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PROBLEM 127 - TRIANGLE MARATHON 101 - 200

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1. In ΔABC

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\le\frac{R}{2r}\sqrt{\frac{2R}{r}-1}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

Using the known identity in triangle $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$ *we write the inequality:*

$$\frac{4R+r}{p}\le\frac{R}{2r}\sqrt{\frac{2R}{r}-1}\Leftrightarrow\left(\frac{4R+r}{p}\right)^2\le\left(\frac{R}{2r}\right)^2\left(\frac{2R}{r}-1\right)\Leftrightarrow p^2R^2(2R-r)\ge4r^3(4R+r)^2$$

which follows from Gerretsen's inequality $p^2\ge16Rr-5r^2$. *It remains to prove that:*

$\Leftrightarrow(16R-5r^2)\cdot R^2(2R-r)\ge4r^3(4R+r)^2\Leftrightarrow34R^4-26R^3r-59R^2r^2-32Rr^3-4r^4\ge0\Leftrightarrow$

$\Leftrightarrow(R-2r)(32R^3+38Rr^2+17Rr^2+2r^3)\ge0$, *obviously from Euler's inequality* $R\ge2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark

The inequality can be developed:

2. In ΔABC

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\le\frac{R}{r}\sqrt{n\cdot\frac{R}{r}-2n+\frac{3}{4}},\ \textbf{where}\ n\ge0.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the known identity in triangle $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$ *we write the inequality:*

$$\frac{4R+r}{p}\le\frac{R}{r}\sqrt{n\cdot\frac{R}{r}-2n+\frac{3}{4}}\Leftrightarrow\left(\frac{4R+r}{p}\right)^2\le\left(\frac{R}{r}\right)^2\left(n\cdot\frac{R}{r}-2n+\frac{3}{4}\right)\Leftrightarrow$$

$\Leftrightarrow p^2R^2\Big[4nR+(3-8n)r\Big]\ge4r^3(4R+r)^2$, *which follows from Gerretsen's inequality:*

$p^2\ge16Rr-5r^2$. *It remains to prove that:*

$$\Leftrightarrow(16Rr-5r^2)\cdot R^2\Big[4nR+(3-8n)r\Big]\ge4r^3(4R+r)^2$$

$$\Leftrightarrow64nR^4+(48-148n)R^3r+(40n-79)R^2r^2-32Rr^3-4R^4\ge0\Leftrightarrow$$

$$\Leftrightarrow(R-2r)\Big[64nR^3+(48-20n)Rr^2+17Rr^2+2r^3\Big]\ge0$$

obviously from Euler's inequality $R\ge2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark

For $n = \frac{1}{2}$ *in inequality* ***2.*** *we obtain inequality* ***1.****, meaning* ***Problem 127***

from TRIANGLE MARATHON 101-200

proposed by George Apostolopoulos - Messolonghi - Greece.

Remark

We can write the double inequality:

3. In ΔABC

$$\sqrt{3} \leq \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \leq \frac{R}{r}\sqrt{n \cdot \frac{R}{r} - 2n + \frac{3}{4}},\ where\ n \geq 0.$$

Proof.

The first inequality follows from the identity $\sum \tan\frac{A}{2} = \frac{4R+r}{p}$ *and Doucet's inequality*

$4R + r \geq p\sqrt{3}$*, the second inequality is* ***2.***

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

Remark

We can propose inequalities in the same format:

4. In ΔABC

$$\mathbf{1 \leq \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \leq \left(\frac{R}{2r}\right)^2}$$

Proof.

The first inequality follows from the identity $\sum \tan^2\frac{A}{2} = \frac{(4R+r)^2 - 2p^2}{p^2}$

and from Doucet's inequality: $(4R+r)^2 \geq 3p^2$.

The second inequality, taking into account the above identity, can be written:

$\frac{(4R+r)^2 - 2p^2}{p^2} \leq \left(\frac{R}{2r}\right)^2 \Leftrightarrow 4r^2(4R+r)^2 - 8r^2p^2 \geq p^2R^2 \Leftrightarrow p^2(R^2+8r^2) \geq 4r^2(4R+r)^2,$

Which follows from Gerretsen's inequality: $p^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$(16Rr - 5r^2)(R^2 + 8r^2) \geq 4r^2(4R+r)^2 \Leftrightarrow 16R^3 - 69R^2r + 96Rr^2 - 44r^3 \geq 0 \Leftrightarrow$

$(r - 2r)(16R^2 - 37Rr + 22r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

5. In ΔABC:

$$\mathbf{\frac{3r}{p} \leq \tan^3\frac{A}{2} + \tan^3\frac{B}{2} + \tan^3\frac{C}{2} \leq \frac{3R}{2p}\left[\left(\frac{3R}{2r}\right)^2 - 8\right].}$$

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Proof.

First we prove the following identity:

Lemma
6. In ΔABC

$$\sum \tan^3 \frac{A}{2} = \frac{(4R+r)^3 - 12p^2R}{p^3}$$

Proof.

We use the identity $(x+y+z)^3 = x^3 + y^3 + z^3 + 3(x+y)(y+z)(z+x)$

we put $x = \tan\frac{A}{2}, y = \tan\frac{B}{2}, z = \tan\frac{C}{2}$ *and then we take into account that*

$$x+y+z = \sum \tan\frac{A}{2} = \frac{4R+r}{p},$$

$$(x+y)(y+z)(z+x) = \prod\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = \frac{4R}{p}.$$

*Let's pass to solving the double inequality **5.**:*

We write the first inequality:

$\frac{(4R+r)^3 - 12p^2R}{p^3} \geq \frac{3r}{p}$, *which follows from Doucet's inequality:* $(4R+r)^2 \geq 3p^2$.

We obtain $\frac{(4R+r)^3 - 12p^2R}{p^3} \geq \frac{(4R+r)\cdot 3p^2 - 12p^2R}{p^3} = \frac{3r}{p}$

We write the second inequality:

$\frac{(4R+r)^3 - 12p^2R}{p^3} \leq \frac{3R}{2p}\left[\left(\frac{3R}{2r}\right)^2 - 8\right] \Leftrightarrow 8r^2(4R+r)^3 - 96p^2Rr^2 \leq 3p^2R(9R^2-32r^2) \Leftrightarrow$

$27p^2R^2 \geq 8r^2(4R+r)^3$, *which follows from Gerretsen's inequality:* $p^2 \geq 16Rr - 5r^2$.

It remains to prove that:

$27(16Rr-5r^2)R^2 \geq 8r^2(4R+r)^3 \Leftrightarrow 432R^4 - 647R^3r - 384R^2r^2 - 96Rr^3 - 8r^4 \geq 0 \Leftrightarrow$

$(R-2r)(432R^3 + 217R^2r + 50Rr^2 + 4r^3) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

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