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INEQUALITY IN TRIANGLE - 242

MARIN CHIRCIU

Prove that in any triangle:

$$\frac{R}{r} \geq \frac{r_a}{r_b + r_c} + \frac{r_b}{r_c + r_a} + \frac{r_c}{r_a + r_b} + \frac{1}{2}$$

Proposed by Adil Abdulallayev - Baku - Azerbaidian,
Marian Ursarescu - Romania

Proof.

Using $r_a = \frac{S}{p-a}$ *we obtain* $\sum \frac{r_a}{r_b + r_c} = \sum \frac{(p-b)(p-c)}{a(p-a)} = \frac{(4R+r)^3 - p^2(8R-r)}{4p^2R}$

We write the inequality $\frac{R}{r} \geq \frac{(4R+r)^3 - p^2(8R-r)}{4p^2R} + \frac{1}{2} \Leftrightarrow p^2(4R^2+6Rr-r^2) \geq r(4R+r)^3$,

which follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$. *It remains to prove that*

$$\Leftrightarrow (16Rr - 5r^2)(4R^2 + 6Rr - r^2) \geq r(4R+r)^3 \Leftrightarrow 14R^2 - 29Rr + 2r^2 \geq 0 \Leftrightarrow$$

$(R - 2r)(14R - r) \geq 0$, *obviously from Euler's inequality:* $R \geq 2r$.

The equality holds for an equilateral triangle

□

Remark

The inequality can be devoloped

Prove that in any triangle:

$$\frac{R}{r} \geq n\left(\frac{r_a}{r_b + r_c} + \frac{r_b}{r_c + r_a} + \frac{r_c}{r_a + r_b}\right) + \frac{4-3n}{2}, \text{ where } 0 \leq n \leq 1.$$

Proposed by Marin Chirciu - Romania

Proof.

Using $r_a = \frac{S}{p-a}$ *we obtain* $\sum \frac{r_a}{r_b + r_c} = \sum \frac{(p-b)(p-c)}{a(p-a)} = \frac{(4R+r)^3 - p^2(8R-r)}{4p^2R}$

We write the inequality:

$$\frac{R}{r} \geq n \cdot \frac{(4R+r)^3 - p^2(8R-r)}{4p^2R} + \frac{4-3n}{2} \Leftrightarrow p^2(4R^2+14nRr-8Rr-nr^2) \geq nr(4R+r)^3$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$.

It remains to prove that:

$$\Leftrightarrow (16Rr - 5r^2)(4R^2 + 14nRr - 8Rr - nr^2) \geq nr(4R+r)^3$$

$$\Leftrightarrow (32 - 32n)R^3 + (88n - 74)R^2r + (20 - 49n)Rr^2 + 2nr^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R - 2r)[(32 - 32n)R^2 + (24n - 10)Rr - nr^2] \geq 0,$$

obviously from Euler's inequality: $R \geq 2r$ *and the condition from the hypothesis* $0 \leq n \leq 1$

The equality holds for an equilateral triangle.

□

Remark

*For $n = 1$ we obtain **INEQUALITY IN TRIANGLE - 242.***

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.
E-mail address: dansitaru63@yahoo.com