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SOLUTION TO PROBLEM UP.048 FROM ROMANIAN MATHEMATICAL MAGAZINE NUMBER 4, SPRING 2017

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UP.048. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that:

$$a^4+b^4+c^4+26abc \leq 1$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

Proof.

Homogenising the inequality we obtain

$$a^4+b^4+c^4+26abc(a+b+c) \leq (a+b+c)^4$$

As $(a+b+c)^4 = \sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc(a+b+c)$, *the above inequality can be written:*

$$\sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc(a+b+c) \geq \sum a^4+26abc(a+b+c) \Leftrightarrow$$

$2\sum bc(b^2+c^2)+3\sum b^2c^2 \geq 7abc(a+b+c)$, *which follows from means inequality and the inequality* $x^2+y^2+z^2 \geq xy+yz+zx$, *with* $x=bc, y=ca, z=ab$.

Indeed:

$$2\sum bc(b^2+c^2)+3\sum b^2c^2 \geq 4\sum b^2c^2+3\sum b^2c^2 = 7\sum b^2c^2 \geq 7\sum bc\cdot ca = 7abc(a+b+c).$$

The equality holds if and only if $a=b=c=\frac{1}{3}$.

□

The problem can be developed:

If $a, b, c > 0, a+b+c+1$ then $a^4+b^4+c^4+\lambda abc \leq \frac{\lambda+1}{27}$, where $\lambda \geq 26$.

Proposed by Marin Chirciu - Romania

Proof.

Homogenising the inequality we obtain:

$$a^b+b^4+c^4+\lambda abc(a+b+c) \leq \frac{\lambda+1}{27}(a+b+c)^4.$$

As $(a+b+c)^4 = \sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc(a+b+c)$, *the above inequality can be written:*

$$\frac{\lambda+1}{27}\cdot\left[\sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc(a+b+c)\right] \geq \sum a^4+\lambda abc(a+b+c) \Leftrightarrow$$

$$(\lambda-26)\sum a^4+(8\lambda+8)\sum bc(b^2+c^2)+(6\lambda+6)\sum b^2c^2 \geq (15\sum \lambda-12)abc(a+b+c)$$

which follows from the condition $\lambda \geq 26$, *means inequality and the inequality*

$x^2+y^2+z^2 \geq xy+yz+zx$, *with* $x=a^2, y=b^2, z=c^2$, *then* $x=bc, y=ca, z=ab$.

Indeed:

$$(\alpha - 26)\sum a^4 + (4\lambda+4)\sum bc(b^2+c^2) + (6\lambda+6)\sum b^2c^2 \geq$$
$$\geq (\lambda - 26)\sum b^2c^2 + (8\lambda+8)\sum b^2c^2 + (6\lambda+6)\sum b^2c^2 =$$
$$= (15\lambda - 12)\sum b^2c^2 \geq (15\lambda - 12)\sum bc\cdot ca$$
$$= (15\lambda - 12)abc(a+b+c).$$

The equality holds if and only if $a=b=c=\dfrac{1}{3}$.

□

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