Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro

PROBLEM 175 - TRIANGLE MARATHON 101 - 200

MARIN CHIRCIU

1. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{(12r)^4}{\sum a^4}.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Remark

*Inequality **1** can be strengthened:*

2. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{(72Rr)^2}{\sum a^4}.$$

Proposed by Marin Chirciu - Romania

Proof.

In order to prove this inequality we will first present two additional results.

Lemma 1

3. In ABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} = \frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4}$$

Proof.

$$\sum \frac{1}{\sin^4 \frac{A}{2}} = \sum \frac{b^2c^2}{(p-b)^2(p-c)^2} = \frac{\sum b^2c^2(p-a)^2}{\prod(p-a)^2} = \frac{p^6 + p^4(2r^2 - 16Rr) + p^2(32R^2r^2 + r^4)}{p^2r^4}$$
$$= \frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4}.$$

□

Lamma 2

4. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{12R^2}{r^2}.$$

Proof.

*Using **Lemma 1** the inequality to prove can be written:*

$$\frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4} \geq \frac{12R^2}{r^2} \Leftrightarrow p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4 \geq 12R^2r^2$$
$$\Leftrightarrow p^2(p^2 + 2r^2 - 16Rr) + 20R^2r^2 + r^4 \geq 0.$$

We distinguish the following cases:

Case 1. *If $p^2 + 2r^2 - 16Rr \geq 0$, the inequality is obvious.*

Case 2. *If* $p^2+2r^2-16Rr<0$, *the inequality can be rewritten:*
$p^2(16Rr-2r^2-p^2)\le 20R^2r^2+r^4$, *which follows form Gerretsen's inequality:*
$16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*
$(4R^2+4Rr+3r^2)(16Rr-2r^2-16Rr+5r^2)\le 20R^2r^2+r^4 \Leftrightarrow 3(4R^2+4Rr+3r^2)\le 20R^2+r^2$
$\Leftrightarrow 2R^2-3Rr-2r^2\ge 0\Leftrightarrow (R-2r)(2R+r)\ge 0$, *which is obvious from Euler's inequality:* $R\ge 2r$.
The equality holds for an equilateral triangle.

□

*Let's pass to solving inequality **2.***

In ΔABC

$$\sum\frac{1}{\sin^4\frac{A}{2}}\ge\frac{(72Rr)^2}{\sum a^4}.$$

*Inequality **2.** is equivalent with:*
$\sum a^4\cdot\sum\frac{1}{\sin^4\frac{A}{2}\ge(72Rr)^2}$, *which follows from using the known identity in triangle*
$\sum a^4=2\Big[p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2\Big]$ *and the inequality* $\sum\frac{1}{\sin^4\frac{A}{2}\ge\frac{12R^2}{r^2}}$
*which we've proved in **Lemma 2.***
It is enough to prove that:

$$2\Big[p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2\Big]\cdot\frac{12R^2}{r^2}\ge(72Rr)^2\Leftrightarrow$$

$p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2\ge 216r^4\Leftrightarrow p^2(p^2-8Rr-6r^2)+r^2(4R+r)^2\ge 216r^4$
which follows from Gerretsen's inequality $p^2\ge 16Rr-5r^2$ *and the remark that* $p^2-8Rr-6r^2>0$.
It remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2-8Rr-6r^2)+r^2(4R+r)^2\ge 216r^4\Leftrightarrow$$
$$(16Rr-5r^2)(8Rr-11r^2)+r^2(4R+r)^2\ge 216r^4\Leftrightarrow$$
$$(16R-5r)(8R-11r)+(4R+r)^2\ge 216r^2\Leftrightarrow 9R^2-13Rr-10r^2\ge 0\Leftrightarrow$$
$$\Leftrightarrow (R-2r)(9R+5r)\ge 0,$$

obvious from Euler's inequality: $R\ge 2r$.
The inequality holds for an equilateral triangle.

Remark

*Inequality **2.** is stronger than inequality **1.**:*

□

5. In ΔABC

$$\sum\frac{1}{\sin^4\frac{A}{2}}\ge\frac{(72Rr)^2}{\sum a^4}\ge\frac{(12r)^4}{\sum a^4}$$

Proof.

*See inequality **2.** and Euler's inequality* $R\ge 2r$.
The inequality holds for an equilateral triangle.

□

6. If $a, b, c > 0$ and $ab + bc + ca = 3$ prove that

$$\sum \frac{a^3 + b^3}{a^2 + ab + b^2} \geq 2$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

Remark

The inequality can be developed:

If $a, b, c > 0$ and $ab + bc + ca = 3$ prove that

$$\sum \frac{a^3 + b^3}{a^2 + nab + b^2} \geq \frac{6}{n + 2}, \text{ where } n \geq 0.$$

Proposed by Marin Chirciu - Romania

Proof.

We have $\frac{a^2 - ab + b^2}{a^2 + nab + b^2} \geq \frac{1}{n + 2} \Leftrightarrow (n+1)(a-b)^2 \geq 0$, *obvious, with equality for* $a = b$.

We obtain $\sum \frac{(a + b)(a^2 - ab + b^2)}{a^2 + nab + b^2} \geq \sum (a+b) \cdot \frac{1}{n + 2} = \frac{2 \sum a}{n + 2} \geq \frac{6}{n + 2}$, *wherefrom the last inequality is equivalent with* $\sum a \geq 3 \Leftrightarrow (\sum a)^2 \geq 9$, *which is true from*

$$(a + b + c)^2 \geq 3(ab + bc + ca) = 9.$$

The equality holds if and only if $a = b = c = 1$.

□

Remark

For $n = 1$ *we obtain* ***Problem 171*** *from TRIANGLE MARATHON 101 -200, proposed by Nguyen Viet Hung - Hanoi - Vietnam*

7. In ΔABC

$$\sum \frac{b^4 + c^4}{\tan^2 \frac{B}{2} + \tan^2 \frac{C}{2}} \geq 48S^2.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Remark

The inequality can be developed:

In ΔABC

$$\sum \frac{b^4 + nc^4}{\tan^2 \frac{B}{2} + n \tan^2 \frac{C}{2}} \geq 48S^2, \text{ where } n \geq 0.$$

Proposed by Marin Chirciu - Romania

Proof.

We have $a^2 \geq 4(p-b)(p-c) \Leftrightarrow a^2 \geq (a+b-c)(a+c-b) \Leftrightarrow a^2 \geq a^2-(b-c)^2 \Leftrightarrow (b-c)^2 \geq 0$

We obtain:

$$b^4 + nc^4 \geq 16(p-a)^2(p-c)^2 + n \cdot 16(p-a)^2(p-b)^2 = 16(p-a)^2\left[(p-c)^2 + n(p-b)^2\right];$$

$$\tan^2 \frac{B}{2} + n \tan^2 \frac{C}{2} = \frac{(p - a)(p - c)}{p(p - b)} + n \cdot \frac{(p - a)(p - b)}{p(p - c)} = \frac{p - a}{p(p - b)(p - c)}\left[(p-c)^2 + n(p-b)^2\right]$$

It follows

$$\frac{b^4+nc^4}{\tan^2\frac{B}{2}+n\tan^2\frac{C}{2}} \geq \frac{16(p-a)^2[(p-c)^2+n(p-b)^2]}{\frac{p-a}{p(p-b)(p-c)}[(p-c)^2+n(p-b)^2]} = 16p(p-a)(p-b)(p-c) = 16S^2$$

We deduce that $\sum \frac{b^4+nc^4}{\tan^2\frac{B}{2}+n\tan^2\frac{C}{2}} \geq \sum 16S^2 = 48S^2.$

The equality holds for an equilateral triangle.

□

Remark

For $n=1$ *we obtain* ***Problem 137*** *from TRIANGLE MARATHON 101 - 200, proposed by George Apostolopoulos - Messolonghi - Greece.*

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.
E-mail address: dansitaru63@yahoo.com