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The discovery of the Batinetu 's inequality

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Abstract

We give a method of discovering the Batinetu 's inequality.

1 Introduction

The Batinetu 's inequality is nice and interesting. The Romanian Mathematical Magazine gave many solutions of this one. In this paper, we give creative methods of the Batinetu 's inequality which are finding solutions, similar problems and generalized problems.

Problem 1. (The Batinetu 's inequality)
Let x, y, z be positive real numbers. Prove that

$$\frac{x}{(y+z)^3} + \frac{y}{(z+x)^3} + \frac{z}{(x+y)^3} \geq \frac{27}{8(x+y+z)^2}.$$

2 Main results

The following are 5 solutions given by [2]
Solution 1 *(Kevin Soto Palacios)*
Applying the Nesbitt 's inequality, we have

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}. \tag{1}$$

We easily prove the following simple inequality

Lemma 2. *Prove that*

$$(x+y+z)^2 \geq 3(xy+yz+zx).$$

Applying the Cauchy 's inequality, we have:

$$\begin{aligned}\frac{x}{(y+z)^3} + \frac{y}{(z+x)^3} + \frac{z}{(x+y)^3} &= \frac{\left(\frac{x}{y+z}\right)^2}{x(y+z)} + \frac{\left(\frac{y}{z+x}\right)^2}{y(z+x)} + \frac{\left(\frac{z}{x+y}\right)^2}{z(x+y)} \\ &\geq \frac{\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right)^2}{2(xy+yz+zx)} \\ &\geq \frac{9 \times 3}{4 \times 2(x+y+z)^2} = \frac{27}{8(x+y+z)^2}.\end{aligned}$$

The equality occurs if and only if $x = y = z$.

Solution 2 (*Soumitra Mandal*)

The Radon 's inequality is stated as follows

Lemma 3. *If* $x_k,\ a_k > 0,\ k \in \{1,\ 2,\ ...,\ n\},\ p > 0,$ *then*

$$\frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} + ... + \frac{x_n^{p+1}}{a_p^n} \geq \frac{(x_1 + x_2 + ... + x_n)^{p+1}}{(a_1 + a_2 + ... + a_n)^p}.$$

Applying the Radon's inequality, we have

$$\begin{aligned}\sum_{cyc}\frac{x}{(y+z)^3} = \sum_{cyc}\frac{x^4}{(xy+xz)^3} &\geq^{Radon} \frac{(x+y+z)^4}{8(xy+yz+xz)^3} \\ &\geq \frac{(x+y+z)^4}{8\left(\frac{(x+y+z)^2}{3}\right)^3} \\ &\geq \frac{27}{8(x+y+z)^2}\end{aligned}$$

The equality occurs if and only if $x = y = z$.

Solution 3 *(Soumava Chakraborty)*

Without loss of generality, suppose that $x \geq y \geq z$. We have

$$\frac{x}{y + z} \geq \frac{y}{z + x} \Leftrightarrow zx + x^2 \geq y^2 + yz.$$

This thing holds true because $x \geq y$.

Similarly,

$$\frac{y}{z + x} \geq \frac{z}{x + y}, \frac{x}{y + z} \geq \frac{y}{z + x} \geq \frac{z}{x + y}.$$

Also,

$$\frac{1}{(y + z)^2} \geq \frac{1}{(z + x)^2} \Leftrightarrow x \geq y.$$

Similarly,

$$\frac{1}{(z + x)^2} \geq \frac{1}{(x + y)^2}, \frac{1}{(y + z)^2} \geq \frac{1}{(z + x)^2} \geq \frac{1}{(x + y)^2}.$$

$$\begin{aligned}LHS &= \left(\frac{x}{y + z}\right) \cdot \frac{1}{(y + z)^2} + \left(\frac{y}{z + x}\right) \cdot \frac{1}{(z + x)^2} + \left(\frac{z}{x + y}\right) \cdot \frac{1}{(x + y)^2} \\ &\overset{Chebyshev}{\geq} \frac{1}{3}\sum\frac{x}{y + z}\sum\frac{1}{(y + z)^2} \overset{Nesbitt}{\geq} \frac{1}{3} \cdot \frac{3}{2}\sum\frac{1}{(y + z)^2} \\ &= \frac{1}{2}\sum\frac{1^3}{(y + z)^2} \overset{Radon}{\geq} \frac{1}{2}.\frac{(1 + 1 + 1)^3}{(2\sum x)^2} = \frac{27}{8(x + y + z)^2} = RHS.\end{aligned}$$

The equality occurs if and only if $x = y = z$.

Solution 4 *(Imad Zak)*

Letting $x + y + z = 3$, the inequality becomes the following problem.

Problem 4. *Prove that*

$$\sum \frac{x}{(3\ -\ x)^3}\ \geq\ \frac{3}{8}$$

with $0\ <\ x\ <\ 3$.

We have:

$$\frac{x}{(3-x)^3}\ -\ \left(\frac{5x}{16}\ -\ \frac{3}{16}\right)\ =\ \frac{\overbrace{(x-1)^2}^{(+)}\overbrace{(5x^2\ -\ 38x\ +\ 81)}^{\Delta<0\Rightarrow(+)}}{16(3-x)^3}\ \geq\ 0.$$

It follows

$$\sum \frac{x}{(3-x)^3}\ \geq\ \sum\left(\frac{5x}{16}\ -\ \frac{3}{16}\right)\ =\ \frac{5(x\ +\ y\ +\ z)}{16}\ -\ \frac{9}{16}\ =\ \frac{6}{16}\ =\ \frac{3}{8}.$$

The equality occurs if and only if $x = y = z$.

Solution 5 *(George Apostolopoulos)*

Let $x + y + z = k > 0$. Consider the function: $f(t)\ =\ \frac{t}{(k\ -\ t)^3},\ t\ >\ 0,\ t\ <\ k.$

We have

$$f'(t)\ =\ \frac{(k\ -\ t)^3\ -\ 3(-1)(k-t)^2 t}{(k\ -\ t)^6}\ =\ \frac{k\ +\ 2t}{(k\ -\ t)^4}$$

$$f''(t)\ =\ \frac{2(k\ -\ t)^4\ -\ 4(-1)(k-t)^3(k\ +\ 2t)}{(k\ -\ t)^8}\ =\ \frac{6k\ +\ 6t}{(k\ -\ t)^5}\ \geq\ 0.$$

Thus, the function f is convex on $(0;\ +\infty)$.

Applying the Jensen 's inequality, we have

$$f(x)\ +\ f(y)\ +\ f(z)\ \geq\ 3f\left(\frac{x\ +\ y\ +\ z}{3}\right).$$

Namely, we have

$$\frac{x}{(y\ +\ z)^3}\ +\ \frac{y}{(z\ +\ x)^3}\ +\ \frac{z}{(x\ +\ y)^3}\ \geq\ 3f\left(\frac{k}{3}\right)\ =\ 3.\frac{\frac{k}{3}}{\left(k\ -\ \frac{k}{3}\right)^3}$$

$$=\ \frac{27}{8k^2}\ =\ \frac{27}{8(x\ +\ y\ +\ z)^2}.$$

The similar problem of problem 1 is as follows:

Problem 5. *Let x, y, z be positive real numbers. Prove that*

$$\frac{x^3}{y\ +\ z}\ +\ \frac{y^3}{z\ +\ x}\ +\ \frac{z^3}{x\ +\ y}\ \geq\ \frac{3}{2}\left(\frac{x\ +\ y\ +\ z}{3}\right)^2$$

We now generalize problem 5 to the following one:

Problem 6. *Let x, y, z be positive real numbers and n be a positive integer number. Prove that*

$$\frac{x^n}{y\ +\ z}\ +\ \frac{y^n}{z\ +\ x}\ +\ \frac{z^n}{x\ +\ y}\ \geq\ \frac{3}{2}\left(\frac{x\ +\ y\ +\ z}{3}\right)^{n-1}$$

We now restrict problem 6 to the following one:

Problem 7. ([1, p. 348 - 349])
Let x, y, z be positive real numbers and n be positive integer number. Prove that

$$\frac{x^n}{y + z} + \frac{y^n}{z + x} + \frac{z^n}{x + y} \geq \frac{1}{2}\left(x^{n-1} + y^{n-1} + z^{n-1}\right)$$

We prove this inequality as follows
- If $n = 1$, we obtain the Nesbitt 's inequality.
- If $n \geq 2$ then from $x, y > 0$, we have

$$\begin{aligned} &(x^{n-2} - y^{n-2})(x - y) \geq 0 \\ &\Rightarrow (x^{n-1} - x^{n-2}y - xy^{n-2} + y^{n-1} \geq 0 \\ &\Rightarrow x^{n-1} + y^{n-1} \geq x^{n-2}y + y^{n-2}x \ (1) \end{aligned}$$

Similarly, we have:

$$\begin{aligned} y^{n-1} + z^{n-1} \geq y^{n-2}z + z^{n-2}y \ (2) \\ z^{n-1} + x^{n-1} \geq z^{n-2}x + x^{n-2}z \ (3) \end{aligned}$$

On the other hand, applying the Cauchy 's inequality to nonnegative real numbers, we have

$$\begin{aligned} &\frac{x^n}{y + z} + \frac{x^{n-2}(y + z)}{4} \geq 2\sqrt{\frac{x^n}{y + z} \cdot \frac{x^{n-2}(y + z)}{4}} \\ &\Rightarrow \frac{x^n}{y + z} + \frac{x^{n-2}y + x^{n-2}z}{4} \geq x^{n-1} \ (4) \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \frac{y^n}{z + x} + \frac{y^{n-2}z + y^{n-2}x}{4} \geq y^{n-1} \ (5) \\ \frac{z^n}{x + y} + \frac{z^{n-2}x + z^{n-2}y}{4} \geq z^{n-1} \ (6) \end{aligned}$$

From (1), (2), (3), (4), (5), (6), we have:

$$\frac{x^n}{y + z} + \frac{y^n}{z + x} + \frac{z^n}{x + y} + \frac{x^{n-1} + y^{n-1}}{4} + \frac{y^{n-1} + z^{n-1}}{4} + \frac{z^{n-1} + x^{n-1}}{4}$$
$$\geq x^{n-1} + y^{n-1} + z^{n-1}.$$

It follows

$$\frac{x^n}{y + z} + \frac{y^n}{z + x} + \frac{z^n}{x + y} \geq \frac{1}{2}\left(x^{n-1} + y^{n-1} + z^{n-1}\right).$$

Using the inductive method, we easily prove that

$$x^{n-1} + y^{n-1} + z^{n-1} \geq 3\left(\frac{x + y + z}{3}\right)^{n-1}.$$

The restricted problem of problem 7 is as follows

Problem 8. ([1, p.350])
Let x, y, z be positive real numbers and let n be a positive integer number. Prove that

$$\frac{x^n}{y+z}+\frac{y^n}{z+x}+\frac{z^n}{x+y} \geq \frac{3}{2}\left(\frac{x^n+y^n+z^n}{x+y+z}\right).$$

Without loss of generality, suppose that $x \leq y \leq z$. We have

$$x+y \leq x+z \leq y+z$$

and

$$\frac{z^n}{x+y} \geq \frac{y^n}{z+x} \geq \frac{x^n}{y+z}.$$

Applying the Chebyshev 's inequality, we obtain:

$$((x+y)+(x+z)+(y+z))\left(\frac{z^n}{x+y}+\frac{y^n}{z+x}+\frac{x^n}{y+z}\right) \geq 3(x^n+y^n+z^n)$$
$$\Rightarrow \frac{x^n}{y+z}+\frac{y^n}{z+x}+\frac{z^n}{x+y} \geq \frac{3}{2}\left(\frac{x^n+y^n+z^n}{x+y+z}\right).$$

We have a lemma as follows

Lemma 9. *Prove that*

$$3(x+y+z+t)^2 \geq 8(xy+xz+xt+yz+yt+zt).$$

Indeed, we have

$$3(x+y+z+t)^2 = 3(x^2+y^2+z^2+t^2)+6xy+6xz+6xt+6yz+6yt+6zt.$$

From this, it follows

$$\begin{aligned} & 3(x+y+z+t)^2 \geq 8(xy+xz+xt+yz+yt+zt) \\ \Leftrightarrow\ & 3x^2+3y^2+3z^2+3t^2-2xy-2xz-2xt-2yz-2yt-2zt \geq 0 \\ \Leftrightarrow\ & (x-y)^2+(x-z)^2+(x-t)^2+(y-z)^2+(y-t)^2+(z-t)^2 \geq 0. \end{aligned}$$

An another generalized problem of problem 1 is as follows

Problem 10. *Let x, y, z, t be positive real numbers. Prove that*

$$\frac{x}{(y+z)^3}+\frac{x}{(y+t)^3}+\frac{x}{(z+t)^3}+\frac{y}{(x+z)^3}+\frac{y}{(x+t)^3}+\frac{y}{(z+t)^3}$$
$$+\frac{z}{(x+y)^3}+\frac{z}{(x+t)^3}+\frac{z}{(y+t)^3}+\frac{t}{(x+y)^3}+\frac{t}{(x+z)^3}+\frac{t}{(y+z)^3}$$
$$\geq \frac{24}{(x+y+z+t)^2}.$$

We prove this problem as follows
Applying the Nesbitt 's inequality, we obtain

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geq \frac{3}{2}.$$

Applying the Cauchy 's inequality, we obtain

$$\frac{x}{(y+z)^3}+\frac{x}{(y+t)^3}+\frac{x}{(z+t)^3}+\frac{y}{(x+z)^3}+\frac{y}{(x+t)^3}+\frac{y}{(z+t)^3}$$
$$+\frac{z}{(x+y)^3}+\frac{z}{(x+t)^3}+\frac{z}{(y+t)^3}+\frac{t}{(x+y)^3}+\frac{t}{(x+z)^3}+\frac{t}{(y+z)^3}$$
$$=\frac{\left(\frac{x}{y+z}\right)^2}{x(y+z)}+\frac{\left(\frac{x}{y+t}\right)^2}{x(y+t)}+\frac{\left(\frac{x}{z+t}\right)^2}{x(z+t)}+\frac{\left(\frac{y}{x+z}\right)^2}{y(x+z)}+\frac{\left(\frac{y}{x+t}\right)^2}{y(x+t)}+\frac{\left(\frac{y}{z+t}\right)^2}{y(z+t)}$$
$$+\frac{\left(\frac{z}{x+y}\right)^2}{z(x+y)}+\frac{\left(\frac{z}{x+t}\right)^2}{z(x+t)}+\frac{\left(\frac{z}{y+t}\right)^2}{z(y+t)}+\frac{\left(\frac{t}{x+y}\right)^2}{t(x+y)}+\frac{\left(\frac{t}{x+z}\right)^2}{t(x+z)}+\frac{\left(\frac{t}{y+z}\right)^2}{t(y+z)}$$
$$\geq\frac{\left(\frac{x}{y+z}\right)^2}{x(y+z)}+\frac{\left(\frac{y}{x+z}\right)^2}{y(x+z)}+\frac{\left(\frac{z}{x+y}\right)^2}{z(x+y)}+\frac{\left(\frac{x}{y+t}\right)^2}{x(y+t)}+\frac{\left(\frac{y}{x+t}\right)^2}{y(x+t)}+\frac{\left(\frac{t}{x+y}\right)^2}{t(x+y)}$$
$$+\frac{\left(\frac{x}{z+t}\right)^2}{x(z+t)}+\frac{\left(\frac{z}{x+t}\right)^2}{z(x+t)}+\frac{\left(\frac{t}{x+z}\right)^2}{t(x+z)}+\frac{\left(\frac{y}{z+t}\right)^2}{y(z+t)}+\frac{\left(\frac{z}{y+t}\right)^2}{z(y+t)}+\frac{\left(\frac{t}{y+z}\right)^2}{t(y+z)}$$
$$\geq\frac{\left(\begin{array}{c}\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}+\frac{x}{y+t}+\frac{y}{x+t}+\frac{t}{x+y}\\+\frac{x}{z+t}+\frac{z}{x+t}+\frac{t}{x+z}+\frac{y}{z+t}+\frac{z}{y+t}+\frac{t}{y+z}\end{array}\right)^2}{4(xy+xz+xt+yz+yt+zt)}$$
$$\geq\frac{\left(\frac{3}{2}\,.\,4\right)^2}{4\,.\,\frac{3}{8}(x+y+z+t)^2}=\frac{24}{(x+y+z+t)^2}.$$

The equality occurs if and only if $x=y=z=t$.
We generalize problem 1 to the following one:

Problem 11. *Let x,y,z,t be positive real numbers. Prove that*

$$\frac{x}{(y+z+t)^3}+\frac{y}{(z+t+x)^3}+\frac{z}{(t+x+y)^3}+\frac{t}{(x+y+z)^3}\geq\frac{64}{27(x+y+z+t)^2}.$$

Indeed, let $x+y+z+t=k>0$. Consider the function $f(t)=\frac{t}{(k-t)^3}$, $t>0$, $t<k$. We have

$$f'(t)=\frac{(k-t)^3-3(-1)(k-t)^2t}{(k-t)^6}=\frac{k+2t}{(k-t)^4}$$
$$f''(t)=\frac{2(k-t)^4-4(-1)(k-t)^3(k+2t)}{(k-t)^8}=\frac{6k+6t}{(k-t)^5}\geq 0.$$

Thus, the function f is convex on the interval $(0;\ +\infty)$.
Applying the Jensen 's inequality, we have

$$f(x)+f(y)+f(z)+f(t)\geq 4f\left(\frac{x+y+z+t}{4}\right).$$

Namely, we have

$$\frac{x}{(y+z+t)^3}+\frac{y}{(z+t+x)^3}+\frac{z}{(t+x+y)^3}+\frac{t}{(x+y+z)^3}$$
$$\geq 4f\left(\frac{k}{4}\right)=4.\frac{\frac{k}{4}}{\left(k-\frac{k}{4}\right)^3}=\frac{64}{27k^2}=\frac{64}{27(x+y+z+t)^2}.$$

The equality occurs if and only if $x = y = z = t$.
The generalized problem of problem 11 is as follows:

Problem 12. *Let* $x_1,\ x_2,\ x_3,\ ...,\ x_{n-1},\ x_n$ *be positive real numbers and* $x_1 + x_2 + ... + x_n = k$. *Prove that*

$$\frac{x_1}{(k-x_1)^3} + \frac{x_2}{(k-x_2)^3} + ... + \frac{x_{n-1}}{(k-x_{n-1})^3} + \frac{x_n}{(k - x_n)^3} \geq \frac{n^3}{(n-1)^3 k^2}.$$

Indeed, consider the function $f(t) = \frac{t}{(k-t)^3}$, $t > 0$, $t < k$. We have

$$f'(t) = \frac{(k-t)^3 - 3(-1)(k-t)^2 t}{(k-t)^6} = \frac{k+2t}{(k-t)^4}$$
$$f''(t) = \frac{2(k-t)^4 - 4(-1)(k-t)^3(k+2t)}{(k-t)^8} = \frac{6k+6t}{(k-t)^5} \geq 0.$$

Thus, the function f is convex on the interval $(0;\ +\infty)$.
Applying the Jensen 's inequality, we have

$$f(x_1) + f(x_2) + ... + f(x_{n-1}) + f(x_n) \geq nf\left(\frac{x_1 + x_2 + ... + x_n}{n}\right).$$

Namely, we have

$$\frac{x_1}{(k-x_1)^3} + \frac{x_2}{(k-x_2)^3} + ... + \frac{x_{n-1}}{(k-x_{n-1})^3} + \frac{x_n}{(k-x_n)^3}$$
$$\geq nf\left(\frac{k}{n}\right) = n.\frac{\frac{k}{n}}{\left(k-\frac{k}{n}\right)^3} = \frac{n^3}{(n-1)^3 k^2}.$$

The equality occurs if and only if $x_1 = x_2 = ... = x_n$.
The generalized problem of problem 12 is as follows:

Problem 13. *Let* $x_1,\ x_2,\ x_3,\ ...,\ x_{n-1},\ x_n$ *be positive real numbers and* $x_1 + x_2 + ... + x_n = k,\ p,\ q \in \mathbb{R},\ p \geq 1,\ q > 0$. *Prove that*

$$\frac{x_1^p}{(k-x_1)^q} + \frac{x_2^p}{(k-x_2)^q} + ... + \frac{x_{n-1}^p}{(k-x_{n-1})^q} + \frac{x_n^p}{(k-x_n)^q}$$
$$\geq \frac{n^{q-p+1}}{(n-1)^q}(x_1 + x_2 + ... + x_n)^{p-q}.$$

Consider the function $f(t) = \frac{t^p}{(k-t)^q}$, $t > 0$, $t < k$. We have

$$f'(t) = \frac{pt^p}{t(k-t)^q} + \frac{qt^p}{(k-t)^{q+1}}$$
$$f''(t) = \frac{p^2t^p}{t^2(k-t)^q} - \frac{pt^p}{t^2(k-t)^q} + \frac{2pqt^p}{t(k-t)^{q+1}} + \frac{q^2t^p}{(k-t)^{q+2}} + \frac{qt^p}{(k-t)^{q+2}} > 0.$$

Thus, the function f is convex on the interval $(0;\ +\infty)$.
Applying the Jensen's inequality, we have

$$f(x_1) + f(x_2) + ... + f(x_{n-1}) + f(x_n) \geq nf\left(\frac{x_1 + x_2 + ... + x_n}{n}\right).$$

Namely, we have

$$\frac{x_1^p}{(k-x_1)^q} + \frac{x_2^p}{(k-x_2)^q} + ... + \frac{x_{n-1}^p}{(k-x_{n-1})^q} + \frac{x_n^p}{(k - x_n)^q}$$
$$\geq nf\left(\frac{k}{n}\right) = n . \frac{\left(\frac{k}{n}\right)^p}{\left(k - \frac{k}{n}\right)^q} = n . \frac{k^p}{n^p} . \frac{n^q}{(n-1)^q k^q} = \frac{n^{q-p+1}}{(n-1)^q} k^{p-q}$$
$$= \frac{n^{q-p+1}}{(n-1)^q}(x_1 + x_2 + ... + x_n)^{p-q}.$$

The equality occurs if and only if $x_1 = x_2 = ... = x_n$.
remarks
- When $n = 3$, $p = 1$, $q = 1$ we obtain the Nesbitt 's inequality.
- When $n = 3$, $p = 1$, $q = 3$ we obtain the Batinetu 's inequality.

3 Conclusion

We have got some interesting discoveries around the Batinetu 's inequality. Its different solutions, similarities and generalizations make us interesting. Do you have any comments on this article ? Please share with us.

References

[1] Nguyen Duc Tan (2003), *The special subject on inequality and its applications in algebra*, The Vietnam Education Publishing House.

[2] http://www.ssmrmh.ro/2017/03/17/solution-sp053-rmm-spring-edition-2017/.

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