Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open acces

SOLUTION TO PROBLEM JP.060. FROM ROMANIAN MATHEMATICAL MAGAZINE NUMBER 4, SPRING 2017

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JP.060. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq\frac{3\sqrt{3}}{2}R.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

We have $\sum\frac{bc}{b+c}\leq\sum\frac{b+c}{4}=p\leq\frac{3\sqrt{3}}{2}R$, *where the last inequality is Mitrinović's inequality.*

The equality holds if and only if the triangle is equilateral.

□

The inequality can be strengthened:

1. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq p.$$

Proof.

$$\sum\frac{bc}{b+c}\leq\sum\frac{b+c}{4}=p.$$

The equality holds if and only if the triangle is equilateral.

*Inequality **1.** is stronger then **JP.060.***

□

2. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq p\leq\frac{3\sqrt{3}}{2}R.$$

Proof.

We have $\sum \frac{bc}{b+c} \leq \sum \frac{b+c}{4} = \leq \frac{3\sqrt{3}}{2}R$, *where the last inequality is Mitrinonvić's inequality.*

The equality holds if and only if the triangle is equilateral.

□

Inequality ***1.*** *can also be strengthened:*

3. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b} + \frac{ab}{a+b} + \frac{ab}{a+b} \leq \frac{3(ab+bc+ca)}{2(a+b+c)}.$$

Proof 1.

We use the known identities in triangle

$$\sum \frac{bc}{b+c} = \frac{p^4 + 2p^2(8R+r^2) + (4R+r)^3}{2p(p^2+r^2+2Rr)} \text{ and } \sum bc = p^2 + r^2 + 4Rr.$$

We write the inequality:

$$\frac{p^4 + 2p^2(8R+r^2) + (4R+r)^3}{2p(p^2+r^2+2Rr)} \leq \frac{3(p^2+r^2+4Rr)}{2 \cdot 2p} \Leftrightarrow$$

$$p^2(p^2 - 14Rr + 2r^2) \geq r^2(8R^2 - 2Rr - r^2).$$

As $(p^2 - 14Rr + 2r^2) > 0$, *see Gerretsen's inquality* $p^2 \geq 16Rr - 5r^2$, *using again Gerretsen's inequality it suffices to prove that*

$$(16Rr - 5r^2)(16Rr - 5r^2 - 14Rr + 2r^2) \geq r^2(8R^2 - 2r - r^2) \Leftrightarrow$$

$$(16R-5r)(2R-3r) \geq r^2(8R^2-2Rr-r^2) \Leftrightarrow 3R^2-7Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(3R-r) \geq 0.$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Proof 2.

The triplets $(a+b, b+c, c+a)$ *and* $\left(\frac{ab}{a+b}, \frac{bc}{b+c}, \frac{ca}{c+a}\right)$ *are ordered the same.*

With Chebyshev's inequality we obtain:

$$(a+b)\cdot\frac{ab}{a+b}+(b+c)\cdot\frac{bc}{b+c}+(c+a)\cdot\frac{ca}{c+a} \geq \frac{1}{3}\Big[(a+b)+(b+c)+(c+a)\Big]\left[\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right]$$

$$\Leftrightarrow (ab+bc+ca) \geq \frac{1}{3} \cdot 2(a+b+c)\cdot\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{ab}{a+b} + \frac{ab}{a+b} + \frac{ab}{a+b} \leq \frac{3(ab+bc+ca)}{2(a+b+c)}.$$

The equality holds if and only if the triangle is equilateral.

Inequality ***3.*** *is stronger then Inequality* ***1.****:*

□

4. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq\frac{3(ab+bc+ca)}{2(a+b+c)}\leq p.$$

Proof.

*We use inequality **3.** and*

$$\frac{3(ab+bc+ca)}{2(a+b+c)}\leq p\Leftrightarrow\frac{3(ab+bc+ca)}{2(a+b+c)}\leq\frac{a+b+c}{2}\Leftrightarrow(a+b+c)^2\geq 3(ab+bc+ca).$$

The equality holds if and only if the triangle is equilateral.

□

We can write the series of inequalities:

5. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq\frac{3(ab+bc+ca)}{2(a+b+c)}\leq\frac{a+b+c}{2}\leq\frac{3(a^2+b^2+c^2)}{2(a+b+c)}.$$

Proof.

*We use inequality **4.** and* $\frac{a+b+c}{2}\leq\frac{3(a^2+b^2+c^2)}{2(a+b+c)}\Leftrightarrow a^2+b^2+c^2\geq ab+bc+ca.$

□

6. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq\frac{3(ab+bc+ca)}{2(a+b+c)}\leq p\leq\frac{3\sqrt{3}}{2}R.$$

Proof.

*We use inequality **4.** and Mitrinović's inequality.*

The equality holds if and only if the triangle is equilateral.

□

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