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THE BEAUTY OF AN EXTREME PROBLEM

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Abstract. The article refers to the method of the exploitation and development of an extreme problem.

The exploitation and development of a problem is the important thought in researching mathematics. There are many methods of creating problem such as finding many solutions, generalized problems, similar problems, etc. We will discover these things through the following one.

Problem 1

Let x, y be the positive real numbers such that $x^2+y^2=1$. Find the minimum value of expression

$$\frac{1}{1-x}+\frac{1}{1-y}?$$

This problem is given by us on the Forum Solving the inequality and it is attractive to many people. The following are some solutions.

Solution 1 (*Nguyen Ngoc Giang - Nguyen Viet Duong - Nguyen Van Tha*)

We have the Lagrange function as follows

$$L=\frac{1}{1-x}+\frac{1}{1-y}+\lambda(1-x^2-y^2)$$

The extreme points are the solutions of the system of equations

$$\begin{cases}\dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial y}=0\\ x^2+y^2=1\end{cases} \Leftrightarrow \begin{cases}\dfrac{1}{(1-x)^2}-2\lambda x &=0\\ \dfrac{1}{(1-y)^2}-2\lambda y &=0\\ x^2+y^2 &=1\end{cases} \Rightarrow \begin{cases}\dfrac{1}{(1-x)^2}-2\lambda x=0 & (1)\\ x(1-x)^2=y(1-y)^2 & (2)\\ x^2+y^2=1 & (3)\end{cases}$$

The equation (2) follows that

$$\begin{aligned}&x(1-2x-x^2)=y(1-2y-y^2)\\ \Leftrightarrow\ &x-2x^2-x^3=y-2y^2-y^3\\ \Leftrightarrow\ &(x-y)-2(x-y)(x+y)+(x-y)(x^2+xy+y^2)\\ \Leftrightarrow\ &(x-y)(2-2(x+y)+xy)=0\end{aligned}$$

$$\Leftrightarrow \left[\begin{array}{c}x-y=0\\ 2-2(x+y)+\dfrac{(x+y)^2-(x^2+y^2)}{2}=0\end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{c} x-y=0 \\ (x+y)^2-4(x+y)+3=0\ (x^2+y^2=1) \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} x=y \\ x+y=1 \\ x+y=3 \end{array}\right.$$

Since $0<x,y<1$ and $x^2+y^2=1$, we follow that $x(1-x)^2=y(1-y)^2$ has only a solution $x=y=\frac{\sqrt{2}}{2}$.

From that we have an extreme point $(x,y)=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$. It follows

$$\left(\frac{1}{1-x}+\frac{1}{1-y}\right)_{min}=\frac{1}{1-\frac{\sqrt{2}}{2}}+\frac{1}{1-\frac{\sqrt{2}}{2}}=\frac{2\sqrt{2}}{\sqrt{2}-1}=4+2\sqrt{2}$$

.

The equality occurs if and only if $x=y=\frac{\sqrt{2}}{2}$

Solution 2(*Nguyen Viet Hung*)

From the given condition we have

$$y^2=1-x^2=(1-x)(1+x)$$

which implies that $\frac{1}{1-x}=\frac{1+x}{y^2}$. Similarly $\frac{1}{1-y}=\frac{1+y}{x^2}$. Now we use the Cauchy-Schwarz inequality to get

$$\begin{aligned}\frac{1}{1-x}+\frac{1}{1-y}&=\frac{1}{x^2}+\frac{1}{y^2}+\frac{x}{y^2}+\frac{y}{x^2}\\ &\ge\frac{4}{x^2+y^2}+\frac{1}{x}+\frac{1}{y}\\ &\ge 4+\frac{4}{x+y}\\ &\ge 4+\frac{4}{\sqrt{2(x^2+y^2)}}\\ &=4+2\sqrt{2}\end{aligned}$$

The equality occurs if and only if $x=y=\frac{\sqrt{2}}{2}$. So

$$\left(\frac{1}{1-x}+\frac{1}{1-y}\right)_{min}=4+2\sqrt{2}.$$

Solution 3 (*Le Nguyen Thanh Nhon*)

We prove the following inequality

$$\frac{1}{1-x}\ge(4+3\sqrt{2})x^2-\frac{\sqrt{2}}{2};\forall x\in(0,1)$$

We have

$$\frac{1}{1-x}-(4+3\sqrt{2})x^2+\frac{\sqrt{2}}{2}=\frac{\frac{1}{2}(4+3\sqrt{2})(x-1+\sqrt{2})(2x-\sqrt{2})^2}{2(1-x)}.$$

Since $0 < x < 1$ we have $\dfrac{\frac{1}{2}(4+3\sqrt{2})(x-1+\sqrt{2})(2x-\sqrt{2})^2}{2(1-x)} \geq 0$

Thus, $\dfrac{1}{1-x} \geq (4+3\sqrt{2})x^2 - \dfrac{\sqrt{2}}{2}; \forall x \in (0,1)$ (4)

Similarly, $\dfrac{1}{1-y} \geq (4+3\sqrt{2})y^2 - \dfrac{\sqrt{2}}{2}; \forall y \in (0,1)$ (5)

Since (4) and (5), it follows

$$\frac{1}{1-x} + \frac{1}{1-y} \geq (4+3\sqrt{2})(x^2+y^2) - 2\frac{2}{\sqrt{2}} = 4+2\sqrt{2}$$

The equality occurs if and only if $x = y = \dfrac{1}{\sqrt{2}}$

Solution 4 *(Ghimisi Dumitrel)*

Since $x^2+y^2 = 1, x, y, > 0$ it follows that there is a number $\alpha \in (0, \frac{\pi}{2})$ such that $x = \cos\alpha, y = \sin\alpha$.

It follows

$$\frac{1}{1-x} + \frac{1}{1-y} = \frac{1}{1-\cos\alpha} + \frac{1}{1-sin\alpha} = f(\alpha).$$

We have

$$f'(\alpha) = \frac{\sin\alpha}{1-2\cos\alpha+\cos^2\alpha} - \frac{\cos\alpha}{1-2\sin\alpha+\sin^2\alpha}$$
$$f'(\alpha) = \frac{(\sin\alpha-\cos\alpha) - 2(\sin^2\alpha-\cos^2\alpha) + \sin^3\alpha - \cos^3\alpha}{(1-2\cos\alpha+\cos^2\alpha)(1-2\sin\alpha+\sin^2\alpha)}$$
$$f'(\alpha) = \frac{(\sin\alpha-\cos\alpha)(1-2\sin\alpha-2\cos\alpha+1+\sin\alpha\cos\alpha)}{(1-2\cos\alpha+\cos^2\alpha)(1-2\sin\alpha+\sin^2\alpha)}$$

Denote by $\sin\alpha+\cos\alpha = p$, it follows $\sin\alpha\cos\alpha = \dfrac{p^2-1}{2}$. We have

$$f'(\alpha) = 0 \Leftrightarrow \left[\begin{array}{l} \sin\alpha = \cos\alpha \\ p^2-4p+3 = 0 \end{array}\right.$$
$$\Leftrightarrow \left[\begin{array}{c} \alpha = \frac{\pi}{4} \\ \sin\alpha+\cos\alpha = 1 \\ \sin\alpha+\cos\alpha = 3 \end{array}\right.$$

Since $0 < \sin\alpha, \cos\alpha < 1, \alpha \in (0, \frac{\pi}{2})$, it follows

$$f'(\alpha) = 0 \Leftrightarrow \alpha = \frac{\pi}{4}$$

From that, we have the variation chat as follows

α	0		$\frac{\pi}{4}$		$\frac{\pi}{2}$
$f'(\alpha)$		-	0	+	
$f(\alpha)$	$+\infty$	↘	$2(2+\sqrt{2})$	↗	$+\infty$

$$f(\frac{\pi}{4}) = \frac{2}{1-\frac{\sqrt{2}}{2}} = \frac{4(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} = 2(2+\sqrt{2})$$

$\min f(\alpha) = 4 + 2\sqrt{2}$ if and only if $\alpha = \frac{\pi}{4}$

It follows that $\left(\frac{1}{1-x} + \frac{1}{1-y}\right)_{min} = 4 + 2\sqrt{2} \Leftrightarrow x = y = \frac{\sqrt{2}}{2}$

The equality occurs if and only if $x = y = \frac{\sqrt{2}}{2}$

Solution 5 *(Marian Dinca)*

$$\begin{aligned}\frac{1}{1-x} + \frac{1}{1-y} &= \frac{1+x}{1-x^2} + \frac{1+y}{1-y^2} \\ &= \frac{1+x}{y^2} + \frac{1+y}{x^2} \\ &= \frac{x^2+x^3}{x^2y^2} + \frac{y^2+y^3}{x^2y^2} \\ &= \frac{1+x^3+y^3}{(xy)^2}\end{aligned}$$

According to Jensen inequality, we obtain

$$x^3 + y^3 = \left(x^2\right)^{\frac{3}{2}} + \left(y^2\right)^{\frac{3}{2}} \geq 2\left(\frac{x^2+y^2}{2}\right)^{\frac{3}{2}} = \frac{\sqrt{2}}{2}.$$

We also have

$$(xy)^2 = x^2y^2 \leq \left(\frac{x^2+y^2}{2}\right)^2 = \frac{1}{4}.$$

From these, we have

$$\frac{1+x^3+y^3}{(xy)^2} \geq \frac{1+\frac{\sqrt{2}}{2}}{\frac{1}{4}} = 4\left(1+\frac{\sqrt{2}}{2}\right) = 4 + 2\sqrt{2}$$

The equality occurs if and only if $x = y = \dfrac{\sqrt{2}}{2}$

The problem is the same as the following one.

Problem 2

Let x, y be the positive real numbers such that $x^2 + y^2 = 1$. Find the minimum value of expression

$$\frac{1}{1 - x^2\sqrt{x}} + \frac{1}{1 - y^2\sqrt{y}}?$$

Solution *(Nguyen Viet Hung)*

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\frac{1}{1 - x^2\sqrt{x}} + \frac{1}{1 - y^2\sqrt{y}} &= 2 + \frac{x^2\sqrt{x}}{1 - x^2\sqrt{x}} + \frac{y^2\sqrt{y}}{1 - y^2\sqrt{y}} \\
&= 2 + \frac{x^4}{x\sqrt{x} - x^4} + \frac{y^4}{y\sqrt{y} - y^4} \\
&\geq 2 + \frac{(x^2 + y^2)^2}{x\sqrt{x} + y\sqrt{y} - (x^4 + y^4)} \\
&\geq 2 + \frac{(x^2 + y^2)^2}{\sqrt{(x^2 + y^2)(x + y)} - \frac{1}{2}(x^2 + y^2)^2} \\
&\geq 2 + \frac{(x^2 + y^2)^2}{\sqrt{(x^2 + y^2)\sqrt{2(x^2 + y^2)}} - \frac{1}{2}(x^2 + y^2)^2} \\
&= 2 + \frac{2}{2\sqrt[4]{2} - 1} = \frac{4\sqrt[4]{2}}{2\sqrt[4]{2} - 1}.
\end{aligned}$$

The equality occurs if and only if $x = y = \dfrac{\sqrt{2}}{2}$.

Thus,

$$\left(\frac{1}{1 - x^2\sqrt{x}} + \frac{1}{1 - y^2\sqrt{y}}\right)_{min} = \frac{4\sqrt[4]{2}}{2\sqrt[4]{2} - 1}.$$

We now give a similar problem as follows

Problem 3

Let x, y be the positive real numbers such that $x^2 + y^2 = 1$. Find the minimum value of expression

$$\frac{1}{1 - x^3} + \frac{1}{1 - y^3}?$$

Solution 1 *(Nguyen Ngoc Giang - Nguyen Viet Duong - Nguyen Van Tha)*

We have the Lagrange function as follows

$$L = \frac{1}{1 - x^3} + \frac{1}{1 - y^3} + \lambda(1 - x^2 - y^2)$$

The extreme point is the solution of the system of equations

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0 \\ x^2 + y^2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{3x^2}{(1-x^3)^2} - 2\lambda x = 0 \\ \dfrac{3y^2}{(1-y^3)^2} - 2\lambda y = 0 \\ x^2+y^2=1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{3x^2}{(1-x^3)^2} - 2\lambda x = 0 & (1) \\ \dfrac{x}{1-2x^3+x^6} = \dfrac{y}{1-2y^3+y^6} & (2) \\ x^2+y^2=1 & (3) \end{cases}$$

The equation (2) follows that

$$x - 2xy^3 + xy^6 = y - 2x^3y + yx^6$$

$$\Leftrightarrow (x-y) - 2xy(y^2-x^2) + xy(y^5-x^5) = 0$$
$$\Leftrightarrow (x-y)(1+2xy(x+y) - xy(x^4+x^3y+x^2y^2+xy^3+y^4)) = 0$$
$$\Leftrightarrow (x-y)\left[1+2xy(x+y) - xy((x^2+y^2)^2 - x^2y^2 + xy(x^2+y^2))\right] = 0$$
$$\Leftrightarrow (x-y)\left[1+2x^2y+2xy^2 - xy(1-x^2y^2+xy)\right] = 0$$
$$\Leftrightarrow (x-y)\left[1+2x^2y+2xy^2 - xy + x^3y^3 - x^2y^2\right] = 0$$
$$\Leftrightarrow (x-y)\left[2x^2y+2xy^2 + (1-xy) + x^2y^2(x-y)\right] = 0$$
$$\Leftrightarrow (x-y)\left[2x^2y+2xy^2 + (1-xy)(1-xy)(1+xy)\right] = 0$$
$$\Leftrightarrow (x-y)\left[2x^2y+2xy^2 + (1-xy)^2(1+xy)\right] = 0$$

Since $0 < x, y < 1$ and $x^2+y^2 = 1$ we follow that $\dfrac{x}{1-2x^3+x^6} = \dfrac{y}{1-2y^3+y^6}$ has only a solution $x = y = \dfrac{\sqrt{2}}{2}$

From that we have an extreme point$(x,y) = (\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2})$. It follows

$$\left(\frac{1}{1-x^3} + \frac{1}{1-y^3}\right)_{min} = \frac{1}{1-\left(\frac{\sqrt{2}}{2}\right)^3} + \frac{1}{1-\left(\frac{\sqrt{2}}{2}\right)^3}$$
$$= \frac{4\sqrt{2}}{2\sqrt{2}-1} = \frac{4\sqrt{2}(2\sqrt{2}+1)}{7}$$
$$= \frac{16+4\sqrt{2}}{7}.$$

Solution 2 *(Imad Zak)*
Let $P(x) = (48+54\sqrt{2})x^5 + (32-13\sqrt{2})x^3 - (48+54\sqrt{2})^2 + 13\sqrt{2} + 17$
We have

$$P(x) = \left(x - \frac{\sqrt{2}}{2}\right)^2 . \left[(48+54\sqrt{2})x^3 + (108+48\sqrt{2})x^2 + (104+68\sqrt{2})x + 26\sqrt{2} + 34\right]$$

$\Rightarrow P(x) \geq 0, \forall x \in (0,1)$ *and* $x^2+y^2 = 1$

We have

$$\frac{1}{1-x^3}-\frac{6}{49}(9\sqrt{2}+8)x^2-\frac{1}{49}(32-13\sqrt{2})=\frac{P(x)}{49(1-x^3)}$$
$$\Rightarrow \frac{1}{1-x^3}\geq \frac{6}{49}(9\sqrt{2}+8)x^2+\frac{1}{49}(32-13\sqrt{2}).$$

From this, we follow

$$\begin{aligned}\frac{1}{1-x^3}+\frac{1}{1-y^3}&\geq \frac{6}{49}(9\sqrt{2}+8)(x^2+y^2)+\frac{2}{49}(32-13\sqrt{2})\\&=\frac{6(9\sqrt{2}+8)+2(32-13\sqrt{2})}{49}\\&=\frac{112+28\sqrt{2}}{49}\\&=\frac{16+4\sqrt{2}}{7}\end{aligned}$$

The equality occurs if and only if $x=y=\dfrac{\sqrt{2}}{2}$

Solution 3 *(Imad Zak)*

We have

$$\sqrt[4]{\frac{x^4+y^4}{2}}\geq\sqrt{\frac{x^2+y^2}{2}}=\sqrt{\frac{1}{2}}\geq\frac{x+y}{2}\Rightarrow\begin{cases}x^4+y^4\geq\dfrac{1}{2} & (3)\\ x+y\geq\sqrt{2} & (4)\end{cases}$$

We have

$$\begin{aligned}\frac{1}{1-x^3}+\frac{1}{1-y^3}&=\left(1+\frac{x^3}{1-x^3}\right)+\left(1+\frac{y^3}{1-y^3}\right)\\&=2+\frac{x^3}{1-x^3}+\frac{y^3}{1-y^3}.\end{aligned}$$

Applying Cauchy-Bouniakowski-Schwarz, we obtain

$$\begin{aligned}\frac{x^4}{x-x^4}+\frac{y^4}{y-y^4}&\geq\frac{(x^2+y^2)^2}{(x+y)-(x^4+y^4)}\\&\geq\frac{1}{\sqrt{2}-\frac{1}{2}}\quad(\textit{by } (3) \textit{ and } (4))\end{aligned}$$

It follows

$$\frac{x^4}{x-x^4}+\frac{y^4}{y-y^4}\geq\frac{2}{2\sqrt{2}-1}=\frac{2(2\sqrt{2}+1)}{7}.$$

Thus,

$$\frac{1}{1-x^3}+\frac{1}{1-y^3}\geq 2+\frac{2(2\sqrt{2}+1)}{7}=\frac{16+4\sqrt{2}}{7}$$

The equality occurs if and only if $x=y=\dfrac{\sqrt{2}}{2}$

We generalize problem 1 to the following one

Problem 4

Let x, y, z be the positive real numbers such that $x^2 + y^2 + z^2 = 1$. Find the minimum value of expression

$$\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z}?$$

Solution *(Tran Hoang)*
Since $x^2 + y^2 + z^2 = 1$ and $x, y, z \neq 1 \Rightarrow 1 - x > 0$
We need to prove:

$$\frac{1}{1-x} \geq \frac{3}{4} + \frac{9+6\sqrt{3}}{4}x^2$$

It is equivalent to

$$\frac{(9+6\sqrt{3})\left(x - \frac{\sqrt{3}}{3}\right)^2\left(x + \frac{2\sqrt{3}-3}{3}\right)}{4(1-x)} \geq 0$$

It holds true. It means that

$$\frac{1}{1-x} \geq \frac{3}{4} + \frac{9+6\sqrt{3}}{4}x^2 \quad (1)$$

Similarly,

$$\frac{1}{1-y} \geq \frac{3}{4} + \frac{9+6\sqrt{3}}{4}y^2 \quad (2)$$

$$\frac{1}{1-z} \geq \frac{3}{4} + \frac{9+6\sqrt{3}}{4}z^2 \quad (3)$$

Adding these inequality (1), (2) and (3), termwise, we obtain

$$\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} \geq \frac{3}{4}.3 + \frac{9+6\sqrt{3}}{4} = \frac{9+3\sqrt{3}}{2}.$$

The equality occurs if and only if $x = y = z = \dfrac{\sqrt{3}}{3}$.
This problem is the same as the following one
Problem 5
Let x, y, z be the positive real numbers such that $x^2 + y^2 + z^2 = 1$. Find the minimum value of expression

$$\frac{1}{1-x\sqrt{x}} + \frac{1}{1-y\sqrt{y}} + \frac{1}{1-z\sqrt{z}}?$$

Solution *(Imad Zak)*
Let $x = a^2, y = b^2, z = c^2$ then we have $a^4 + b^4 + c^4 = 1$
Let $A = a\sqrt[4]{3}, B = b\sqrt[4]{3}, C = c\sqrt[4]{3} \Rightarrow A^4 + B^4 + C^4 = 3$
We need to find the minimum value of expression

$$\frac{m}{m-A^3} + \frac{m}{m-B^3} + \frac{m}{m-C^3}; (m = \sqrt[4]{27})$$

Let $f(x) = \dfrac{m}{m-x^3}, g(x) = \dfrac{3m}{4(m-1)^2}x^4 + \dfrac{4m^2-7m}{4(m-1)^2}$
We have $f(x) - g(x) = \dfrac{m(x-1)^2.P(x)}{4(m-1)^2(m-x^3)}$, where

$$P(x) = 3x^5 + 6x^4 + 9x^3 + 3(4-m)x^2 + 2(4-m)x + 4 - m > 0$$

We have

$$\begin{aligned} f(x) \geq g(x) \Rightarrow & \frac{m}{m-A^3} + \frac{m}{m-B^3} + \frac{m}{m-C^3} \\ & \geq \frac{3m}{4(m-1)^2}(A^4+B^4+c^4) + \frac{12m^2-21m}{4(m-1)^2} \\ & = \frac{12m^2-12m}{4(m-1)^2} = \frac{3m}{m-1}. \end{aligned}$$

Thus, $\left(\frac{1}{1-x\sqrt{x}} + \frac{1}{1-y\sqrt{y}} + \frac{1}{1-z\sqrt{z}}\right)_{min} = \frac{3\sqrt[4]{27}}{\sqrt[4]{27}-1}$.

The equality occurs if and only if $x = y = \frac{\sqrt{3}}{3}$

We have some creative methods of a problem. Do you have any comments on this article. Please share with us.

The last are some exercises.

Problem 6

Let $x_1, x_2, ..., x_n$ be the positive real numbers such that $x_1^2 + x_2^2 + ... + x_n^2 = 1$. Find the minimum value of expression

$$\frac{1}{1-x_1} + \frac{1}{1-x_2} + ... + \frac{1}{1-x_n}?$$

Problem 7

Let $x_1, x_2, x_3, ..., x_n$ be the positive real numbers such that $x_1^2 + x_2^2 + ... + x_n^2 = 1$. Find the minimum value of expression

$$\frac{1}{1-x_1\sqrt{x_1}} + \frac{1}{1-x_2\sqrt{x_2}} + ... + \frac{1}{1-x_n\sqrt{x_n}}?$$

Problem 8

Let $x_1, x_2, x_3, ..., x_n$ be the positive real numbers such that $x_1^2 + x_2^2 + ... + x_n^2 = 1$. Find the minimum value of expression

$$\frac{1}{1-x_1\sqrt[n]{x_1}} + \frac{1}{1-x_2\sqrt[n]{x_2}} + ... + \frac{1}{1-x_n\sqrt[n]{x_n}}?$$

Problem 9

Let $x_1, x_2, x_3, ..., x_n$ be the positive real numbers such that $x_1^2 + x_2^2 + ... + x_n^2 = 1$. Find the minimum value of expression

$$\frac{1}{1-x_1^n} + \frac{1}{1-x_2^n} + ... + \frac{1}{1-x_n^n}?$$

Problem 10

Let $x_1, x_2, x_3, ..., x_n$ be the positive real numbers such that $x_1^2 + x_2^2 + ... + x_n^2 = 1$. Find the minimum value of expression

$$\frac{1}{1-x_1^k\sqrt[n]{x_1}} + \frac{1}{1-x_2^k\sqrt[n]{x_2}} + ... + \frac{1}{1-x_n^k\sqrt[n]{x_n}}?$$

References

[1] Forum Nice problem - beauty solution (Vietnamese)
[2] Forum Solving the inequality (Vietnamese).

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