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# PROBLEM 175 - TRIANGLE MARATHON 101 - 200

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**1. In $\Delta ABC$**

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{(12r)^4}{\sum a^4}.$$

***Proposed by George Apostolopoulos - Messolonghi - Greece***

**Remark**

*Inequality **1.** can be strengthened:*

**2. In $\Delta ABC$**

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{(72Rr)^2}{\sum a^4}.$$

***Proposed by Marin Chirciu - Romania***

*Proof.*

*In order to prove this inequality we will first present two additional results.*

**Lemma 1**

**3. In $\Delta ABC$**

$$\sum \frac{1}{\sin^4 \frac{A}{2}} = \frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4}.$$

*Proof.*

$$\sum \frac{1}{\sin^4 \frac{A}{2}} = \sum \frac{b^2c^2}{(p-b)^2(p-c)^2} = \frac{\sum b^2c^2(p-a)^2}{\prod(p-a)^2} = \frac{p^6 + p^4(2r^2 - 16Rr) + p^2(32R^2r^2 + r^4)}{p^2r^4}$$

$$= \frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4}.$$

□

**Lemma 2**

**4. In $\Delta ABC$**

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{12R^2}{r^2}.$$

*Proof.*

*Using **Lemma 1** the inequality we have to prove can be written:*

$$\frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4} \geq \frac{12R^2}{r^2} \Leftrightarrow p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4 \geq 12R^2r^2$$

$$\Leftrightarrow p^2(p^2 + 2r^2 - 16Rr) + 20R^2r^2 + r^4 \geq 0$$

*We distinguish the cases:*

***Case 1.*** *If* $p^2+2r^2-16Rr\geq 0$, *the inequality is obvious.*

***Case 2.*** *If* $p^2+2r^2-16Rr<0$, *the inequality can be rewritten:*

$p^2(16Rr-2r^2-p^2)\leq 20R^2r^2+r^4$, *which follows from Gerretsen's inequality:*

$16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$(4R^2+4Rr+3r^2)(16Rr-2r^2-16Rr+5r^2)\leq 20R^2r^2+r^4\Leftrightarrow 3(4R^2+4Rr+3r^2)\leq 20R^2+r^2$

$\Leftrightarrow 2R^2-3Rr-2r^2\geq 0\Leftrightarrow (R-2r)(2R+r)\geq 0$, *obvious from Euler's inequality:* $R\geq 2r$.

*The equality holds for an equilateral triangle.*

□

*Let's pass to solving inequality* ***2.***

**In $\Delta ABC$**

$$\sum\frac{1}{\sin^4\frac{A}{2}}\geq\frac{(72Rr)^2}{\sum a^4}.$$

*Inequality* ***2.*** *is equivalent with:*

$\sum a^4\cdot\sum\frac{1}{\sin^4\frac{A}{2}}\geq(72Rr)^2$, *which follows from using the known identity in triangle.*

$\sum a^4=2[p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2]$ *and the inequality* $\sum\frac{1}{\sin^4\frac{A}{2}}\geq\frac{12R^2}{r^2}$

*which we have proved it in* ***Lemma 2.***

*It is enough to prove that:*

$$2[p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2]\cdot\frac{12R^2}{r^2}\geq(72Rr)^2\Leftrightarrow$$

$p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2\geq 216r^4\Leftrightarrow p^2(p^2-8Rr-6r^2)+r^2(4R+r)^2\geq 216r^4$,

*which follows from Gerretsen's inequality* $p^2\geq 16Rr-5r^2$ *and the remark that* $p^2-8Rr-6r^2>0$.

*It remains to prove that:*

$$(16Rr-5r^2)(16Rr-5r^2-8Rr-6r^2)+r^2(4R+r)^2\geq 216r^4\Leftrightarrow$$

$$(16Rr-5r^2)(8Rr-11r^2)+r^2(4R+r)^2\geq 216r^4\Leftrightarrow$$

$$(16R-5r)(8R-11r)+(4R+r)^2\geq 216r^2\Leftrightarrow 9R^2-13Rr-10r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(9R+5r)\geq 0$, *obvious from Euler's inequality:* $R\geq 2r$.

*The equality holds for an equilateral triangle.*

□

**Remark**

*Inequality* ***2.*** *is stronger than inequality* ***1.****:*

**5. In $\Delta ABC$**

$$\sum\frac{1}{\sin^4\frac{A}{2}}\geq\frac{(72Rr)^2}{\sum a^4}\geq\frac{(12r)^4}{\sum a^4}.$$

*Proof.*

*See inequality* ***2.*** *and Euler's inequality* $R\geq 2r$.

*The equality holds for an equilateral triangle.*

□

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