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PROBLEM 177

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1. In ΔABC

$$\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\le\frac{3R}{2r}.$$

Proposed by George Apostolopoulos - Messolonghi - Grece

Proof.

Using Hölder's inequality, we obtain

$$\left(\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\right)^3\le$$

$$\le(a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)(1+1+1)=$$

$= 2p\cdot\frac{4R+r}{2pr}\cdot 3=3\left(1+\frac{4R}{r}\right)\le\left(\frac{3R}{2r}\right)^3$, *where the last inequality is equivalent with*

$$9R^3\ge 8r^2(4R+r)\Leftrightarrow 9R^3-32Rr^2-8r^3\ge 0\Leftrightarrow(R-2r)(9R^2+18Rr+4r^2)\ge 0$$

true from Euler's inequality: $R\ge 2r$.

The equality holds for an equilateral triangle.

□

Remark

*Inequality **1.** can be strengthened:*

2. In ΔABC

$$\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\le 1+\frac{R}{r}$$

Proposed by Marin Chirciu - Romania

Proof.

Using Hölder's inequality we obtain

$$\left(\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\right)^3\le$$

$$\le(a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)(1+1+1)=$$

$= 2p\cdot\frac{4R+r}{2pr}\cdot 3=3\left(1+\frac{4R}{r}\right)\le\left(1+\frac{R}{r}\right)^3$, *where the last inequality is equivalent with*

$$(R+r)^3 \geq 3r^2(4R+r) \Leftrightarrow R^3+3R^2r-9Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(R^2+5r+r^2) \geq 0$$

true from Euler's inequality: $R \geq 2r$.

The equality holds for an equilateral triangle.

□

Remark

*Inequality **2.** is stronger the inequality **1.***

3. In ΔABC

$$\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-a}} \leq 1+\frac{R}{r} \leq \frac{3R}{2r}.$$

Proof.

*See inequality **2.** and* $1+\frac{R}{r} \leq \frac{3R}{2r} \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds for an equilateral triangle.

□

*Inequality **2** can be developed*

4. In ΔABC

$$\sqrt[4]{\frac{a}{b+c-a}}+\sqrt[4]{\frac{b}{c+a-b}}+\sqrt[4]{\frac{c}{a+b-c}} \leq 1+\frac{R}{r}.$$

Proof.

Using Hölder's inequality we obtain

$$\left(\sqrt[4]{\frac{a}{b+c-a}}+\sqrt[4]{\frac{b}{c+a-b}}+\sqrt[4]{\frac{c}{a+b-c}}\right)^4 \leq$$

$$\leq (a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)(1+1+1)(1+1+1)=$$

$= 2p\cdot\frac{4R+r}{2pr}\cdot 3\cdot 3 = 9\left(1+\frac{4R}{r}\right) \leq \left(1+\frac{R}{r}\right)^4$, *where the last inequality is equivalent with*

$$(R+r)^4 \geq 9r^3(4R+r) \Leftrightarrow R^4+4R^3r+6R^2r^2-32Rr^3-8r^4 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(R^3+6R^3r+18Rr^2+4r^3) \geq 0$$

which is true form Euler's inequality: $R \geq 2r$

The equality holds for an equilateral triangle.

□

5. In ΔABC

$$\sqrt[4]{\frac{a}{b+c-a}}+\sqrt[4]{\frac{b}{c+a-b}}+\sqrt[4]{\frac{c}{a+b-c}} \leq 1+\frac{R}{r} \leq \frac{3R}{2r}.$$

Proof.

*See **4.** and Euler's inequality $R \geq 2r$.*

□

Let's generalise inequality ***1.***

6. In ΔABC

$$\boldsymbol{\sqrt[n]{\frac{a}{b+c-a}}+\sqrt[n]{\frac{b}{c+a-b}}+\sqrt[n]{\frac{c}{a+b-c}}\le\frac{3R}{2r}}, \textit{\textbf{ where }} \boldsymbol{n\in\mathbb{N}, n\ge 2}$$

Proposed by Marin Chirciu - Romania

Proof.

Using Hölder's inequality we obtain

$$\left(\sqrt[n]{\frac{a}{b+c-a}}+\sqrt[n]{\frac{b}{c+a-b}}+\sqrt[n]{\frac{c}{a+b-c}}\right)^n\le$$

$$\le(a+b+c)\Big(\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\Big)(1+1+1)\dots(1+1+1)$$

$$=2p\cdot\frac{4R+r}{2pr}\cdot 3^{n-2}=3^{n-2}\cdot\Big(1+\frac{4R}{r}\Big)\le\Big(\frac{3R}{2r}\Big)^n,$$ *where the last inequality is equivalent with*

$$9R^n\ge 2^n r^{n-1}\Big(4R+r)\Leftrightarrow 9R^n-2^{n+2}Rr^{n-1}-2^n r^n\ge 0$$

Denoting $\frac{R}{r}=t\ge 2$ *it remains to prove that*

$$9t^n-2^{n+2}t-2^n\ge 0\Leftrightarrow(t-2)(9t^{n-1}+9\cdot 2t^{n-2}+9\cdot 2^2\cdot t^{n-3}+\ldots+9\cdot 2^{n-3}t^2+9\cdot 2^{n-2}t+2^{n-1})\ge 0,$$

Obviously because $t\ge 2$.

The equality holds for an equilateral triangle.

□

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