Adding three expressions, we get desired inequality.

Solution 2 by Soumava Chakraborty-Kolkata-India





$$2(\sin C + a\cos C)(\sin C + b\cos C) - 1 - ab \quad (*)$$

$$= 2\left\{\left(\sqrt{a^2+1}\cos\theta\right)\sin C + \left(\sqrt{a^2+1}\sin\theta\right)\cos C\right\}$$

$$\left\{\left(\sqrt{b^2+1}\cos\theta\right)\sin C + \left(\sqrt{b^2+1}\sin\phi\right)\cos C\right\} -$$

$$-\underbrace{\left(\sqrt{a^2+1}\cos\theta\right)\left(\sqrt{b^2+1}\cos\phi\right)}_{1} - \underbrace{\left(\sqrt{a^2+1}+\sin\theta\right)}_{a}\underbrace{\left(\sqrt{b^2+1}\sin\phi\right)}_{b}$$

$$= 2\sqrt{a^2+1}\sqrt{b^2+1}\sin(\theta + C)\sin(\phi + C) -$$

$$-\sqrt{a^2+1}\sqrt{b^2+1}(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$= \sqrt{a^2+1}\sqrt{b^2+1}\{\cos(\theta - \phi) - \cos(\theta + \phi + 2C)\} -$$

$$-\sqrt{a^2+1}\sqrt{b^2+1}\cos(\theta - \phi) \quad (*)$$

$$= -\sqrt{(a^2+1)(b^2+1)}\cos(\theta + \phi + 2C)$$

$$\therefore |2(\sin C + a\cos C)(\sin C + b\cos C) - 1 - ab| \quad (**)$$

$$= \sqrt{(a^2+1)(b^2+1)}|\cos(\theta + \phi + 2C)| \le \sqrt{(a^2+1)(b^2+1)}$$

$$\overset{G\le A}{\le} \frac{(a^2+1)(b^2+1)}{2} \quad (**)$$

$$\therefore LHS \le \frac{\sum(a^2+1+b^2+1)}{2} = \frac{2\sum a^2 + 6}{2} = \sum a^2 + 3 = RHS$$

173. If in $\Delta\ ABC, a \neq b \neq c \neq a$ then:

$$\sum \frac{1+a^2}{a\sin A\,(a-b)(a-c)} > \frac{3}{2S\sqrt[3]{abc}}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = 2R\sum \frac{1}{a^2(a-b)(a-c)} + 2R\sum \frac{1}{(a-b)(a-c)}$$

$$\text{Now, } \sum \frac{1}{(a-b)(a-c)} = \frac{-(b-c)-(c-a)-(a-b)}{(a-b)(b-c)(c-a)} = 0$$

$$\sum \frac{1}{a^2(a-b)(a-c)} = \frac{-\{b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)\}}{a^2b^2c^2(a-b)(b-c)(c-a)}$$

$$= -\frac{b^3(c^2-a^2)+c^2a^2(c-a)-b^2(c^3-a^3)}{a^2b^2c^2(a-b)(b-c)(c-a)}$$

$$= -(c-a)\frac{b^2c(b-c)+ab^2(b-c)-a^2(b^2-c^2)}{a^2b^2c^2\prod(a-b)}$$

$$= -(c-a)(b-c)\frac{c(b^2-a^2)+ab(b-a)}{a^2b^2c^2\prod(a-b)}$$

$$= \frac{(a-b)(b-c)(c-a)(\sum ab)}{a^2b^2c^2(a-b)(b-c)(c-a)} \overset{A-G}{>} \frac{3\sqrt[3]{a^2b^2c^2}}{a^2b^2c^2}$$

$$\therefore LHS > \frac{3\sqrt[3]{a^2b^2c^2}}{a^2b^2c^2}\cdot 2R \therefore \textit{it suffices to prove that: } 2R\left(\frac{3\sqrt[3]{a^2b^2c^2}}{a^2b^2c^2}\right) \geq \frac{3}{2S\sqrt[3]{abc}}$$

$$\Leftrightarrow \frac{2R(abc)}{a^2b^2c^2} \geq \frac{1}{2S} \Leftrightarrow \frac{2R}{4RS} \geq \frac{1}{2S} \Leftrightarrow \frac{1}{2S} \geq \frac{1}{2S}$$ *(true) (Proved)*

174. Prove that if in $\Delta ABC; a \neq b \neq c \neq a$ then:

$$\frac{2p+a}{(b-a)(c-a)\sin A} + \frac{2p+b}{(c-b)(a-b)\sin B} + \frac{2p+c}{(a-c)(b-c)\sin C} > \frac{2}{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Daniel Sitaru – Romania

$$\sum\frac{2p+a}{(b-a)(c-a)\sin A}=\sum\frac{2p+a}{(b-a)(c-a)\cdot\frac{2S}{bc}}=$$

$$=\frac{1}{2S}\sum\frac{(2p+a)bc}{(b-a)(c-a)}=$$

$$=\frac{1}{2S}\left(\frac{(2p+a)bc}{(a-b)(a-c)}+\frac{(2p+b)ac}{(b-a)(b-c)}+\frac{(2p+c)ab}{(c-a)(c-b)}\right)=$$

$$=\frac{1}{2S}\cdot\frac{(2p+a)bc(b-c)-(2p+b)ac(a-c)+(2p+c)ab(a-b)}{(a-b)(a-c)(b-c)}=$$

$$=\frac{1}{2S}\cdot\frac{2p(b^2c-bc^2-a^2c+ac^2+a^2b-ab^2)}{b^2c-bc^2-a^2c+ac^2+a^2b-ab^2}=\frac{2p}{2S}=\frac{p}{S}=\frac{p}{pr}=\frac{1}{r}>\frac{2}{R}$$

Solution 2 by Myagmarsuren Yadamsuren – Mongolia

$$\Delta\ ABC:\ \sum\frac{2s+a}{(b-a)(c-a)\sin A}>\frac{2}{R}\ (*)$$

$$\left.\begin{matrix}(b-a)(c-a)<2bc\\(c-b)(a-b)<2ca\\(a-c)(b-c)<2ab\end{matrix}\right\}\text{(Assure)}$$

$$(b-a)(c-a)=bc-a(b+c)+a^2<2bc$$

$$a\cdot\big(a-(b+c)\big)<bc\ \textbf{(True)};\ b+c>a;\ a\cdot\big(a-(b+c)\big)<0$$

$$bc>0;\ (*)\Rightarrow$$

$$LHS>\frac{2s+a}{2bc\cdot\sin A}+\frac{2s+b}{2ca\cdot\sin B}+\frac{2s+c}{2ab\cdot\sin C}=\frac{6s+2s}{4s}=$$

$$=\frac{8s}{4s}=\frac{2s}{s}=\frac{2}{r}>\frac{2}{r}\ \textbf{(True)}$$

175. In $\Delta\ ABC$ the following relationship holds:

$$\frac{1}{\sin^4\frac{A}{2}}+\frac{1}{\sin^4\frac{B}{2}}+\frac{1}{\sin^4\frac{C}{2}}\geq\frac{(12r)^4}{a^4+b^4+c^4}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$(a^4+b^4+c^4)\left(\csc^4\frac{A}{2}+\csc^4\frac{B}{2}+\csc^4\frac{C}{2}\right)\geq(12r)^4$$

1. De la siguiente desigualdad en un triángulo ABC:

$$\frac{a^2+b^2+c^2}{4S}=\cot A+\cot B+\cot C\geq\sqrt{3}\rightarrow$$

$$\rightarrow a^2+b^2+c^2\geq 4S\sqrt{3}\ \wedge\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\frac{p}{r}\geq 3\sqrt{3}\rightarrow p\geq 3\sqrt{3}r$$

Por la desigualdad Cauchy:

$$\Rightarrow 3(a^4+b^4+c^4)\geq(a^2+b^2+c^2)^2\geq 48S^2\rightarrow a^4+b^4+c^4\geq 16S^2=$$

$$=16p^2r^2\geq 16\times 27r^4=432r^4\ \ldots\ (A)$$

2. Asimismo se cumple otra desigualdad en un triángulo ABC:

$$\csc^4\frac{A}{2}+\csc^4\frac{B}{2}+\csc^4\frac{C}{2}\geq\frac{\left(\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}\right)^2}{3}\geq 48\ \ldots\ (B)$$

Multiplicando $(A\times B)$ ...

$$\Rightarrow(a^4+b^4+c^4)\left(\csc^4\frac{A}{2}+\csc^4\frac{B}{2}+\csc^4\frac{C}{2}\right)\geq(432\times 48)r^4=(12r)^4\ \ldots$$

176. In $\Delta\ ABC$ the following relationship holds:

$$\prod(5m_a+3m_b)(3m_a+5m_b)<64\prod(2s+a)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt{(5m_a+3m_b)(3m_a+5m_b)} \overset{G-A}{\leq} \frac{8(m_a+m_b)}{2}$$

$$\Rightarrow (5m_a+3m_b)(3m_a+5m_b) \leq 16(m_a+m_b)^2 \quad \textbf{(1)}$$

$$\textbf{Now, } m_a^2 = \frac{2b^2+2c^2-a^2}{4} = \frac{(b+c)^2+(b-c)^2-a^2}{4} = \frac{(b+c)^2+(b-c+a)(b-c-a)}{4}$$

$$< \frac{(b+c)^2}{4} \quad \begin{pmatrix} \because a+b>c, \therefore a+b-c>0 \\ \because b<c+a, \therefore b-c-a<0 \end{pmatrix} \Rightarrow m_a < \frac{b+c}{2}$$

$$\textbf{Similarly, } m_b < \frac{c+a}{2} \textbf{ and } m_c < \frac{a+b}{2}$$

$$\therefore (5m_a+3m_b)(3m_a+5m_b) \leq 16(m_a+m_b)^2 \quad \textbf{(from (1))}$$

$$\underset{(a)}{<} 16\left(\frac{b+c}{2}+\frac{c+a}{2}\right)^2 = 4(a+b+c+c)^2 = 4(2s+c)^2$$

$$\textbf{Similarly, } (5m_b+3m_c)(3m_b+5m_c) < 4(2s+a)^2 \quad \textbf{(b)}$$

$$\textbf{and } (5m_c+3m_a)(3m_c+5m_a) < 4(2s+b)^2 \quad \textbf{(c)}$$

$$\textbf{(a)} \times \textbf{(b)} \times \textbf{(c)} \Rightarrow \prod(5m_a+3m_b)(3m_a+5m_b)$$

$$< 4(2s+c)^2 \cdot 4(2s+a)^2 \cdot 4(2s+b)^2 = 64\prod(2s+a)^2$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\textbf{1. } \Delta ABC: \left.\begin{matrix} a+b>c \\ b+c>a \\ c+a>b \end{matrix}\right\} \Rightarrow 2s+a = (b+a)+(a+c) > b+c$$

$$\textbf{Similarly: } \begin{matrix} 2s+b > a+c \\ 2s+c > b+a \end{matrix}$$

$$b+c \geq 2\sqrt{bc};\ a+c \geq 2\sqrt{ac};\ a+b \geq 2\sqrt{ab}$$

$$64 \cdot \prod(2s+a)^2 > 64 \cdot \prod\left(2\sqrt{ab}\right)^2 = 64 \cdot 8^2 \cdot a^2b^2c^2 = 16^3a^2b^2c^2$$

$$\textbf{2. } \prod(5m_a+3m_b)\cdot(3m_a+5m_b) = \prod(15m_a^2+34m_a\cdot m_b+15m_c^2) =$$

$$\underset{Cauchy}{\geq}\prod\left(15m_a^2+34\cdot\left(\frac{m_a^2+m_b^2}{2}\right)+15m_b^2\right)=\prod 32^3(m_a^2+m_b^2)=$$

$$=32^3\cdot\prod\left(\frac{b^2+c^2-a^2}{4}+\frac{a^2+c^2-b^2}{4}\right)=32^3\cdot\prod\frac{2^3c^2}{4^3}=16^3a^2b^2c^2$$

$$LHS\leq 16^3a^2b^2c^2<RHS$$

Solution 3 by Soumitra Mandal - Chandar Nagore – India

$$(5m_a+3m_b)+(3m_a+5m_b)=15(m_a^2+m_b^2)+34m_am_b$$

$$<\frac{15}{4}(4c^2+a^2+b^2)+\frac{34}{4}(2c^2+ab)$$

$$\because m_a^2=\frac{2b^2+2c^2-a^2}{4},\ m_b^2=\frac{a^2+c^2}{2}-\frac{b^2}{4}\text{ and } m_am_b\leq\frac{2c^2+ab}{4}$$

need to prove, $32c^2+\frac{15}{4}(a^2+b^2)+\frac{17}{2}ab<4(2s+c)^2$

$$\Leftrightarrow 16c^2+\frac{ab}{2}<\frac{a^2+b^2}{4}+16c(a+b),\text{ which is true}$$

since, $c<a+b$ ***and*** $a^2+b^2\geq 2ab$

$$\therefore\prod_{cyc}(5m_a+3m_b)(3m_a+5m_b)<64\prod_{cyc}(2s+c)^2$$

177. In $\Delta\, ABC$:

$$\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\leq\frac{3R}{2r}.$$

Proposed by George Apostolopoulos – Messolonghi - Greece

Solution by Kevin Soto – Palacios – Huarmey – Peru

Probar en un triángulo ABC***:*** $\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\leq\frac{3R}{2r}.$

Por la desigualdad de Holder:

$$(a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)(1+1+1)\geq$$

$$\geq\left(\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\right)^3 \dots (A)$$

Ahora recordar las siguientes identidades y desigualdad en un triángulo ABC, se cumple lo siguiente:

1. $(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)=r(4R+r)$

2. $S^2=s^2r^2=s(s-a)(s-b)(s-c)$; ***3.*** $R\geq 2r$

Por la tanto, en (A) se tiene lo siguiente: $s\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)=$

$$=s^2\left(\frac{(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)}{s(s-a)(s-b)(s-c)}\right)=\frac{(4R+r)}{r}\leq$$

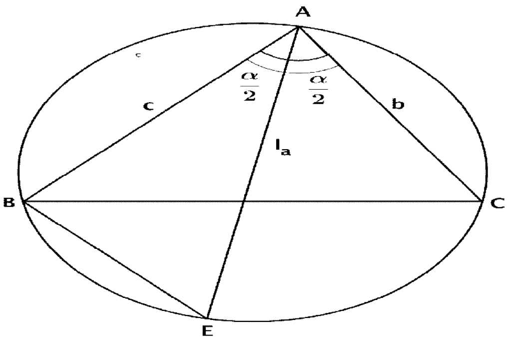
$\leq\frac{9R}{2r}\leq\frac{9R^2}{4r^2}\leq\frac{9R^3}{8r^3}$***. Por la transitividad tenemos en … (A):***

$$\frac{27R^3}{8r^3}\geq(a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)(3)\geq$$

$$\geq\left(\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\right)^3$$

$$\Rightarrow\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\leq\frac{3R}{2r}$$

178.



$AB = c; AC = b; BC = a$. **Prove that:** $BE = \frac{abc}{l_a(b+c)}$

Proposed by Adil Abdullayev – Baku – Azerbaidian

Solution by Daniel Sitaru – Romania

$$AE \cap BC = \{F\}, l_a \cdot FE = BF \cdot FC = \rho(F)$$

$$FE = \frac{a^2bc}{l_a(b+c)^2}, \frac{b}{\sin\left(C + \frac{A}{2}\right)} = \frac{l_a}{\sin C}$$

$$\frac{FE}{BE} = \frac{\sin\frac{A}{2}}{\sin\left(C + \frac{A}{2}\right)} \rightarrow \frac{a^2bc}{l_a(b+c)^2} = \frac{BE\sin\frac{A}{2}}{\sin\left(C + \frac{A}{2}\right)}$$

$$BE = \frac{a^2b^2c\sin C}{l_a^2(b+c)^2\sin\frac{A}{2}} = \frac{a^2b^2c \cdot \frac{2S}{ab}}{l_a^2(b+c)^2\sin\frac{A}{2}} = \frac{abc}{l_a(b+c)}$$

179. **In acute-angled** $\Delta\ A_1B_1C_1, \Delta\ A_2B_2C_2, \Delta\ A_3B_3C_3$:

$$2\left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2 < \sum a_1^2 + \sum a_2^2 + \sum a_3^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey - Peru

Probar en un triángulo $\Delta\ A_1B_1C_1, \Delta\ A_1B_2C_2, \Delta\ A_3B_3C_3$:

$$2\left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3}\right)^2<\sum a_1^2+\sum a_2^2+\sum a_3^2$$

Por la desigualdad de Cauchy:

$$2\left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3}\right)^2\le 2(S_1+S_2+S_3)(1+1+1)=6(S_1+S_2+S_3)$$

1. Ahora bien en un triángulo xyz se cumple la siguiente desigualdad:

$\sum a_x^2\ge 4\sqrt{3}S_x$***. Por la tanto:***

$\sum a_1^2+\sum a_2^2+\sum a_3^2\ge 4\sqrt{3}(S_1+S_2+S_3)>6(S_1+S_2+S_3)$ ***… (LQQD)***

Solution 2 by Soumitra Mandal - Chandar Nagore – India

In acute $\Delta\ A_kB_kC_k$ where $k=1,2,3$

$$2\left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3}\right)^2<\sum_{cyc}a_1^2+\sum_{cyc}a_2^2+\sum_{cyc}a_3^2$$

We know, $\left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3}\right)^2<\left(\sum_{k=1}^3 S_k\right)$

now, $\frac{a_1b_1c_1}{8}\ge(p_1-a_1)(p_1-b_1)(p_1-c_1)$ ***where*** $2p_1=a_1+b_1+c_1$

$\therefore\ \frac{1}{4}\sqrt{a_1b_1c_1(a_1+b_1+c_1)}\ge S_1$***. Similarly,*** $\frac{1}{4}\sqrt{a_2b_2c_2(a_2+b_2+c_2)}\ge S_2$

and $\frac{1}{4}\sqrt{a_3b_3c_3(a_3+b_3+c_3)}\ge S_3$***. Now we know,***

$xy+yz+zx\ge\sqrt{3xyz(x+y+z)}$ ***so,***

$$\sum_{k=1}^3 S_k\le\frac{1}{4\sqrt{3}}\left(\sum_{k=1}^3(a_kb_k+b_kc_k+c_ka_k)\right)\le\frac{1}{4\sqrt{3}}\left(\sum_{k=1}^3(a_k^2+b_k^2+c_k^2)\right)$$

$$\therefore\ 2\left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3}\right)^2\le 6\left(\sum_{k=1}^3 S_k\right)\le\frac{\sqrt{3}}{2}\left(\sum_{k=1}^3(a_k^2+b_k^2+c_k^2)\right)$$

$$<\sum_{cyc}a_1^2+\sum_{cyc}a_2^2+\sum_{cyc}a_3^2$$

Solution 3 by Soumava Chakraborty-Kolkata-India

In any 3 triangles, $A_1B_1C_1, A_2B_2C_2, A_3B_3C_3$,

$$2\left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3}\right)^2 \overset{(i)}{<} \sum a_1^2 + \sum a_2^2 + \sum a_3^2$$

$$2\left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3}\right)^2 \le 6(S_1+S_2+S_3) \quad \textit{(1)}$$

$$\left(\because \left(\sum x\right)^2 \le 3\sum x^2\right)$$

we shall prove that in any $\Delta\, ABC$ with area S,

$$\sum ab \ge 4\sqrt{3}S \Leftrightarrow s^2 + r(4R+r) \ge 4\sqrt{3}S$$

But $s^2 + r(4R+r) \ge s\left(3\sqrt{3}r\right) + r\left(s\sqrt{3}\right)$

($\because s \ge 3\sqrt{3}r$ ***and*** $4R + r \ge s\sqrt{3}$ ***(Trucht))***

$$\Rightarrow s^2 + r(4R+r) \ge 4\sqrt{3}rs = 4\sqrt{3}S$$

$$\therefore 6(S_1+S_2+S_3) < \left(4\sqrt{3}S_1 + 4\sqrt{3}S_2 + 4\sqrt{3}S_3\right)$$

$$\le \sum a_1b_1 + \sum a_2b_2 + \sum a_3b_3 \underset{(2)}{\le} \sum a_1^2 + \sum a_2^2 + \sum a_3^2$$

So, (1), (2) proves (i). So, the stronger inequality (*) would be

$$2\left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3}\right)^2 < \sum a_1 b_1 + \sum a_2b_2 + \sum a_3b_3$$

180. In $\Delta\, A_1B_1C_1, \Delta\, A_2B_2C_2$:

$$3\left(\sum a_1^2 + \sum a_2^2\right) + 6\sqrt{\left(\sum a_1^2\right)\left(\sum a_2^2\right)} \ge 16 s_1 s_2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nirapada Pal – India

$$3\left(\sum a_1^2 + \sum a_2^2\right) + 6\sqrt{\left(\sum a_1^2\right)\left(\sum a_2^2\right)} \ge 9\left[\frac{\sum a_1^2}{3} + \frac{\sum a_2^2}{3}\right] + 6\sum a_1a_2 \ge$$

$$\geq 9\left[\left(\frac{\sum a_1}{3}\right)^2+\left(\frac{\sum a_2}{3}\right)^2\right]+18\frac{\sum a_1a_2}{3}\geq 4(s_1^2+s_2^2)+18\,\frac{\sum a_1}{3}\cdot\frac{\sum a_2}{3}\geq$$

$$\geq 8s_1s_2+8s_1s_2=16s_1s_2$$

Solution 2 by Myagmarsuren Yadamsuren – Darkhan – Mongolia

$$3\cdot\left(\sum a_1^2+\sum a_2^2+2\cdot\sqrt{\sum a_1^2}\cdot\sqrt{\sum a_2^2}\right)=$$

$$=3\cdot\left(\sqrt{\sum a_1^2}+\sqrt{\sum a_2^2}\right)^2=3\cdot\left(\frac{\sqrt{(1^2+1^2+1^2)\cdot\sum a_1^2}}{\sqrt{3}}+\frac{\sqrt{(1^2+1^3+1^2)\cdot\sum a_2^2}}{\sqrt{2}}\right)^2\geq$$

$$\geq\left(\sum a_1+\sum a_2\right)^2=4\cdot(s_1+s_2)^2\geq 4\cdot\left(2\cdot\sqrt{s_1s_2}\right)^2=16s_1\cdot s_2$$

Solution 3 by Vijay Rana – Kapurthala – India

$$LHS=3\left(\sum a_1^2+\sum a_2^2\right)+6\sqrt{\sum a_1^2\sum a_2^2}\geq 12\sqrt{\sum a_1^2\cdot\sum a_2^2}$$

$$\geq 12\sqrt{\left(\sum a_1a_2\right)^2}=12\sum a_1\,a_2\ \rightarrow \textbf{by Cauchy}$$

$$=4\cdot\left(3\sum a_1a_2\right)\geq 4\cdot\sum a_1\sum a_2\rightarrow \textbf{by Chebyshev's inequality}$$

$$=4\cdot 2s_1\cdot 2s_2=16s_1s_2=RHS$$

Solution 4 by SK Rejuan-West Bengal-India

$$\Delta\ A,B,C\ \&\ \Delta\ A_2B_2C_2$$

$$3\left(\sum a_1^2+\sum a_2^2\right)\geq 3\left\{\frac{(\sum a_1)^2}{3}+\frac{(\sum a_2)^2}{2}\right\}$$

by mth power theorem

$$\Rightarrow 3\left(\sum a_1^2+\sum a_2^2\right)\geq\left(\sum a_1\right)^2+\left(\sum a_2\right)^2\geq 4\left(\sum a_1\right)\left(\sum a_2\right)$$

by AM ≥ GM

$$\Rightarrow 3\left(\sum a_1^2+\sum a_2^2\right)\geq 4\cdot\sum a_1\cdot\sum a_2=4\cdot 2s_1\cdot 2s_2=16s_1s_2$$

$$\Rightarrow 3\left(\sum a_1^2 + \sum a_2^2\right) \geq 16 s_1 s_2 \quad (1)$$

181. In $\Delta\, ABC$:

$$\text{(a) } \left(\frac{3R}{2r}\right)^2 \geq \frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{c}} + \frac{\sqrt{c}}{\sqrt{a}} + 6$$

$$\text{(b) } \frac{\sqrt{a}}{a^3 \cdot b} + \frac{\sqrt{b}}{b^3 \cdot c} + \frac{\sqrt{c}}{c^3 \cdot a} \leq \frac{1}{16r^3}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: A) $\left(\frac{3R}{2r}\right)^2 \geq \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 6$

Recordar las siguientes identidades en un triángulo ABC:

$$\frac{R}{2r} = \frac{abc}{8(s-a)(s-b)(s-c)}, r_a = \frac{S}{s-a}, r_b = \frac{S}{s-b}; r_c = \frac{S}{s-c}$$

Dado que: a, b, c ***son lados de un triángulo*** $\rightarrow$

$$\rightarrow (b+c-a)\cdot(a+b-c)\cdot(c+a-b) > 0.$$

Realizamos lo siguientes cambios de variables:

$$s-a = x > 0, s-b = y > 0, s-c = z > 0 \rightarrow c = x+y, a = y+z, b = z+x$$

La desigualdad es equivalente: $\left(\frac{3(x+y)(y+z)(z+x)}{8xyz}\right)^2 \geq \frac{y}{x} + \frac{z}{y} + \frac{x}{z} + 6$

$$\Rightarrow \frac{9}{64}\left(\frac{x+y}{z} + \frac{y+z}{x} + \frac{z+x}{y} + 2\right)^2 = \frac{9}{64}\left(\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right) + \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + 2\right)\right)^2 \geq$$

$$\geq \frac{9}{64}\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z} + 5\right)^2 \geq \frac{y}{x} + \frac{z}{y} + \frac{x}{z} + 6$$

Desde que: $x, y, z > 0 \rightarrow$ ***Sea:*** $m = \frac{y}{x} + \frac{z}{y} + \frac{x}{z} \geq 3; m \geq 3 \rightarrow$

→ ***(Válido por: MA ≥ MG)***

Por la tanto: $9(m+5)^2 \geq 64(m+6) \Leftrightarrow (m-3)(9m+53) \geq 0 \rightarrow$

→ ***lo cual es cierto ya que:*** $m \geq 3$ ***... (LQQD)***

Probar en un triángulo ABC: B) $\frac{r_a}{a^3b}+\frac{r_b}{b^3c}+\frac{r_c}{c^3a} \leq \frac{1}{16r^3}$

1) Tener en cuenta las siguientes identidades y desigualdades en un triángulo ABC

$$Sr=(s-a)(s-b)(s-c),\ abc=4sRr,\ r_a=\frac{S}{s-a},\ r_b=\frac{S}{s-b},\ r_c=\frac{S}{s-c},$$

$R \geq 2r$. ***La desigualdad es equivalente:***

$$\frac{r_a}{a^3b}+\frac{r_b}{b^3c}+\frac{r_c}{c^3a}=\frac{S}{a^3b^3c^3}\left(\frac{b^2c^3(s-b)(s-c)+c^2a^3(s-c)(s-a)+a^2b^3(s-a)(s-b)}{(s-a)(s-b)(s-c)}\right) \text{ ... (A)}$$

Desde que:

$$b^2c^3(s-b)(s-c)=\frac{b^2c^3(a+c-b)(a+b-c)}{4}=\frac{b^2c^3(a^2-(b-c)^2)}{4} \leq \frac{b^2c^3a^2}{4} \text{ ... (I)}$$

Por la tanto:

$$c^2a^3(s-c)(s-a) \leq \frac{c^2a^3b^2}{4} \text{ ... (II)} \wedge a^2b^3(s-a)(s-b) \leq \frac{a^2b^3c^2}{4} \text{ ... (III)}$$

Luego sumando (I) + (II) + (III), tenemos en (A):

$$\frac{S}{a^3b^3c^3}\left(\frac{b^2c^3(s-b)(s-c)+c^2a^3(s-c)(s-a)+a^2b^3(s-a)(s-b)}{(s-a)(s-b)(s-c)}\right) \leq$$

$$\leq \frac{S}{4abc}\left(\frac{a+b+c}{(s-a)(s-b)(s-c)}\right)=\frac{1}{4R}\left(\frac{a+b+c}{abc}\right)$$

$$\Rightarrow \frac{r_a}{a^3b}+\frac{r_b}{b^3c}+\frac{r_c}{c^3a} \leq \frac{1}{4r} \times \frac{2s}{4sRr}=\frac{1}{8Rr^2} \leq \frac{1}{16r^3} \text{ ... (LQQD)}$$

182. In $\Delta\ ABC$ the following relationship holds:

$$\frac{\sin A \sin B}{m_am_b}+\frac{\sin B \sin C}{m_bm_c}+\frac{\sin C \sin A}{m_cm_a} \geq \frac{1}{R^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Given inequality} \Leftrightarrow \frac{ab}{m_am_b}+\frac{bc}{m_bm_c}+\frac{ca}{m_cm_a}\geq 4$$

$$\because m_am_b\leq\frac{2c^2+ab}{4}, \therefore \frac{ab}{m_am_b}\geq\frac{4ab}{2c^2+ab}. \text{ Similarly, } \frac{bc}{m_bm_c}\geq\frac{4bc}{2a^2+bc} \text{ and}$$

$$\frac{ca}{m_cm_a}\geq\frac{4ca}{2b^2+ca} \therefore \text{it suffices to prove that: } \frac{ab}{2c^2+ab}+\frac{bc}{2a^2+bc}+\frac{ca}{2b^2+ca}\geq 1$$

$$\Leftrightarrow ab(2a^2+bc)(2b^2+ca)+bc(2b^2+ca)(2c^2+ab)+$$

$$+ca(2c^2+ab)(2a^2+bc)-(2c^2+ab)(2a^2+bc)(2b^2+ca)\geq 0$$

$$\Leftrightarrow 2a^4bc+2ab^4c+2abc^4-6a^2b^2c^2\geq 0$$

$$\Leftrightarrow a^3+b^3+c^3\geq 3abc \rightarrow \text{true by AM-GM}$$

183. In a triangle ABC with $BC=a, CA=b, AB=c$. l_a, l_b, l_c is length of bisector down from A, B, C.

$$\text{Prove that: } \frac{1}{l_al_b}+\frac{1}{l_bl_c}+\frac{1}{l_cl_a}\geq\frac{4}{3}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Hoang Le Nhat Tung – Hanoi – Vietnam

*** We have:**

$$l_a=\frac{2bc\cdot\cos\frac{A}{2}}{b+c}=\frac{2bc}{b+c}\cdot\sqrt{\frac{1+\cos A}{2}}=\frac{2bc}{b+c}\cdot\sqrt{\frac{1+\frac{b^2+c^2-a^2}{2bc}}{2}}=$$

$$=\frac{2bc}{b+c}\cdot\sqrt{\frac{(b+c)^2-a^2}{4bc}}$$

$$\Leftrightarrow l_a=\frac{2bc}{b+c}\cdot\sqrt{\frac{(a+b+c)(b+c-a)}{4bc}}=\frac{\sqrt{bc(a+b+c)(b+c-a)}}{b+c} \quad (2)$$

+ Similar: $l_b = \frac{\sqrt{ca(a+b+c)(c+a-b)}}{c+a}$ **(3)**

- Since (2), (3): $\Rightarrow l_a l_b = \frac{\sqrt{bc(a+b+c)(b+c-a)}}{b+c} \cdot \frac{\sqrt{ca(a+b+c)(c+a-b)}}{c+a}$

$$\Leftrightarrow l_a l_b = \frac{c(a+b+c)\cdot\sqrt{ab(b+c-a)(c+a-b)}}{(b+c)(c+a)} \Leftrightarrow \frac{1}{l_a l_b} = \frac{(b+c)(c+a)}{c(a+b+c)\sqrt{ab(b+c-a)(c+a-b)}} \quad (4)$$

+ Similar: $\frac{1}{l_b l_c} = \frac{(c+a)(a+b)}{a(a+b+c)\sqrt{bc(c+a-b)(a+b-c)}}$ **(5)**

$$\frac{1}{l_c l_a} = \frac{(a+b)(b+c)}{b(a+b+c)\sqrt{ca(a+b-c)(b+c-a)}} \quad (6)$$

- Since (4), (5), (6):

$$\Rightarrow \frac{1}{l_a l_b} + \frac{1}{l_b l_c} + \frac{1}{l_c l_a} = \frac{(b+c)(c+a)}{c(a+b+c)\sqrt{ab(b+c-a)(c+a-b)}} +$$

$$+ \frac{(c+a)(a+b)}{a(a+b+c)\sqrt{bc(c+a-b)(a+b-c)}} + \frac{(a+b)(b+c)}{b(a+b+c)\sqrt{ca(a+b-c)(b+c-a)}} \quad (7)$$

- Since (1), (7). We need to prove:

$$\frac{(b+c)(c+a)}{c(a+b+c)\sqrt{ab(b+c-a)(c+a-b)}} + \frac{(c+a)(a+b)}{a(a+b+c)\sqrt{bc(c+a-b)(a+b-c)}} +$$

$$+ \frac{(a+b)(b+c)}{b(a+b+c)\sqrt{ca(a+b-c)(b+c-a)}} \geq \frac{4}{3}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)$$

$$\Leftrightarrow \frac{(b+c)(c+a)}{c(a+b+c)\sqrt{ab(b+c-a)(c+a-b)}} + \frac{(c+a)(a+b)}{a(a+b+c)\sqrt{bc(c+a-b)(a+b-c)}} +$$

$$+ \frac{(a+b)(b+c)}{b(a+b+c)\sqrt{ca(a+b-c)(b+c-a)}} \geq \frac{4(a+b+c)}{3abc}$$

$$\Leftrightarrow (b+c)(c+a)\cdot\sqrt{\frac{ab}{(b+c-a)(c+a-b)}} + (c+a)(a+b)\cdot\sqrt{\frac{bc}{(c+a-b)(a+b-c)}} +$$

$$+ (a+b)(b+c)\cdot\sqrt{\frac{ca}{(a+b-c)(b+c-a)}} \geq \frac{4(a+b+c)^2}{3} \quad (8)$$

- Since Inequality AM-GM for 3 positive real numbers we have:

$$(b+c)(c+a)\cdot\sqrt{\frac{ab}{(b+c-a)(c+a-b)}}+(c+a)(a+b)\cdot\sqrt{\frac{bc}{(c+a-b)(a+b-c)}}+$$

$$+(a+b)(b+c)\cdot\sqrt{\frac{ca}{(a+b-c)(b+c-a)}}\geq 3\cdot\sqrt[3]{\frac{(a+b)^2(b+c)^2(c+a)^2\cdot abc}{(b+c-a)(c+a-b)(a+b-c)}}\quad (9)$$

We will prove:

$$3\cdot\sqrt[3]{\frac{(a+b)^2(b+c)^2(c+a)^2\cdot abc}{(b+c-a)(c+a-b)(a+b-c)}}\geq\frac{4(a+b+c)^2}{3}\qquad (10)$$

$$\Leftrightarrow\frac{27(a+b)^2(b+c)^2(c+a)^2\cdot abc}{(b+c-a)(c+a-b)(a+b-c)}\geq\frac{64(a+b+c)^6}{27}$$

$$\Leftrightarrow 27^2\cdot abc(a+b)^2(b+c)^2(c+a)^2>64(b+c-a)(c+a-b)(a+b-c)(a+b+c)^6$$

(11)

- Since Inequality AM-GM for 2 positive real numbers we have

$$(a+b)(b+c)(c+a)\geq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}=8\sqrt{(abc)^2}=8abc\Leftrightarrow$$

$$\Leftrightarrow abc\leq\frac{(a+b)(b+c)(c+a)}{8}\qquad (12)$$

- Therefore, since (12):

$$\Rightarrow(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc\leq$$

$$\leq(a+b)(b+c)(c+a)+\frac{(a+b)(b+c)(c+a)}{8}=\frac{9(a+b)(b+c)(c+a)}{8}$$

$$\Leftrightarrow(a+b+c)(ab+bc+ca)\leq\frac{9(a+b)(b+c)(c+a)}{8}$$

$$\Leftrightarrow(a+b)(b+c)(c+a)\geq\frac{8(a+b+c)(ab+bc+ca)}{9}$$

$$\Leftrightarrow(a+b)^2(b+c)^2(c+a)^2\geq\frac{64(a+b+c)^2(ab+bc+ca)^2}{81}\qquad (13)$$

- Since Inequality AM-GM for 2 positive real numbers we have:

$$(ab)^2+(bc)^2+(ca)^2=\frac{(ab)^2+(bc)^2}{2}+\frac{(bc)^2+(ca)^2}{2}+\frac{(ca)^2+(ab)^2}{2}\geq$$

$$\geq\frac{2ab\cdot bc}{2}+\frac{2bc\cdot ca}{2}+\frac{2ca\cdot ab}{2}=ab^2c+abc^2+a^2bc=abc(a+b+c)$$

$$\Leftrightarrow (ab)^2+(bc)^2+(ca)^2+2abc(a+b+c)\geq 3abc(a+b+c)$$

$$\Leftrightarrow (ab+bc+ca)^2\geq 3abc(a+b+c)$$

- Since (13).(14): $\Rightarrow (a+b)^2(b+c)^2(c+a)^2\geq \frac{64(a+b+c)^2\cdot 3abc(a+b+c)}{81}$

$$\Leftrightarrow 27(a+b)^2(b+c)^2(c+a)^2\geq 64abc(a+b+c)^3 \qquad \textbf{(15)}$$

$$\Leftrightarrow 27^2abc(a+b)^2(b+c)^2(c+a)^2\geq 27\cdot 64a^2b^2c^2(a+b+c)^3 \qquad \textbf{(16)}$$

*** Since (11), (16). We need to prove:**

$$27\cdot 64\, a^2b^2c^2(a+b+c)^3\geq 64(b+c-a)(c+a-b)(a+b-c)(a+b+c)^6$$

$$\Leftrightarrow 27a^2b^2c^2\geq (b+c-a)(c+a-b)(a+b-c)(a+b+c)^3 \qquad \textbf{(17)}$$

- Put $\begin{cases} b+c-a=2x\\ c+a-b=2y;\ (x,y,z>0)\\ a+b-c=2z\end{cases} \Leftrightarrow \begin{cases} a=y+z\\ b=z+x;\Rightarrow a+b+c=2(x+y+z)\\ c=x+y\end{cases}$

(17) $:\Leftrightarrow 27(y+z)^2(z+x)^2(x+y)^2\geq (2x)\cdot(2y)\cdot(2z)\cdot\big(2(x+y+z)\big)^3$

$$\Leftrightarrow 27(x+y)^2(y+z)^2(z+x)^2\geq 64xyz(x+y+z)^3 \qquad \textbf{(18)}$$

$+(a,b,c)\Rightarrow(x,y,z)$ **since (15)** $:\Rightarrow$

$$\Rightarrow 27(x+y)^2(y+z)^2(z+x)^2\geq 64xyz(x+y+z)^3$$

$\Rightarrow$ **Inequality (18) True** $\Rightarrow$ **(17) True.**

+ Since (16), (17): $\Rightarrow$

$$\Rightarrow 27^2abc(a+b)^2(b+c)^2(c+a)^2\geq 64(b+c-a)(c+a-b)(a+b-c)(a+b+c)^6$$

$\Rightarrow$ **Inequality (11) True** $\Rightarrow$ **(10) True.**

+ Since (9).(10):

$$\Rightarrow (b+c)(c+a)\cdot\sqrt{\frac{ab}{(b+c-a)(c+a-b)}}+(c+a)(a+b)\cdot\sqrt{\frac{bc}{(c+a-b)(a+b-c)}}+$$

$$+(a+b)(b+c)\cdot\sqrt{\frac{ca}{(a+b-c)(b+c-a)}}\geq\frac{4(a+b+c)^2}{3}$$

$\Rightarrow$ **Inequality (8) True.**

+ Since (7), (8) $:\Rightarrow \frac{1}{l_a l_b}+\frac{1}{l_b l_c}+\frac{1}{l_c l_a}\ge\frac{4}{3}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$

$\Rightarrow$ ***Inequality (1) True and we get the desired result.***

+ Equality occurs if $: a = b = c$

Solution 2 by Aditya Narayan Sharma-Kanchrapara-India

$\frac{1}{w_a w_b}+\frac{1}{w_b w_c}+\frac{1}{w_c w_a}\ge\frac{4}{3}\sum_{cyc}\frac{1}{ab}$. ***Note that,*** $w_a = \frac{2bc}{b+c}\cos\frac{A}{2}$

while, $\cos\frac{A}{2}=\frac{1}{2}\sqrt{\frac{(b+c+a)(b+c-a)}{bc}}\therefore w_a = \frac{\sqrt{(b+c-ca)(b+c-a)}}{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}$

$$\therefore\frac{1}{w_a}=\frac{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}{\sqrt{(b+c+a)(b+c-a)}}\ge\frac{2}{\sqrt{(b+c+a)(b+c-a)}}$$

$\therefore\frac{1}{w_a w_b}\ge\frac{4}{(a+b+c)}\cdot\frac{1}{\sqrt{(b+c-a)(c+a-b)}}$. ***Then,***

$$\sum_{cyc}\frac{1}{w_a w_b}\ge\frac{4}{a+b+c}\sum_{cyc}\frac{1}{\sqrt{(b+c-a)(c+a-b)}}$$

Since, $b+c-a>0, c+a-b>0$

$$\therefore(b+c-a)+(c+a-b)\ge 2\sqrt{(b+c-a)(c+a-b)}$$

$$\therefore\frac{1}{\sqrt{(b+c-a)(c+a-b)}}\ge\frac{1}{c}\therefore\sum_{cyc}\frac{1}{w_a w_b}\ge\frac{4}{a+b+c}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\therefore\sum_{cyc}\frac{1}{w_a w_b}\ge\frac{4}{a+b+c}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

From Cauchy – Schwarz: $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9$

$$\Rightarrow(a+b+c)^3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9(a+b+c)^2$$

$$\Rightarrow 9(a+b+c)^2 \le 3abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\Rightarrow (a+b+c)\frac{(a+b+c)}{2} \le 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\therefore \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\frac{1}{(a+b+c)} \ge \frac{1}{3}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

$$\therefore \sum_{cyc}\frac{1}{w_a w_b} \ge \frac{4}{3}\sum_{cyc}\frac{1}{ab}$$

184. In $\Delta\ ABC$:

$$\left(\frac{a^2+b^3+c^3-ab(a+b)-bc(b+c)-ca(c+a)}{4[ABC]}\right)^2+\left(\frac{2[ABC]}{r}\right)^2 \le 36R^2$$

where $[ABC]$ represents the area of $\Delta\ ABC$.

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto – Palacios – Huarmey – Peru

Probar en un triángulo ABC, donde S es el área:

$$\left(\frac{a^2+b^3+c^3-ab(a+b)-bc(b+c)-ca(c+a)}{4S}\right)^2+\left(\frac{2S}{r}\right)^2 \le 36R^2$$

Parimos de la suma de Cosenos en un triángulo ABC:

$$\cos A+\cos B+\cos C=\frac{ab^2+ac^2-a^3}{2abc}+\frac{ba^2+bc^2-b^3}{2abc}+\frac{ca^2+cb^2-c^3}{2abc}$$

$$\cos A+\cos B+\cos C=\frac{-\left(a^3+b^3+c^3-ab(a+b)-bc(b+c)-ca(c+a)\right)}{2abc}$$

$$-2R(\cos A+\cos B+\cos C)=-\left(\frac{a^3+b^3+c^3-ab(a+b)-bc(b+c)-ca(c+a)}{4S}\right)$$

La desigualdad es equivalente:

$$\left(-2R(\cos A+\cos B+\cos C)\right)^2+4p^2=4R^2(\cos A+\cos B+\cos C)^2+4p^2$$

$$\Rightarrow 4R^2(\cos A+\cos B+\cos C)^2+4p^2\le 4R^2\times\frac{9}{4}+27R^2=$$

$36R^2$...*(LQQD)*

185. **Prove that in any triangle ABC:**

$$(b+c)\cos^2 A+(c+a)\cos^2 B+(a+b)\cos^2 C\ge\frac{a+b+c}{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$(b+c)\cos^2 A+(c+a)\cos^2 B+(a+b)\cos^2 C\ge\frac{a+b+c}{2}$***. Teniendo en cuenta el teorema de las proyecciones:*** $a=b\cos C+c\cos B$ ***... (I),***

$b=a\cos C+c\cos A$ ***... (II),*** $c=a\cos B+b\cos A$ ***... (III)***

$$a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C$$

Por la desigualdad de Cauchy:

$$(b+c)\cos^2 A+(c+a)\cos^2 B+(a+b)\cos^2 C\ge$$

$$\ge\frac{(|b+c||\cos A|+|c+a||\cos B|+|a+b||\cos C|)^2}{(b+c)+(c+a)+(a+b)}$$

De la desigualdad triangular:

$$|x|+|y|+|z|\ge|x+y+z|\to x=(b+c)\cos A$$

$$y=(c+a)\cos B,\qquad z=(a+b)\cos C$$

Luego: $|(b+c)\cos A|+|(c+a)\cos B|+|(a+b)\cos C|\ge$

$$\ge|(b+c)\cos A+(c+a)\cos B+(a+b)\cos C|$$

$$\Rightarrow|(b+c)\cos A|+|(c+a)\cos B|+|(a+b)\cos C|\ge|a+b+c|=a+b+c$$

Finalmente: $\rightarrow (b+c)\cos^2 A + (c+a)\cos^2 B + (a+b)\cos^2 C \geq$

$$\geq \frac{(|b+c||\cos A| + |c+a||\cos B| + |a+b||\cos C|)^2}{(b+c)+(c+a)+(a+b)} = \frac{a+b+c}{2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS = \sum (2s-a)(1-\sin^2 A) = \sum \{2s - a - \sin^2 A\,(2s) + a\sin^2 A\} =$$

$$= 6s - \sum a - 2s\sum \frac{a^2}{4R^2} + \sum \frac{a^3}{4R^2}$$

$$= 4s - \frac{s(s^2-4Rr-r^2)}{R^2} + \frac{1}{4R^2}\left\{3abc + (2s)\left(\sum a^2 - \sum ab\right)\right\}$$

$$= 4s - \frac{s(s^2-4Rr-r^2)}{R^2} + \frac{1}{4R^2}\{12Rrs + 2s(s^2-12Rr-3r^2)\}$$

$= 4s - \frac{s(s^2-4Rr-r^2)}{R^2} + \frac{s(s^2-6Rr-3r^2)}{2R^2}$ $\therefore$ ***it suffices to prove that:***

$$3 - \frac{s^2-4Rr-r^2}{R^2} + \frac{s^2-6Rr-3r^2}{2R^2} \geq 0 \Leftrightarrow s^2 \leq 6R^2 + 2Rr - r^2$$

Gerretsen $\Rightarrow s^2 \leq 4R^2 + 4Rr + 3r^2$ $\therefore$ ***it suffices to prove that:***

$R^2 - Rr - 2r^2 \geq 0 \Leftrightarrow (R-2r)(R+r) \geq 0 \rightarrow$ ***true,*** $\because R \geq 2R$ ***(Euler)***

186. In $\Delta\ ABC$:

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \leq \frac{\sqrt{3}}{2}\cdot\left(\frac{R}{r}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\csc A + \csc B + \csc C \leq \frac{\sqrt{3}}{2}\left(\frac{R}{r}\right)^2$ ***... (A)***

Para ello demostraremos previamente la siguiente desigualdad:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{\sqrt{3}}{2r}\to\frac{ab+bc+ca}{abc}=\frac{ab+bc+ca}{4pRr}\le\frac{\sqrt{3}}{2r}$$

$\Rightarrow(ab+bc+ca)\le\sqrt{3}(2p)R$, ***además se sabe que:***

$3\sqrt{3}R\ge 2p\ \wedge R\ge 2r$. ***Por la tanto se puede afirmar:***

$\sqrt{3}(2p)R\ge\frac{(2p)^2}{3}=\frac{(a+b+c)^2}{3}\ge ab+bc+ca$ ***... (LQQD)***

La desigualdad es equivalente en ... (A):

$2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le 2R\ \times\frac{\sqrt{3}}{2r}=\frac{\sqrt{3}R}{r}=\frac{\sqrt{3}R^2}{Rr}\le\frac{\sqrt{3}}{2}\left(\frac{R}{r}\right)^2$ ***... (LQQD)***

187. ROMANIAN INEQUALITY – 2

In ΔABC:

$$\max(h_a,h_b,h_c)\ge\min(w_a,w_b,w_c)$$

Proposed by L. Panaitopol – Romania

Solution by Soumava Chakraborty-Kolkata-India

WLOG, we can assume $a\ge b\ge c$***. Then,*** $h_a\le h_b\le h_c$ ***and*** $w_a\le w_b\le w_c$

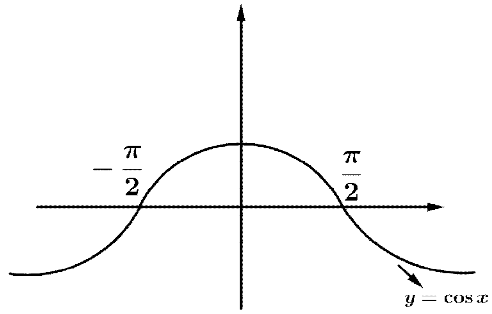
So, we need to prove: $h_c\ge w_a$

$$\Leftrightarrow b\sin A\ge\frac{2bc\cos\frac{A}{2}}{b+c}\Leftrightarrow 2b\sin\frac{A}{2}\cos\frac{A}{2}\ge\frac{2bc\cos\frac{A}{2}}{b+c}$$

$$\Leftrightarrow\sin\frac{A}{2}\ge\frac{c}{b+c}=\frac{\sin C}{\sin B+\sin C}=\frac{\sin C}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$\Leftrightarrow\sin\frac{A}{2}\ge\frac{\sin C}{2\cos\frac{A}{2}\cos\frac{B-C}{2}}\Leftrightarrow\cos\frac{B-C}{2}\ge\frac{\sin C}{\sin A}=\frac{c}{a}$$

(Here, $-\frac{\pi}{2}<\frac{B-C}{2}<\frac{\pi}{2}\Rightarrow\cos\frac{B-C}{2}>0$***)***



$$\Leftrightarrow \cos^2\frac{B-C}{2} \ge \frac{c^2}{a^2} \Leftrightarrow \sec^2\frac{B-C}{2} \le \frac{a^2}{c^2} \Leftrightarrow 1+\tan^2\frac{B-C}{2} \le \frac{a^2}{c^2}$$

$$\Leftrightarrow 1+\frac{(b-c)^2}{(b+c)^2}\cot^2\frac{A}{2} \le \frac{a^2}{c^2} \quad (1)$$

Now, $\because a \ge b \ge c, \therefore A \ge B \ge C$

If $A < 60°, B \le A < 60°$ ***and*** $C \le A < 60° \Rightarrow A+B+C < 180° \rightarrow$ ***impossible***

$$\therefore A \ge 60° \Rightarrow \frac{A}{2} \ge 30° \Rightarrow 30° \le \frac{A}{2} < 90° \Rightarrow \tan\frac{A}{2} \ge \tan 30° = \frac{1}{\sqrt{3}} \Rightarrow \cot^2\frac{A}{2} \le 3$$

$$\therefore 1+\frac{(b-c)^2}{(b+c)^2}\cot^2\frac{A}{2} \le 1+\frac{(b-c)^2}{(b+c)^2}\cdot 3 \quad (2)$$

From (1) and (2), it suffices to prove:

$$1+\frac{3(b-c)^2}{(b+c)^2} \le \frac{a^2}{c^2} \Leftrightarrow (b+c)^2+3(b-c)^2 \le \frac{a^2}{c^2}(b+c)^2$$

$$\Leftrightarrow \frac{a^2}{c^2}(b+c)^2 \ge 4(b^2+c^2-bc) \quad (3)$$

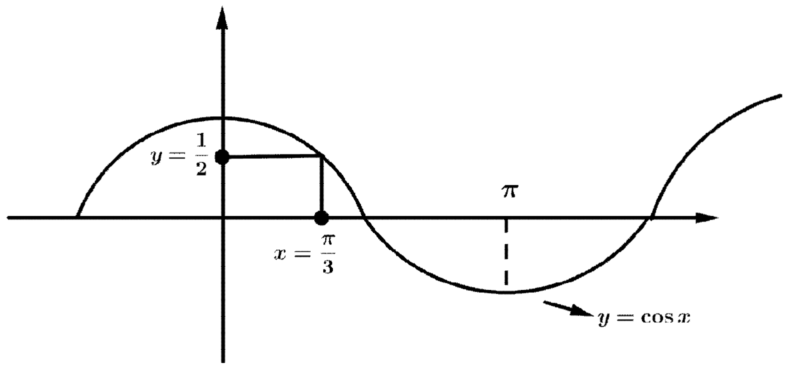
$$\because a \ge b \ge c, \therefore (b+c)^2 \ge (2c)^2 = 4c^2 \Rightarrow \frac{a^2}{c^2}(b+c)^2 \underset{(4)}{\ge} \frac{a^2}{c^2}\cdot(4c^2) = 4a^2$$

(3), (4) $\Rightarrow$ ***it suffices to prove that:***

$$a^2 \ge b^2+c^2-bc \Leftrightarrow bc \ge b^2+c^2-a^2$$

$$\Leftrightarrow \frac{1}{2} \ge \frac{b^2+c^2-a^2}{2bc} \Leftrightarrow \cos A \le \frac{1}{2} \quad (5)$$

$\because A \geq \frac{\pi}{3}$ ***(proved earlier)*** $\therefore \cos A \leq \frac{1}{2}$***, as is clear from the adiacent graph*** $\Rightarrow$ ***(5) is true (Hence proved)***



188. In $\Delta\ ABC$ the following relationship holds:

$$\sum \frac{a^4 + a^2b^2 + b^4}{a^2 + b^2} \geq 3S\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC: \sum \frac{a^4+b^4+a^2b^2}{a^2+b^2} \geq 3S(\csc A + \csc B + \csc C)$

Recordar lo siguiente en un triángulo $ABC: S = \frac{abc}{4R} = \frac{\sin Abc}{2} \to \frac{bc}{2S} = \csc A$, $\frac{ca}{2S} = \csc B$, $\frac{ab}{2S} = \csc C$. ***Ahorea probaremos lo siguiente:***

$$\frac{a^4 + b^4 + a^2b^2}{a^2 + b^2} \geq \frac{3(a^2 + b^2)}{4} \to$$

$$\to 4(a^4 + b^4 + a^2b^2) \geq 3(a^2 + b^2)^2 \to (a^2 - b^2)^2 \geq 0$$

Por la tanto la desigualdad es equivalente:

$$\sum \frac{a^4 + b^4 + a^2b^2}{a^2 + b^2} = 3\sum \frac{(a^2 + b^2)}{4} = 3\left(\frac{a^2 + b^2 + c^2}{3}\right) \geq\geq 3S\left(\frac{bc}{2S} + \frac{ca}{2S} + \frac{ab}{2S}\right)$$

Lo cual nos resulta: $a^2 + b^2 + c^2 \geq ab + bc + ca$ ***... (LQQD)***

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

A, B, C ***son ángulos interiores de un triángulo acutángulo.***

Probar que: $\frac{\csc A + \csc B + \csc C}{\cot A + \cot B + \cot C} \leq 2 \Rightarrow \csc A + \csc B + \csc C \leq$

$\leq 2(\cot A + \cot B + \cot C)$ ***(1)***

Multiplicamos $(\sin A \sin B \sin C)$ ***a la expresión ... (1):***

$\sin B \sin C + \sin A \sin C + \sin A \sin B \leq$

$\leq 2(\cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin B \sin A)$ ***... (2)***

Sabemos que: $A + B + C = \pi \Rightarrow \sin(A + B) = \sin C$ ***... (3) (elevando al cuadrado tenemos):*** $(\sin A \cos B + \cos A \sin B)^2 = \sin^2 C \Rightarrow$

$\Rightarrow \sin^2 A(1 - \sin^2 B) + 2\sin A \sin B(\cos A \cos B) + \sin^2 B(1 - \sin^2 A) = \sin^2 C$

$\sin^2 C = \sin^2 A + \sin^2 B + 2\sin A \sin B(\cos A \cos B - \sin A \sin B) \Rightarrow$

$\Rightarrow 2\sin A \sin B \sin C = \sin^2 A + \sin^2 B - \sin^2 C$ ***... (4)***

$\Rightarrow 2\sin C \sin B \cos A = \sin^2 C + \sin^2 B - \sin^2 A$ ***... (5)*** $\wedge$

$\wedge\ 2\sin C \sin A \cos B = \sin^2 C + \sin^2 A - \sin^2 B$ ***... (6)***

Sumando (4) + (5) + (6):

$2(\cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin B \sin A) =$

$= \sin^2 A + \sin^2 B + \sin^2 C$ ***... (7). De (7)*** $\wedge$ ***(2):***

$\sin^2 A + \sin^2 B + \sin^2 C \geq \sin B \sin C + \sin A \sin C + \sin A \sin B \Rightarrow$

$\Rightarrow (\sin A - \sin B)^2 + (\sin B - \sin C)^2 + (\sin A - \sin C)^2 \geq 0$

Solution 3 by Kevin Soto Palacios – Huarmey – Peru

A, B, C ***son los ángulos interiores de un triángulo acutángulo.***

Probar que: $\frac{\csc A + \csc B + \csc C}{\cot A + \cot B + \cot C} \leq 2 \Rightarrow$

$\Rightarrow 2(\cot A + \cot B + \cot C) \geq \csc A + \csc B + \csc C$

En la desigualdad inicial es equivalente y tendriamos demostrar que:

$$2\left(\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}\right)-\frac{1}{\sin A}-\frac{1}{\sin B}-\frac{1}{\sin C}\ge 0 \ldots \textit{(1)}$$

Sabemos que: $a=2R\sin A$, $b=2R\sin B \wedge c=2R\sin C\Rightarrow$

$$\Rightarrow \frac{2R}{a}=\frac{1}{\sin A},\frac{2R}{b}=\frac{1}{\sin A},\frac{2R}{b}=\frac{1}{\sin B}\ Y\ \frac{2R}{c}=\frac{1}{\sin C}\ldots \textit{(2)}$$

De (2) $\wedge$ ***(1):*** $4R\left(\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}\right)-2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\Rightarrow$

$$\Rightarrow \frac{2R(2\cos A\,bc+2\cos B\,ac+2\cos C\,ab)}{abc}-\frac{2R(bc+ac+ab)}{abc}$$

$$\Rightarrow 2R\frac{(b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2-bc-ca-ab)}{abc}\Rightarrow$$

$$\Rightarrow \frac{2R}{abc}(a^2+b^2+c^2-ab-bc-ac)\ge 0$$

Solution 4 by Soumitra Mandal - Chandar Nagore – India

$$\sum_{cyc}\frac{a^4+a^2b^2+b^4}{a^2+b^2}\ge\sum_{cyc}\frac{1}{4(a^2+b^2)}\{3(a^2+b^2)^2+(a^2-b^2)^2\}$$

$$\ge\frac{3}{4}\sum_{cyc}(a^2+b^2)\ [\because (a^2-b^2)^2\ge 0, (b^2-c^2)^2\ge 0\ and\ (c^2-a^2)^2\ge 0]$$

$$=\frac{3}{2}\sum_{cyc}ab=3S\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)$$

(Proved) $\because 2S=ab\sin C=bc\sin A=ca\sin B$

Solution 5 by Anas Adlany - El Jadida – Morocco

We have $\sum\frac{a^4+a^2b^2+b^4}{a^2+b^2}\ge 3S\sum\frac{1}{\sin(A)}\Leftrightarrow\sum\frac{a^4+a^2b^2+b^4}{a^2+b^2}\ge 3\sum\frac{S}{\sin(A)}\Leftrightarrow$

$$\sum\frac{a^4+a^2b^2+b^4}{a^2+b^2}\ge\frac{3}{2}\cdot\sum ab\Leftrightarrow$$

But, $a^4+a^2b^2+b^4=(a^2+b^2)^2-a^2b^2$

$$\geq (a^2+b^2)^2 - \frac{(a^2+b^2)^2}{16} = \frac{15}{16}(a^2+b^2)^2.$$ ***Hence,***

$$\frac{a^4+a^2b^2+b^4}{a^2+b^2} \geq \frac{15}{16}(a^2+b^2) \geq \frac{15}{8}ab \geq \frac{3}{2}ab.$$ ***Hence proved.***

Solution 6 by Myagmarsuren Yadamsuren – Darkhan – Mongolia

1. $$\sum_{\Delta}\frac{a^4+b^2a^2+b^4}{a^2+b^2} = \sum\frac{\frac{1}{2}\cdot(a^2+b^2)^2+a^4+b^4}{a^2+b^2} =$$

$$= \sum\left(\frac{a^2+b^2}{2}\right) + \sum\frac{a^4+b^4}{a^2+b^2} \geq \frac{1}{2}\sum(a^2+b^2) + \frac{\left(2(a^2+b^2+c^2)\right)^2}{2\cdot(a^2+b^2+c^2)} =$$

$$= (a^2+b^2+c^2) + \frac{1}{2}\cdot(a^2+b^2+c^2) = \frac{3}{2}(a^2+b^2+c^2) \geq \frac{3}{2}(ab+bc+ca)$$

$$= 3\left(\frac{ab\cdot\sin C}{2}\cdot\frac{1}{\sin C} + \frac{bc\cdot\sin A}{2}\cdot\frac{1}{\sin A} + \frac{ca\cdot\sin B}{2}\cdot\frac{1}{\sin B}\right) =$$

$$= 3\cdot S\cdot\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right)$$

2. $$\sum\frac{a^4+a^2b^2+b^4}{a^2+b^2} = \sum\frac{(a^2+b^2)^2-a^2b^2}{a^2+b^2} =$$

$$= \sum(a^2+b^2) - \sum\frac{a^2b^2}{a^2+b^2} \geq (a^2+b^2+c^2)\cdot 2 - \sum\frac{(ab)^2}{2ab} =$$

$$= (a^2+b^2+c^2)\cdot 2 - \frac{1}{2}(ab+bc+ca) \geq \frac{3}{2}(ab+bc+ca) =$$

$$= 3S\cdot\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right)$$

189. If in $\Delta\ ABC$, $A, B, C \in \left(0, \frac{2\pi}{3}\right)$ then:

$$\sum\sqrt{\cot\left(\frac{2\pi-3A}{6}\right)\cot\left(\frac{2\pi-3B}{6}\right)} \leq \prod\cot\left(\frac{2\pi-3C}{6}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum\sqrt{\cot\left(\frac{2\pi-3A}{6}\right)\cot\left(\frac{2\pi-3B}{6}\right)}\le\prod\cot\left(\frac{2\pi-3C}{6}\right)$$

where A,B,C ***are the angles of a*** $\Delta\ ABC|A,B,C\in\left(0,\frac{2\pi}{3}\right)$

Let $\cot\left(\frac{2\pi-3A}{6}\right)=x,\cot\left(\frac{2\pi-3B}{6}\right)=y$ ***and*** $\cot\left(\frac{2\pi-3C}{6}\right)=y$

$$\because 0<A<\frac{2\pi}{3};\therefore 0<3A<2\pi\Rightarrow -2\pi<-3A<0$$

$$\Rightarrow 0<2\pi-3A<2\pi\Rightarrow 0<\frac{2\pi-3A}{6}<\frac{\pi}{3}$$

Similarly, $0<\frac{2\pi-3B}{6},\frac{2\pi-3C}{6}<\frac{\pi}{3}\therefore x,y,z>0$

$$LHS=\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\overset{C-B-S}{\le}\sqrt{\sum x}\sqrt{\sum x}=\sum x$$

$$=\cot\left(\frac{2\pi-3A}{6}\right)+\cot\left(\frac{2\pi-3B}{6}\right)+\cot\left(\frac{2\pi-3C}{6}\right)$$

$$=\tan\left(\frac{\pi}{2}-\frac{2\pi-3A}{6}\right)+\tan\left(\frac{\pi}{2}-\frac{2\pi-3B}{6}\right)+\tan\left(\frac{\pi}{2}-\frac{2\pi-3C}{6}\right)$$

$$=\tan\left(\frac{\pi}{6}+\frac{A}{2}\right)+\tan\left(\frac{\pi}{6}+\frac{B}{2}\right)+\tan\left(\frac{\pi}{6}+\frac{C}{2}\right)\quad\textbf{(1)}$$

Now, $\left(\frac{\pi}{6}+\frac{A}{2}\right)+\left(\frac{\pi}{6}+\frac{B}{2}\right)+\left(\frac{\pi}{6}+\frac{C}{2}\right)=\pi$

$$\Rightarrow\sum\tan\left(\frac{\pi}{6}+\frac{A}{2}\right)=\prod\tan\left(\frac{\pi}{6}+\frac{A}{2}\right)\overset{(2)}{=}\prod\cot\left(\frac{2\pi-3A}{6}\right)$$

(1), (2) $\Rightarrow LHS\le\prod\cot\left(\frac{2\pi-3A}{6}\right)=RHS$ ***(Proved)***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

If in any $\Delta\ ABC\ A,B,C\in\left(0,\frac{2\pi}{3}\right)$ ***then***

$$\sum_{cyc}\sqrt{\cot\left(\frac{2\pi-3A}{6}\right)\cot\left(\frac{2\pi-3B}{6}\right)}\le\prod_{cyc}\left(\frac{2\pi-3A}{6}\right)$$

If A, B, C are the angles of a $\Delta\ ABC$ then

$\frac{\pi}{6}+\frac{A}{2},\frac{\pi}{6}+\frac{B}{2}$ ***and*** $\frac{\pi}{6}+\frac{C}{2}$ ***also forms the angles of a $\Delta\ A'B'C'$ where***

$A'=\frac{\pi}{6}+\frac{A}{2}, B'=\frac{\pi}{6}+\frac{B}{2}$ ***and*** $C'=\frac{\pi}{6}+\frac{C}{2}$***. Then we have***

$$\sum_{cyc}\tan A'\le\prod_{cyc}\tan A'\Rightarrow\sum_{cyc}\cot\left(\frac{\pi}{2}-A'\right)=\prod_{cyc}\left(\frac{\pi}{2}-A'\right)$$

$$\Rightarrow\sum_{cyc}\cot\left(\frac{\pi}{2}-\frac{\pi}{6}-\frac{A}{2}\right)=\prod_{cyc}\cot\left(\frac{\pi}{2}-\frac{\pi}{6}-\frac{A}{2}\right)\Rightarrow\sum_{cyc}\cot\left(\frac{2\pi-3A}{6}\right)=$$

$$=\prod_{cyc}\cot\left(\frac{2\pi-3A}{6}\right)$$

$$\therefore\sum_{cyc}\sqrt{\cot\left(\frac{2\pi-3A}{6}\right)\cot\left(\frac{2\pi-3B}{6}\right)}\le\prod_{cyc}\cot\left(\frac{2\pi-3A}{6}\right)$$

190. In $\Delta\ ABC$:

$$\frac{x}{Rx+2r(y+z)}+\frac{y}{Ry+2r(z+x)}+\frac{z}{Rz+2r(x+y)}\le\frac{3}{R+4r},\quad x,y,z>0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nirapada Pal-India

Let $f(a)=\frac{a}{ap+q}$***. Where*** $p=R-2r$ ***and*** $q=2r(x+y+z)$

Now $f''(a)=-\frac{2q}{(ap+q)^4}<0$***. So f is concave. Hence we have,***

$$\sum\frac{x}{Rx+2(y+z)}=3\left[\frac{f(x)+f(y)+f(z)}{3}\right]\le 3f\left(\frac{x+y+z}{3}\right)=$$

$= \frac{3}{R+4r} < \frac{3}{R+2r}$. ***Inspired by Abdallah El Farissi***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

In $\Delta\, ABC$ ***and*** $x, y, z > 0$

$$\sum_{cyc} \frac{x}{Rx + 2r(y+z)} \le \frac{3}{R+4r}$$

Let $f(u) = \frac{u}{(R-2r)u+2rd}$ ***where*** $d = x+y+z$ ***and*** $u \in (0,\infty)$

$f'(u) = \frac{2rd}{\{(R-2r)u+2rd\}^2}$ ***and*** $f''(u) = -\frac{4rd(R-2r)}{\{(R-2r)u+2rd\}^3} < 0$

hence f ***is concave. Applying Jensen's Inequality***

$$\sum_{cyc} f(3) \le 3f\left(\frac{x+y+z}{3}\right)$$

where $x, y, z > 0$

$$\sum_{cyc} \frac{x}{x(R-2r) + 2r(x+y+z)} \le 3\frac{\frac{x+y+z}{3}}{(R-2r)\frac{x+y+z}{3} + 2r(x+y+z)}$$

$$\therefore \sum_{cyc} \frac{x}{Rx + 2r(y+z)} \le \frac{3}{R+4r}$$

Solution 3 by Marian Dincă – Romania

$$\sum_{ciclic} \frac{x}{Rx + 2r(y+z)} \le \frac{3}{R+4r}$$

because is homogene in variables x, y, z ***to consider:*** $x+y+z = 1$

we obtain:

$$\sum_{ciclic} \frac{x}{Rx + 2r(y+z)} = \sum_{ciclic} \frac{x}{Rx + 2r(1-x)} = \sum_{ciclic} \frac{x}{(R-2r)x + 2r}$$

Let: $f(x)=\frac{x}{Rx+2r(1-x)}$ ***and*** $g(x)=\frac{f(x)-f\left(\frac{1}{3}\right)}{x-\frac{1}{3}}=$

$$=\frac{\frac{x}{Rx+2r(1-x)}-\frac{1}{R+4r}}{x-\frac{1}{3}}=\frac{(R+4r)x-Rx-2r(1-x)}{[Rx+2r(1-x)](R+4r)\left(x-\frac{1}{3}\right)}=$$

$$=\frac{2r(3x-1)}{[Rx+2r(1-x)](R+4r)\left(x-\frac{1}{3}\right)}=\frac{6r}{[(R-2r)x+2r](R+4r)}$$

obviously the function g ***decreasing***

$$f(x)-f\left(\frac{1}{3}\right)+f(y)-f\left(\frac{1}{3}\right)+f(z)-f\left(\frac{1}{3}\right)=$$

$$=g(x)\left(x-\frac{1}{3}\right)+g(y)\left(y-\frac{1}{3}\right)+g(z)\left(z-\frac{1}{3}\right)\le$$

$$\le\frac{1}{3}\big(g(x)+g(y)+g(z)\big)\left(x-\frac{1}{3}+y-\frac{1}{3}+z-\frac{1}{3}\right)=0$$

use Cebyshev for sequences: $\{g(x),g(y),g(z)\}$ ***and*** $\left\{x-\frac{1}{3},y-\frac{1}{3},z-\frac{1}{3}\right\}$

$$x\ge y\ge z\Rightarrow x-\frac{1}{3}\ge y-\frac{1}{3}\ge z-\frac{1}{3}\text{ and } g(x)\le g(y)\le g(z)$$

result: $f(x)+f(y)+f(z)\le 3f\left(\frac{1}{3}\right)=\frac{3}{R+4r}$

191. In $\Delta\ ABC$ the following relationship holds:

$$\frac{\sqrt{a^4+b^4}}{a^3+b^3}+\frac{\sqrt{b^4+c^4}}{b^3+c^3}+\frac{\sqrt{c^4+a^4}}{c^3+a^3}\le\frac{\sqrt{6}\cdot R}{8\cdot r^2}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC***:*** $\sqrt{\frac{a^4+b^4}{(a^3+b^3)^2}}+\sqrt{\frac{b^4+c^4}{(b^3+c^3)^2}}+\sqrt{\frac{c^4+a^4}{(c^3+a^3)^2}}\le\frac{\sqrt{6}R}{8r^2}$ ***... (α)***

Desde que: $(a-b)^4 \geq 0 \rightarrow a^4+b^4+6a^2b^2-4a^3b-4b^3a \geq 0$

$\Rightarrow 2(a^4+b^4+a^2b^2-2a^3b-2b^3a+2a^2b^2) \geq a^4+b^4 \rightarrow 2(a^2-ab+b^2)^2 \geq a^4+b^4$ ***... (A)***

$\Rightarrow (a+b)^2 \geq 4ab$ ***... (B), luego, multiplicando ... (A) × (B):***

$\Rightarrow 2(a+b)^2(a^2-ab+b^2)^2 \geq 4ab(a^4+b^4) \rightarrow (a^3+b^3)^2 \geq 2ab(a^4+b^4)$

Análogamente para los demás términos:

$$(b^3+c^3)^2 \geq 2bc(b^4+c^4) \wedge (c^3+a^3)^2 \geq 2ca(c^4+a^4)$$

Tener en cuenta lo siguiente: $R \geq 2r$,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq \sqrt{\frac{3}{x^2}+\frac{3}{y^2}+\frac{3}{z^2}}, \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}$$

Por consiguiente, se tiene en ... (α)

$$\sqrt{\frac{a^4+b^4}{(a^3+b^3)^2}}+\sqrt{\frac{b^4+c^4}{(b^3+c^3)^2}}+\sqrt{\frac{c^4+a^4}{(c^3+a^3)^2}} \leq \frac{1}{\sqrt{2ab}}+\frac{1}{2\sqrt{bc}}+\frac{1}{\sqrt{2ac}} \leq$$

$$\leq \frac{1}{\sqrt{2}}\sqrt{\frac{3}{ab}+\frac{3}{bc}+\frac{3}{ca}}=\frac{\sqrt{6}}{2}\sqrt{\frac{1}{2Rr}} \leq \frac{\sqrt{6}\times R}{R\times 4r} \leq \frac{\sqrt{6}R}{8r^2}$$ ***... (LQQD)***

192. In $\Delta\ ABC$ the following relationship holds:

$$(R+r)^5\cdot\left(\sum a^5\right)\cdot\left(\sum\frac{1}{(Ra+rb)^5}\right) \geq 9$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC***:*** $3(R+r)^5\left(\sum a^5\right)\left(\sum\frac{1}{(Ra+rb)^5}\right) \geq 27$

Por la desigualdad de Holder:

$$\left(\sum(R+r)^5\right)\left(\sum a^5\right)(1+1+1)(1+1+1)(1+1+1) \geq$$

$$\geq \left((Ra+ra)+(Rb+rb)+(Rc+rc)\right)^5$$

$$\Rightarrow \sum(R+r)^5\left(\sum a^5\right)(27) \geq \left((Ra+rb)+(Rb+Rc)+(Rc+ra)\right)^5 \text{ ... (A)}$$

Por otro lado, por MA ≥ MG:

$$\left((Ra+rb)+(Rb+Rc)+(Rc+ra)\right)^5\left(\sum \frac{1}{(Ra+rb)^5}\right) \geq 729 \text{ ... (B)}$$

Multiplicando: (A × B): $(R+r)^5\left(\sum a^5\right)\left(\sum \frac{1}{(Ra+rb)^5}\right) \geq 9$ ***... (LQQD)***

Solution 2 by Mihalcea Andrei Ștefan – Romania

$$\sum \frac{1^6}{(Ra+rb)^5} \overset{Radon}{\geq} \frac{3^6}{(R+r)^5(\sum a)^5}; \sum a^5 \overset{Holder}{\geq} \frac{(\sum a)^5}{3^4}$$

$$\Rightarrow LHS \geq (R+r)^5 \cdot \frac{(\sum a)^5}{3^4} \cdot \frac{3^6}{(R+r)^5(\sum a)^5} = 9$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{1}{(Ra+rb)^5} \geq \frac{(1+1+1)^6}{\sum(Ra+rb)^5} \quad \text{(1) (Radon)}$$

Now, $\sum(Ra+rb)^5 < (Ra+rb+Rb+rc+Rc+ra)^5$

$$= \left(R\sum a + r\sum a\right)^5 = (R+r)^5\left(\sum a\right)^5$$

$$\therefore \frac{1}{\sum(Ra+rb)^5} > \frac{1}{(R+r)^5(\sum a)^5} \quad \text{(2)}$$

$$\therefore \sum \frac{1}{(Ra+rb)^5} > \frac{3^6}{(R+r)^5(\sum a)^5} \quad \text{(from (1), (2))}$$

$$\therefore LHS > \frac{3^6(\sum a^5)}{(\sum a)^5} \geq \frac{3^6 \cdot \frac{1}{3^4}(\sum a)^5}{(\sum a)^5} \quad \text{(Chebyshev)}$$

$$= 9 = RHS \quad \text{(Proved)}$$

Solution 4 by Myagmarsuren Yadamsuren – Darkhan – Mongolia

$$(R+2)^n \cdot \sum_{\Delta} a^n \cdot \sum_{\Delta} \frac{1}{(a \cdot R + b \cdot r)^n} \geq 9$$

$$\sum\frac{1}{(a\cdot R+b\cdot r)^n}=\frac{1}{(a\cdot R+b\cdot r)^n}+\frac{1}{(br+cr)^n}+\frac{1}{(c\cdot R+ar)^n}$$

$$\left.\begin{array}{r}a\cdot R+br=x_1\\ b\cdot R+c\cdot r=x_2\\ c\cdot R+a\cdot r=x_3\end{array}\right\} f(x)=\frac{1}{x^n}\Rightarrow f''(x)\ge 0$$

$$\sum\frac{1}{(a\cdot R+b\cdot r)^n}\ge\frac{3}{\left(\frac{(a+b+c)\cdot R+(a+b+c)\cdot r}{3}\right)^n}=$$

$$=\frac{3^{n+1}}{(a+b+c)^n\cdot(R+r)^n}\ (*)$$

$$\sum a^n\overset{Chebysev}{\ge}\frac{1}{3^{n-1}}\cdot(a+b+c)^4\quad (**)$$

$$(*);(**)\Rightarrow(R+r)^n\cdot\frac{1}{3^{n-1}}\cdot\left(\sum a\right)^n\cdot\frac{3^{n+1}}{(\sum a)^n(R+r)^n}=9$$

193. In $\Delta\ ABC$ the following relationship holds:

$$\frac{h_a\cdot\tan\frac{A}{2}}{a^3}+\frac{h_b\cdot\tan\frac{B}{2}}{b^3}+\frac{h_c\cdot\tan\frac{C}{2}}{c^3}\le\frac{1}{8\cdot r^2}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{h_a}{a^3}\tan\frac{A}{2}+\frac{h_b}{b^3}\tan\frac{B}{2}+\frac{h_c}{c^3}\tan\frac{C}{2}\le\frac{1}{8r^2}$

1. *Tener en cuenta las siguientes identidades en un triángulo ABC:*

$$h_a=\frac{2S}{a},h_b=\frac{2S}{b},h_c=\frac{2S}{c}$$

$$\tan\frac{A}{2}=\frac{(s-b)(s-c)}{S};\tan\frac{B}{2}=\frac{(s-c)(s-a)}{S},\tan\frac{C}{2}=\frac{(s-a)(s-b)}{S}$$

2. *Recordar la siguiente desigualdad en un triángulo ABC:*

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2}\ \dots\ \textit{(demostrado anteriormente)}$$

El lado izquierdo es equivalente:

$$\Rightarrow \frac{2(s-b)(s-c)}{a^4}+\frac{2(s-c)(s-a)}{b^4}+\frac{2(s-a)(s-b)}{c^4} \dots (A)$$

Desde que: a, b, c ***son lados de un triángulo***

$$\to (s-a),(s-b),(s-c) > 0$$

Por: MA ≥ MG $\to \frac{(s-b)+(s-c)}{2} \geq \sqrt{(s-b)(s-c)},$

$$\frac{a^2}{4} \geq (s-b)(s-c), \frac{1}{2a^2} \geq \frac{2(s-b)(s-c)}{a^4}$$

Por consiguiente tenemos en ... (A):

$$\Rightarrow \frac{2(s-b)(s-c)}{a^4}+\frac{2(s-c)(s-a)}{b^4}+\frac{2(s-a)(s-b)}{c^4} \leq$$

$$\leq \frac{1}{2a^2}+\frac{1}{2b^2}+\frac{1}{2c^2} \leq \frac{1}{8r^2} \dots \textit{(LQQD)}$$

194. In $\Delta\, ABC$ the following relationship holds:

$$\frac{1}{a}\sqrt{\frac{a}{b+c-a}}+\frac{1}{b}\sqrt{\frac{b}{c+a-b}}+\frac{1}{c}\sqrt{\frac{c}{a+b-c}} \geq \frac{\sqrt{3}}{R}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{1}{a}\sqrt{\frac{a}{b+c-a}}+\frac{1}{b}\sqrt{\frac{b}{c+a-b}}+\frac{1}{c}\sqrt{\frac{c}{a+b-c}} \geq \frac{\sqrt{3}}{R}$

La desigualdad es equivalente:

$$\sqrt{\frac{1}{a(b+c-a)}}+\sqrt{\frac{1}{b(c+a-b)}}+\sqrt{\frac{1}{c(a+b-c)}} \geq \frac{\sqrt{3}}{R} \dots (A)$$

Desde que: a, b, c ***son lados de un triángulo, se cumple:***

$(b+c-a),(c+a-b),(a+b-c) > 0$. ***Por: MA ≥ MG***

$$a+(b+c-a) \geq 2\sqrt{a(b+c-a)} \to \frac{b+c}{2} \geq \sqrt{a(b+c-a)} \to$$

$\rightarrow \frac{2}{b+c} \le \frac{1}{\sqrt{a(b+c-a)}}$. ***Análogamente para los demás términos:***

$$\frac{2}{c+a} \le \frac{1}{\sqrt{b(c+a-b)}}, \frac{2}{a+b} \le \frac{1}{\sqrt{c(a+b-c)}}$$

Luego tenemos en ... (A):

$$\sqrt{\frac{1}{a(b+c-a)}} + \sqrt{\frac{1}{b(c+a-b)}} + \sqrt{\frac{1}{c(a+b-c)}} \ge 2\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \ge$$

$$\ge \frac{9}{a+b+c} \ge \frac{9}{3\sqrt{3}R} = \frac{\sqrt{3}}{R} \ldots \textbf{\textit{(LQQD)}}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\textbf{\textit{In }} \Delta\ ABC: \frac{1}{a} \cdot \sqrt{\frac{a}{b+c-a}} + \frac{1}{b} \cdot \sqrt{\frac{b}{c+a-b}} + \frac{1}{c} \cdot \sqrt{\frac{c}{a+b-c}} \ge \frac{\sqrt{3}}{R} \quad (*)$$

$$\sum_{\Delta} \frac{1}{a} \cdot \sqrt{\frac{a}{b+c-a}} \overset{Cauchy}{\ge} 3 \cdot \sqrt[3]{\frac{1}{abc} \cdot \sqrt{\frac{abc}{(a+b-c)(b+c-a)(c+a-b)}}} \ge$$

$$\overset{Ravi}{\ge} \frac{3}{\sqrt[3]{abc}} \ge \frac{\sqrt{3}}{R} \quad \textbf{\textit{(ASSURE)}}$$

$$\Rightarrow 3\sqrt{3} \cdot R^3 \ge abc \Rightarrow \frac{3\sqrt{3}}{4} \cdot R^2 \ge S \ \textbf{\textit{(True)}}$$

$$s = p \cdot r \overset{Euler}{\le} p \cdot \frac{R}{2} = \frac{R}{2} \cdot \left(\frac{a+b+c}{2}\right) = \frac{R}{2} \cdot R \cdot (\sin A + \sin B + \sin C)$$

$$\le \frac{R^2}{2} \cdot \frac{3\sqrt{3}}{2} = \frac{3\sqrt{3}}{4} \cdot R^2$$

195. In acute – angled $\Delta\ ABC$:

$$\sum \cot\frac{A}{2}\cot\frac{B}{2} \ge 4\sum \frac{1}{1 + \tan\frac{A}{2}\tan\frac{B}{2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC: $\sum \cot\frac{A}{2}\cot\frac{B}{2} \geq 4\sum \frac{1}{1+\tan\frac{A}{2}\tan\frac{B}{2}}$

1. Realizamos los siguientes combios de variables:

$$x = \cot\frac{A}{2}\cot\frac{B}{2} > 0, y = \cot\frac{B}{2}\cot\frac{C}{2} > 0, z = \cot\frac{C}{2}\cot\frac{A}{2} > 0$$

2. En un triángulo ABC, ***si:*** $A + B + C = \frac{\pi}{2}$, ***se cumple lo siguiente:***

$$\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1 \to \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

Por desigualdad de Cauchy: $\to x + y + z \geq 9$

La desigualdad es equivalente:

$$x + y + z \geq \frac{4x}{x+1} + \frac{4y}{y+1} + \frac{4z}{z+1} \to \left(x + \frac{4}{x+1}\right) + \left(y + \frac{4}{y+1}\right) + \left(z + \frac{4}{z+1}\right) \geq 12$$

Lo cual es cierto; ya que por: $MA \geq MG$

$$\Rightarrow \left(\frac{x+1}{4} + \frac{4}{x+1}\right) + \left(\frac{y+1}{4} + \frac{4}{y+1}\right) + \left(\frac{z+1}{4} + \frac{4}{z+1}\right) \geq 6 \ldots \textbf{(A)}$$

$$\Rightarrow \frac{3x}{4} + \frac{3y}{4} + \frac{3z}{4} - \frac{3}{4} \geq \frac{27}{4} - \frac{3}{4} = 6 \ldots \textbf{(B)}$$

Summando: (A) + (B) $\to \left(x + \frac{4}{x+1}\right) + \left(y + \frac{4}{y+1}\right) + \left(z + \frac{4}{z+1}\right) \geq 12 \ldots$

Solution 2 by Soumava Chakraborty – Kolkata – India

Let $f(x) = \ln\left(\tan\frac{x}{2}\right), \forall x \in \left(0, \frac{\pi}{2}\right);\ f''(x) = \frac{\sec^2\left(\frac{x}{2}\right)}{2} - \frac{\sec^4\left(\frac{x}{2}\right)}{4\tan^2\left(\frac{x}{2}\right)}$

$$= \frac{\sec^2\left(\frac{x}{2}\right)}{2}\left\{1 - \frac{\sec^2\left(\frac{x}{2}\right)}{2\tan^2\left(\frac{x}{2}\right)}\right\} = \frac{\sec^2\left(\frac{x}{2}\right)}{4\tan^2\frac{x}{2}}\left\{2\tan^2\frac{x}{2} - \left(1 + \tan^2\frac{x}{2}\right)\right\}$$

$$= \frac{\sec^2\left(\frac{x}{2}\right)}{4\tan^2\left(\frac{x}{2}\right)}\left\{\tan^2\left(\frac{x}{2}\right) - 1\right\} < 0 \left(\begin{array}{c} \because 0 < \frac{x}{2} < \frac{\pi}{4}, \\ \therefore 0 < \tan^2\left(\frac{x}{2}\right) < 1 \\ \Rightarrow \tan^2\left(\frac{x}{2}\right) - 1 < 0 \end{array}\right)$$

$\therefore \forall x \in \left(0, \frac{\pi}{2}\right), f(x) = \ln\left(\tan\frac{x}{2}\right)$ ***is concave. Applying Jensen's inequality,***

$$\sum \ln\left(\tan\frac{A}{2}\right) \le 3\ln\left(\frac{A+B+C}{6}\right) \Rightarrow \ln\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right) \le \ln\left(\frac{1}{\sqrt{3}}\right)^3$$

$$\Rightarrow \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \le \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9} \Rightarrow \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \overset{(1)}{\ge} \frac{9}{\sqrt{3}} = 3\sqrt{3}$$

Let $u = \cot\frac{A}{2}, v = \cot\frac{B}{2}, w = \cot\frac{C}{2}$***. Then given inequality*** $\Leftrightarrow$

$$\Leftrightarrow \sum uv \ge 4\sum \frac{1}{1+\frac{1}{uv}} \Leftrightarrow uv + vw + wu \ge 4\left(\frac{uv}{1+uv} + \frac{vw}{1+vw} + \frac{wu}{1+wu}\right)$$

$$\Leftrightarrow (uv + vw + wu)(1+uv)(1+vw)(1+wu) \ge$$

$$4uv(1+vw)(1+wu) + 4vw(1+wu)(1+uv) + 4wu(1+uv)(1+vw)$$

$$\Leftrightarrow u^2v^2w^2(uv+vw+wu) + uvw(u^2v + v^2w + w^2u + uv^2 + vw^2 + wu^2) + u^2v^2 + v^2w^2 + w^2u^2$$

$$\overset{(2)}{\ge} 6uvw(u+v+w) + 3(uv+vw+wu) + 9u^2v^2w^2$$

(expanding and re-arranging) . Now, from (1), $uvw \ge 3\sqrt{3}$ ***(i)***

Again, $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

$\Rightarrow \frac{1}{uv} + \frac{1}{vw} + \frac{1}{wu} = 1 \Rightarrow u + v + w = uvw$ ***(ii)Now,***

$$\frac{1}{9}u^2v^2w^2(uv+vw+wu) \underset{(a)}{\overset{(i)}{\ge}} \frac{27}{9}(uv+vw+wu) = 3(uv+vw+wu)$$

Also, $uvw(u^2v + v^2w + w^2u + uv^2 + vw^2 + wu^2) =$

$$= uvw\left(w(u^2+v^2) + u(v^2+w^2) + v(w^2+u^2)\right)$$

$$\underset{(b)}{\overset{A-G}{\geq}} uvw(2wuv+2uvw+2vwu)=6u^2v^2w^2\overset{(ii)}{=}6uvw(u+v+w)$$

Also, $u^2v^2+v^2w^2+w^2u^2\overset{(c)}{\geq}uvw(u+v+w)\overset{(ii)}{=}u^2v^2w^2$

(a), (b), (c) $\Rightarrow$ ***it is sufficient to prove that***

$$\frac{8}{9}u^2v^2w^2(uv+vw+wu)\geq 8u^2v^2w^2 \quad \textit{(from (2))}$$

$$\Leftrightarrow uv+vw+wu\geq 9 \quad (3)$$

But $uv+vw+wu\overset{A-G}{\geq}3\sqrt[3]{u^2v^2w^2}\overset{(i)}{\geq}3\sqrt[3]{27}=9\Rightarrow$ ***(3) is true (Proved)***

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\tan\frac{A}{2}x>0 \quad \tan\frac{B}{2}=y>0 \quad \tan\frac{C}{2}=z>0;\ xy+yz+zx=1$$

$$\cot\frac{A}{2}=\frac{1}{x};\cot\frac{B}{2}=\frac{1}{y};\cot\frac{C}{2}=\frac{1}{z};\ \sum\frac{1}{xy}\geq 4\sum\frac{1}{1+xy} \quad \textit{(ASSURE)}$$

$$\sum\frac{4}{1+xy}=\sum\frac{4}{2xy+yz+zx}=\frac{1}{4}\cdot\sum\frac{\overbrace{16}^{(1+1+1+1)^4}}{xy+xy+yz+zx}\leq$$

$$\leq\frac{1}{4}\cdot\sum\left(\frac{2}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)=$$

$$=\frac{1}{4}\cdot\left[\left(\frac{2}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)+\left(\frac{1}{xy}+\frac{2}{yz}+\frac{1}{zx}\right)+\left(\frac{1}{xy}+\frac{1}{yz}+\frac{2}{zx}\right)\right]=$$

$$=\frac{1}{4}\cdot\left(\frac{4}{xy}+\frac{4}{yz}+\frac{4}{xz}\right)=\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}$$

Solution 4 by Ravi Prakash-New Delhi-India

$$\frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}\Rightarrow\tan\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{C}{2}\right)\Rightarrow\sum\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)=1$$

Let $x=\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right), y=\tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right), z=\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)$

Consider $\sum \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right) - 4\sum \frac{1}{1+\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}$

$$= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 4\left(\frac{1}{2x+y+z} + \frac{1}{x+2y+z} + \frac{1}{x+y+2z}\right)$$

$$= \frac{y+z-2x}{x(1+x)} + \frac{z+x-2y}{y(1+y)} + \frac{x+y-2z}{z(1+z)}$$

$$= (y-z)\left[\frac{1}{x+x^2} - \frac{1}{y+y^2}\right] + (z-x)\left[\frac{1}{x+x^2} - \frac{1}{z+z^2}\right] +$$

$$+(y-z)\left(\frac{1}{z+z^2} - \frac{1}{y+y^2}\right)$$

$$= \frac{(y-x)^2(1+x+y)}{xy(1+x)(1+y)} + \frac{(z-x)^2(1+z+x)}{xz(1+x)(1+z)} + \frac{(y-z)^2(1+y+z)}{yz(1+y)(1+z)} \geq 0$$

Solution 5 by Saptak Bhattacharya-Kolkata-India

By Cauchy Schwarz on 4 variables

$$\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}}, \sqrt{\tan\frac{A}{2}\tan\frac{B}{2}}, \sqrt{\tan\frac{B}{2}\tan\frac{C}{2}}, \sqrt{\tan\frac{C}{2}\tan\frac{A}{2}}$$

and their reciprocals,

$$\frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}} + \sum \frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}} \geq \frac{16}{\tan\frac{A}{2}\tan\frac{B}{2} + \sum \tan\frac{A}{2}\tan\frac{B}{2}}$$

$$= \frac{16}{1+\tan\frac{A}{2}\tan\frac{B}{2}} \quad (i)$$

$\left[\because \sum \tan\frac{A}{2}\tan\frac{B}{2} = 1\right]$. ***Similarly,***

$$\frac{1}{\tan\frac{B}{2}\tan\frac{C}{2}} + \sum \frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}} \geq \frac{16}{1+\tan\frac{B}{2}\tan\frac{C}{2}} \quad (ii); \frac{1}{\tan\frac{C}{2}\tan\frac{A}{2}} + \sum \frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}} \geq \frac{16}{1+\tan\frac{C}{2}\tan\frac{A}{2}} \quad (iii)$$

Adding (i), (ii) and (iii);

$$4\sum \frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}} \geq 16\sum \frac{1}{1+\tan\frac{A}{2}\tan\frac{B}{2}}; \sum \frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}} \geq 4\sum \frac{1}{1+\tan\frac{A}{2}\tan\frac{B}{2}} \text{ (Proved)}$$

196. **In $\Delta\ A_1B_1C_1, \Delta\ A_2B_2C_2$ the following relationship holds:**

$$\sum a_1^2\sqrt{\sin B_2\sin C_2}\le\frac{9\sqrt{3}}{2}R_1^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\textit{Given inequality}\Leftrightarrow\sum a_1^2\sqrt{b_2c_2}\le 9\sqrt{3}R_1^2R_2\quad (1)$$

$$\textit{WLOG, we may assume } a_1\le b_1\le c_1 \textit{ and } a_2\le b_2\le c_2$$

$$\therefore\sqrt{b_2c_2}\ge\sqrt{c_2a_2}\ge\sqrt{a_2b_2}\quad\text{and}\quad a_1^2\le b_1^2\le c_1^2$$

Applying Chebyshev's Inequality,

$$\sum a_1^2\sqrt{b_2c_2}\le\frac{1}{3}\left(\sum a_1^2\right)\left(\sum\sqrt{b_2c_2}\right)\overset{C-B-S}{\le}\frac{1}{3}\left(\sum a_1^2\right)\sqrt{\sum a_2}\sqrt{\sum a_2}$$

$$=\frac{1}{3}\left(\sum a_1^2\right)\left(\sum a_2\right)\le\frac{1}{3}(9R_1^2)(3\sqrt{3}R_2)=9\sqrt{3}R_1^2R_2\Rightarrow \textit{(1) is true}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\textit{In } \Delta\ A_iB_iC_i\ i=1,2 \textit{ show that } \frac{a_1^2}{c_2}+\frac{b_1^2}{a_2}+\frac{c_1^2}{b_2}\ge\frac{a_1b_1c_1}{R_1R_2}.\ \textit{We know,}$$

$$\sum_{cyc}a_1^2\le 9R_1^2 \text{ and } \sum_{cyc}a_2^2\le 9R_2^2$$

$$\frac{a_1^2}{c_2}+\frac{b_1^2}{a_2}+\frac{c_1^2}{b_2}=\frac{a_1^3}{a_1c_2}+\frac{b_1^3}{b_1a_2}+\frac{c_1^3}{c_1b_2}\ge\frac{(a_1+b_1+c_1)^3}{3(a_1c_2+b_1a_2+c_1b_2)}\quad \textit{[Holder]}$$

$$\overset{Cauchy-Schwarz}{\ge}\frac{(a_1+a_2+a_3)^3}{\sqrt[3]{\left(\sum_{cyc}a_1^2\right)\left(\sum_{cyc}a_2^2\right)}}\ge\frac{27a_1b_1c_1}{3\cdot 3R_1\cdot 3R_2}=\frac{a_1b_1c_1}{R_1R_2}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{a_1^2}{c_2}+\frac{b_1^2}{a_2}+\frac{c_1^2}{b_2}\ge\frac{a_1b_1c_1}{R_1R_2}$$

$$\sum_{\Delta}\frac{a_1^2}{c_2}=\sum_{\Delta}\frac{a_1^2}{2R_2\cdot\sin C_2}=\frac{1}{2\cdot R_2}\cdot\sum_{\Delta}\frac{a_1^2}{\sin C_2}\geq$$

$$\geq\frac{1}{2\cdot R_2}\cdot\frac{(\sum a_1)^2}{\sin A_2+\sin B_2+\sin C_2}\geq\frac{1}{2R_2}\cdot\frac{4p_1^2}{\frac{3\sqrt{3}}{2}}=$$

$$=\frac{4p_1^2}{R_2\cdot 3\sqrt{3}}=\frac{4p_1\cdot p_1}{R_2\cdot 3\sqrt{3}}\geq\frac{4p_1\cdot r_1\cdot 3\sqrt{3}}{R_2\cdot 3\sqrt{3}}=\frac{4S_1}{R_2}=\frac{4\cdot\frac{a_1\cdot b_1\cdot c_1}{4\cdot R_1}}{R_2}=\frac{a_1b_1c_1}{R_1R_2}$$

197. **In $\Delta\, ABC$ the following relationship holds:**

$$\frac{m_a}{m_c}+\frac{m_b}{m_a}+\frac{m_c}{m_b}\geq\frac{(s^2+r^2+4Rr)^2}{R^2(5s^2-3r^2-12Rr)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{m_a}{m_c}+\frac{m_b}{m_a}+\frac{m_c}{m_b}\geq\frac{(s^2+r^2+4Rr)^2}{R^2(5s^2-3r^2-12Rr)}$

1. Siendo: a,b,c ***lados de un triángulo*** ABC ***se cumple las siguiente identidades:***

$$ab+bc+ca=s^2+r^2+4Rr,\ a^2+b^2+c^2=2(s^2-r^2-4Rr)$$

Así que: $2(a^2+b^2+c^2)+ab+bc+ca=5s^2-3r^2-12Rr$

2. Recordar las siguiente desigualdades de las mdeianas en un triángulo ABC***:***

$$4m_am_b\leq 2c^2+ab,\ 4m_bm_c\leq 2a^2+bc,\ 4m_cm_a\leq 2b^2+ca$$

$m_a \ge \frac{b^2+c^2}{4R}, m_b \ge \frac{c^2+a^2}{4R}, m_c \ge \frac{a^2+b^2}{4R}, m_a + m_b + m_c \ge \frac{a^2+b^2+c^2}{2R}$. ***La desigualdad es equivalente:*** $\frac{m_a}{m_c} + \frac{m_b}{m_a} + \frac{m_c}{m_b} \ge \frac{1}{R^2} \times \frac{(ab+bc+ca)^2}{2(a^2+b^2+c^2)+ab+bc+ca}$

Aplicando la desiguladad de Cauchy:

$$\frac{m_a}{m_c} + \frac{m_b}{m_a} + \frac{m_c}{m_b} \ge \frac{4(m_a + m_b + m_c)^2}{4m_c m_a + 4m_b m_a + 4m_c m_a} \ge$$

$$\ge \frac{4\left(\frac{a^2+b^2+c^2}{2R}\right)^2}{2(a^2+b^2+c^2)+ab+bc+ca} \ge \frac{1}{R} \times \frac{(ab+bc+ca)^2}{2(a^2+b^2+c^2)+ab+bc+ca}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$LHS \ge 3$ ***(by AM-GM)*** $\therefore$ ***it suffices to prove:*** $3 \ge \frac{(\sum ab)^2}{R^2(5s^2-3r^2-12Rr)}$

$$\Leftrightarrow 3R^2\{2s^2 + 3(s^2 - 4Rr - r^2)\} \ge \left(\sum ab\right)^2$$

$$\Leftrightarrow 12R^2s^2 + 9R^2\left(\sum a^2\right) \ge 2\left(\sum ab\right)^2 \quad \textbf{(1)}$$

Now, $4R^2s^2 \ge \sum a^2b^2$ ***(Goldstone's Inequality)*** $\Rightarrow 12R^2s^2 \ge 3\sum a^2b^2$ ***(2)***

Again, $9R^2 \ge \sum a^2$ ***(Leibniz)*** $\Rightarrow 9R^2(\sum a^2) \ge (\sum a^2)^2 \overset{(3)}{\ge} (\sum ab)^2$

(2) + (3) $\Rightarrow 12R^2s^2 + 9R^2(\sum a^2) \overset{(4)}{\ge} 3\sum a^2b^2 + (\sum ab)^2$

(1), (4) $\Rightarrow$ ***it is sufficient to prove that:*** $3\sum a^2b^2 + (\sum ab)^2 \ge 2(\sum ab)^2$

$$\Leftrightarrow 3\sum a^2b^2 \ge \left(\sum ab\right)^2 = \sum a^2b^2 + 2abc(a+b+c)$$

$$\Leftrightarrow 2\sum a^2b^2 \ge 2abc(a+b+c) \Leftrightarrow \sum a^2b^2 \ge abc(a+b+c)$$

which is true $\because \sum x^2 \ge \sum xy$, ***where*** $x = ab, y = bc, z = ca$ ***(Proved)***

198. Let ABC be a triangle and let K be its symmedian point. Prove that

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(a) $\frac{1}{KA^2}+\frac{1}{KB^2}+\frac{1}{KC^2}\geq\frac{(a^2+b^2+c^2)(a+b+c)^2}{3a^2b^2c^2}$

(b) $\frac{KB\cdot KC}{a}+\frac{KC\cdot KA}{b}+\frac{KA\cdot KB}{c}\leq\frac{3abc}{a^2+b^2+c^2}$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo ABC un triángulo y $K\rightarrow$ Symmedian Point. Prove that:

1) $\frac{1}{KA^2}+\frac{1}{KB^2}+\frac{1}{KC^2}\geq\frac{(a^2+b^2+c^2)(a+b+c)^2}{3a^2b^2c^2}$

Recordar la siguientes identidades en: K "Symmedian Point":

$$KA=\frac{bc\sqrt{2b^2+2c^2-a^2}}{a^2+b^2+c^2},\ KB=\frac{ca\sqrt{2c^2+2a^2-b^2}}{a^2+b^2+c^2},$$

$KC=\frac{ab\sqrt{2a^2+2b^2-c^2}}{a^2+b^2+c^2}$. **Se puede observar claramente que:**

$$2b^2+2c^2-a^2=(b-c)^2+(b+c)^2-a^2=(b-c)^2+(b+c+a)(b+c-a)\geq 0$$

Por lo tanto: $(2b^2+2c^2-a^2),(2c^2+2a^2-b^2),(2a^2+2b^2-c^2)>0$

La desigualdad es equivalente:

$$\frac{(a^2+b^2+c^2)^2}{b^2c^2(2b^2+2c^2-a^2)}+\frac{(a^2+b^2+c^2)^2}{c^2a^2(2c^2+2a^2-b^2)}+\frac{(a^2+b^2+c^2)^2}{a^2b^2(2a^2+2b^2-c^2)}\geq$$

$\geq\frac{(a^2+b^2+c^2)(a+b+c)^2}{3a^2b^2c^2}$. **Por la desigualdad de Cauchy:**

$$\Rightarrow 3(a^2+b^2+c^2)\left(\frac{a^2}{2b^2+2c^2-a^2}+\frac{b^2}{2c^2+2a^2-b^2}+\frac{c^2}{2a^2+2b^2-c^2}\right)\geq 3(a^2+b^2+c^2)\left(\frac{(a+b+c)^2}{3(a^2+b^2+c^2)}\right)$$

$$\Rightarrow 3(a^2+b^2+c^2)\left(\frac{a^2}{2b^2+2c^2-a^2}+\frac{b^2}{2c^2+2a^2-b^2}+\frac{c^2}{2a^2+2b^2-c^2}\right)\geq(a+b+c)^2$$

2) $\frac{KB\cdot KC}{a}+\frac{KC\cdot KA}{b}+\frac{KA\cdot KB}{c}\leq\frac{3abc}{a^2+b^2+c^2}$

La desigualdad es equivalente:

$$\sum\frac{KB\cdot KC}{a}=\frac{abc}{(a^2+b^2+c^2)^2}\sum\sqrt{(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)}\leq$$

$$\leq \frac{3abc}{a^2+b^2+c^2}$$

$$\Rightarrow \sum \sqrt{(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)} \leq 3(a^2+b^2+c^2) \ldots \textbf{(A)}$$

Realizamos los siguientes cambios de variables:

$$x = 2b^2+2c^2-a^2 > 0,\ y = 2c^2+2a^2-b^2 > 0,$$

$$z = 2a^2+2b^2-c^2 > 0 \rightarrow x+y+z = 3(a^2+b^2+c^2).$$

Por último, reemplazando en … (A):

$$\Rightarrow \sqrt{xy}+\sqrt{yz}+\sqrt{zx} \leq x+y+z \Leftrightarrow$$

$$\Leftrightarrow \left(\sqrt{x}-\sqrt{y}\right)^2+\left(\sqrt{y}-\sqrt{z}\right)^2+\left(\sqrt{z}-\sqrt{x}\right)^2 \geq 0 \ldots \textbf{(LQQD)}$$

199. In $\Delta\, ABC$ the following relationship holds:

$$\frac{\sin A}{(m_b\cdot m_c)^2}+\frac{\sin B}{(m_c\cdot m_a)^2}+\frac{\sin C}{(m_a\cdot m_b)^2} \leq \frac{\sqrt{3}}{54\cdot r^4}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Daniel Sitaru – Romania

Lemma(Gheorghe Szollosy-Romania):

$$\text{In } \Delta\, ABC: \sum \frac{1}{m_b m_c} \leq \frac{\sqrt{3}}{s}$$

Proof:

$$m_a = \frac{\sqrt{2(b^2+c^2)-a^2}}{2} \geq \frac{\sqrt{(a+b+c)(b+c-a)}}{2} = \sqrt{s(s-a)}$$

$$\sum \frac{1}{m_b m_c} \leq \sum \frac{1}{\sqrt{s(s-a)}}\cdot\frac{1}{\sqrt{s(s-a)}} = \frac{1}{s}\sum \frac{1}{\sqrt{(s-a)(s-a)}} =$$

$$= \frac{1}{s} \cdot \frac{\sum \sqrt{s-a}}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{1}{S\sqrt{s}} \sum \sqrt{s-a} \overset{CBS}{\leq} \frac{1}{S\sqrt{s}} \sqrt{3(s-a+s-b+s-c)} =$$

$$= \frac{1}{S\sqrt{s}} \sqrt{3s} = \frac{\sqrt{3}}{S}$$

Solution:

$$a \leq b \leq c \to \frac{1}{(m_b m_c)^2} \leq \frac{1}{(m_a m_b)^2} \leq \frac{1}{(m_a m_c)^2}$$

$$\sum \frac{1}{m_b m_c} \leq \frac{\sqrt{3}}{S} \quad \textit{(Lemma)}$$

$$\sum \frac{\sin A}{(m_b m_c)^2} \overset{CHEBYSHEV}{\leq} \frac{1}{3} \sum \sin A \sum \frac{1}{(m_b m_c)^2} \overset{CBS}{\leq} \frac{1}{3} \cdot \frac{3\sqrt{3}}{2} \cdot \frac{3}{S^2} \cdot \frac{1}{3} =$$

$$= \frac{\sqrt{3}}{2 \cdot r^2 \cdot s^2} \overset{MITRINOVIC}{\leq} \frac{\sqrt{3}}{2r^2 \cdot 27r^2} = \frac{\sqrt{3}}{54r^4}$$

200. In $\Delta\ ABC$ the following relationship holds:

$$\frac{a^{m+2}}{(b+c)^m} + \frac{b^{m+2}}{(c+a)^m} + \frac{c^{m+2}}{(a+b)^m} \geq \frac{\sqrt{3}S}{2^{m-2}}, m > 0$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum a^2 \cdot \frac{a^m}{(b+c)^m} = a^2 \cdot \left(\frac{a}{b+c}\right)^m + b^2 \cdot \left(\frac{b}{a+c}\right)^m + c^2 \cdot \left(\frac{c}{a+b}\right)^m$$

$$a \geq b \geq c; a^2 \geq b^2 \geq c^2; \quad \left.\begin{matrix} a+b \geq c+a \geq b+c \\ \frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b} \\ a \geq b \geq c \\ a^2 \geq b^2 \geq c^2 \end{matrix}\right\} \Rightarrow$$

$$\sum a^2 \cdot \left(\frac{a}{b+c}\right)^m \overset{Chebyshev}{\geq} \frac{1}{3} \cdot \sum (a^2) \cdot \sum \left(\frac{a}{b+c}\right)^m \overset{Chebyshev}{\geq}$$

$$\geq \frac{1}{3}\cdot\frac{1}{3^{m-1}}\cdot\sum a^2\cdot\left(\sum\frac{a}{b+c}\right)^m = \frac{1}{3^m}\cdot\sum a^2\cdot\left(\sum\frac{a}{b+c}\right)^m \geq$$

$$\geq \left(\begin{matrix}\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}\ (Nesbitt)\\ a^2+b^2+c^2\geq 4\sqrt{3}\cdot S\ (Weizenbock's)\end{matrix}\right)\geq$$

$$\geq \frac{1}{3^m}\cdot 4\sqrt{3}\cdot S\cdot\left(\frac{3}{2}\right)^m = \frac{\sqrt{3}\cdot S}{2^{m-2}} \Rightarrow (S=\Delta)$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

In any $\Delta\ ABC$,

$$\frac{a^{m+2}}{(b+c)^m}+\frac{b^{m+2}}{(c+a)^m}+\frac{c^{m+2}}{(a+b)^m}\geq\frac{\sqrt{3}\Delta}{2^{m-2}}\ \textit{where } \Delta = \textit{area of } \Delta\ ABC$$

Weitzenbock's Inequality: $a^2+b^2+c^2\geq 4\sqrt{3}\Delta$

Result: let $a_i, b_i\in(0,\infty)$ ***where*** $i=1,2,3,\dots,n$ ***and*** $m>0$ ***then***

$$\frac{a_1^{m+1}}{b_1^m}+\frac{a_2^{m+1}}{b_2^m}+\cdots+\frac{a_n^{m+1}}{b_n^m}\geq\frac{(a_1+a_2+\cdots+a_n)^{m+1}}{(b_1+b_2+\cdots+b_n)^m}$$

Now,

$$\sum_{cyc}\frac{a^{m+2}}{(b+c)^m}=\sum_{cyc}\frac{a^{2(m+1)}}{(ab+ac)^m}\geq\frac{(a^2+b^2+c^2)^{m+1}}{2^m(ab+bc+ca)^m}$$

[applying above result]

$$=\frac{1}{2^m}\cdot\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^m\cdot(a^2+b^2+c^2)\geq\frac{4\sqrt{3}\Delta}{2^m}=\frac{\sqrt{3}\Delta}{2^{m-2}}$$

Solution 3 by Soumava Chakraborty – Kolkata – India

WLOG, we may assume $a\geq b\geq c$

$$\therefore ac+a^2\geq b^2+bc\Rightarrow a(c+a)\geq b(b+c)$$

$$\Rightarrow\frac{a}{b+c}\geq\frac{b}{c+a}\Rightarrow\left(\frac{a}{b+c}\right)^m\geq\left(\frac{b}{c+a}\right)^m$$

$$\Rightarrow a\cdot\left(\frac{a}{b+c}\right)^m \ge b\left(\frac{b}{c+a}\right)^m \Rightarrow \frac{a^{m+1}}{(b+c)^m} \ge \frac{b^{m+1}}{(c+a)^m} \quad (1)$$

Again, $ab+b^2 \ge c^2+ac \Rightarrow b(a+b) \ge c(c+a)$

$$\Rightarrow \frac{b}{c+a} \ge \frac{c}{a+b} \Rightarrow \left(\frac{b}{c+a}\right)^m \ge \left(\frac{c}{a+b}\right)^m$$

$$\Rightarrow b\cdot\left(\frac{b}{c+a}\right)^m \ge c\cdot\left(\frac{c}{a+b}\right)^m \Rightarrow \frac{b^{m+1}}{(c+a)^m} \ge \frac{c^{m+1}}{(a+b)^m} \quad (2)$$

Now, $LHS = a\cdot\frac{a^{m+1}}{(b+c)^m} + b\cdot\frac{b^{m+1}}{(c+a)^m} + c\cdot\frac{c^{m+1}}{(a+b)^m}$

$$\overset{Chebyshev}{\ge} \frac{1}{3}\left(\sum a\right)\left\{\frac{a^{m+1}}{(b+c)^m}+\frac{b^{m+1}}{(c+a)^m}+\frac{c^{m+1}}{(a+b)^m}\right\} \text{ (using (1), (2))}$$

$$\overset{Radon}{\ge} \frac{1}{3}\left(\sum a\right)\frac{(\sum a)^{m+1}}{(2\sum a)^m} = \frac{(\sum a)(\sum a)(\sum a)^m}{3\cdot 2^m(\sum a)^m} = \frac{4s^2}{2^m\cdot 3}$$

$$= \frac{s\cdot s}{3\cdot 2^{m-2}} \ge \frac{(3\sqrt{3}r)s}{3\cdot 2^{m-2}} = \frac{\sqrt{3}s}{2^{m-2}} = RHS \text{ (Proved)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru