RMM - Inequalities Marathon 101 - 200

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101. If $a, b, c \in (0, \infty), a^2 + b^2 + c^2 = 10$ then:

$$27\left(\frac{1}{c} + \frac{5}{b} - \frac{1}{a}\right) \le 25\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^3$$

Proposed by Daniel Sitaru-Romania

Solution by Dat Vo-Quynh Luu – VietNam:

$$25\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^3 - 27\left(\frac{1}{c} + \frac{5}{b} - \frac{1}{a}\right)$$

$$= 25\left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^3 - \frac{27}{abc}\right] + \frac{27}{abc}[25 - (5ac + ab - bc)]$$

$$= 25\left[3\sum \frac{1}{c}\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \frac{1}{2}(a + b + c)\sum\left(\frac{1}{a} - \frac{1}{b}\right)^2\right]$$

$$+ \frac{27}{abc}\left[\frac{5}{2}(a^2 + b^2 + c^2) - (5ac + ab - bc)\right]$$

$$= 25\left[3\sum \frac{1}{c}\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \frac{1}{2}(a + b + c)\sum\left(\frac{1}{a} - \frac{1}{b}\right)^2\right]$$

$$+ \frac{27}{2abc}[3(a - c - b)^2 + 2(a - c + b)^2] \ge 0$$

102. Let x, y, z be positive real numbers such that $xyz = x + 27y + 125z$.

Prove that:

$$x + y + z \ge 27$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Marian Dincă – Romania

$$xyz = x + 27y + 125z$$

$$\Leftrightarrow \frac{1}{yz} + \frac{27}{xz} + \frac{125}{xy} = 1$$

$$1 = \left(\frac{1}{yz} + \frac{27}{xz} + \frac{125}{xy}\right) \geq \frac{1^3}{\left(\frac{y+z}{2}\right)^2} + \frac{3^3}{\left(\frac{x+z}{2}\right)^2} + \frac{5^3}{\left(\frac{x+y}{2}\right)^2} \geq$$

$$\geq \frac{(1+3+5)^3}{\left(\frac{y+z}{2} + \frac{x+z}{2} + \frac{x+y}{2}\right)^2}$$

$AM - GM$ and Radon inequality result:

$$1 \geq \frac{(1+3+5)^3}{\left(\frac{y+z}{2} + \frac{x+z}{2} + \frac{x+y}{2}\right)^2} = \frac{9^3}{(x+y+z)^2}$$

$$(x+y+z)^2 \geq 9^3 \Leftrightarrow x+y+z \geq 27$$

103. If $x, y, z \in \mathbb{R}$ then:

$$(x+y)^2 + (y+z)^2 + (z+x)^2 + 14 \geq 2(3x+4y+5z)$$

and shows when equality holds.

Proposed by Dorin Mărghidanu-Romania

Solution 1 by Imad Zak-Saida-Lebanon:

$$\begin{cases} (x+z)^2 - 4(x+z) + 4 = (x+z-2)^2 \\ (y+z)^2 - 6(y+z) + 9 = (y+z-3)^2 \\ (x+y)^2 - 2(x+y) + 1 = (x+y-1)^2 \end{cases}$$

By adding we get:

$$\sum (x+y)^2 + 14 - 4(x+z) - 6(y+z) - 2(x+y) \geq 0$$

$$\sum (x+y)^2 + 14 \geq 6x + 10z + 8y = 2(3x+4y+5z)$$

Equality holds when:

$$\begin{cases} x+z=2 \\ y+z=3 \\ x+y=1 \end{cases} \Rightarrow \Rightarrow \begin{cases} x=0 \\ y=1 \\ z=2 \end{cases}$$

Solution 2 by Daniel Sitaru-Romania

$$f(x,y,z)=\sum(x+y)^2-6x-8y-10z+14$$

$$\left.\begin{aligned} f'_x &= 2(x+y)+2(z+x)-6=0 \\ f'_y &= 2(x+y)+2(y+z)-8=0 \\ f'_z &= 2(x+z)+2(y+z)-10=0 \end{aligned}\right\} \Rightarrow A(0,1,2) \text{ - critical point}$$

$$H_f(0,1,2)=\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

$$\Delta_1=1>0;\ \Delta_2=12>0;\ \Delta_3=32>0$$

$$A(0,1,2) \text{ - minimal point}$$

$$\sum(x+y)^2-6x-8y-10z+14=f(x,y,z)\geq f(0,1,2)=0$$

104. If $a,b,c,d\in\mathbb{R}, a\leq b\leq c\leq d$ then:

$$e^a-e^c+e^b-e^d\geq 2\left(\sqrt{e^{a+b}}-\sqrt{e^{c+d}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal - Chandar Nagore – India

$$e^a-e^c+e^b-e^d\geq 2\left(\sqrt{e^{a+b}}-\sqrt{e^{c+d}}\right)$$

$$\Leftrightarrow e^a+e^b-2\sqrt{e^{a+b}}\geq e^c+e^d-2\sqrt{e^{c+d}}$$

$$\Leftrightarrow \left(\sqrt{e^b}-\sqrt{e^a}\right)^2\geq\left(\sqrt{e^c}-\sqrt{e^d}\right)^2 \Leftrightarrow \sqrt{e^b}-\sqrt{e^a}\geq\sqrt{e^c}-\sqrt{e^d}$$

$$\Leftrightarrow \sqrt{e^d}-\sqrt{e^a}\geq\sqrt{e^c}-\sqrt{e^b} \qquad (1)$$

Let $f:\mathbb{R}\to\mathbb{R}$ ***be defined as*** $f(x)=\sqrt{e^x}\quad\forall\, x\in\mathbb{R}$

$$f'(x)=\frac{\sqrt{e^x}}{2}\Rightarrow f''(x)=\frac{\sqrt{e^x}}{4}>9\quad\forall x\in\mathbb{R}$$

$f(x)$ ***is a convex function:*** $\sqrt{e^d}-\sqrt{e^a}\ge\sqrt{e^c}-\sqrt{e^b}$

Hence statement (1) is true.

$$e^a-e^c+e^b-e^d\ge 2\left(\sqrt{e^{a+b}}-\sqrt{e^{c+d}}\right)$$

(proved)

105. If $a,b,c,\eta\in\mathbb{R}$ **then:**

$$|(a-b)(b-c)(c-a)|\le\sum|(a-b)(c+b+\eta)(c+a+\eta)|$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

Let

$$x=b+c+\eta$$

$$y=c+a+\eta$$

$$z=a+b+\eta$$

Now, $a-b=x-y$ ***and analogous:***

$$RHS=\sum|(a-b)(c+b+\eta)(c+a+\eta)|=\sum|(x-y)xy|\ge$$

$$\ge|(x-y)xy+(y-z)yz+(z-x)xz|=\left|x^2(y-z)+y^2(z-x)+z^2(x-y)\right|=$$

$$=\left\|\begin{vmatrix}1&1&1\\x&y&z\\x^2&y^2&z^2\end{vmatrix}\right\|=|(x-y)(y-z)(z-x)|=|(a-b)(b-c)(c-a)|$$

106. If $a, b, c \in \mathbb{R}^*; a \neq b \neq c \neq a$ then:

$$\frac{\sqrt[6]{a^2+b^2}}{\sqrt[3]{|a|}+\sqrt[3]{|b|}}+\frac{\sqrt[10]{b^2+c^2}}{\sqrt[5]{|b|}+\sqrt[5]{|c|}}+\frac{\sqrt[14]{c^2+a^2}}{\sqrt[7]{|c|}+\sqrt[7]{|a|}}<3$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Pal – Kolkata-India

$$\left((|x|^2)^{\frac{1}{n}}+(|y|^2)^{\frac{1}{n}}\right)^n=\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)^n=$$

$$=|x|^2+\binom{n}{1}|x|^{\frac{2(n-1)}{n}}|y|^{\frac{2}{n}}+\cdots+|y|^2>|x|^2+|y|^2=x^2+y^2$$

$\sum_{i=1}^{n-1}\binom{n}{i}|x|^{\frac{2(n-i)}{n}}|y|^{\frac{2i}{n}}>0$ - ***because all terms are positive here***

$$\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)^n>x^2+y^2\Rightarrow(x^2+y^2)^{\frac{1}{n}}<|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}$$

Putting $(x,y,n)=(a,b,6),(b,c,10),(c,a,14)$ ***we get***

$$(a^2+b^2)^{\frac{1}{6}}<|a|^{\frac{1}{3}}+|b|^{\frac{1}{3}},(b^2+c^2)^{\frac{1}{10}}<|b|^{\frac{1}{5}}+|c|^{\frac{1}{5}},(c^2+a^2)^{\frac{1}{14}}<|c|^{\frac{1}{7}}+|a|^{\frac{1}{7}}$$

$$\Rightarrow\frac{\sqrt[6]{a^2+b^2}}{|a|^{\frac{1}{3}}+|b|^{\frac{1}{3}}}<1,\frac{\sqrt[10]{b^2+c^2}}{\sqrt[5]{|b|}+\sqrt[5]{|c|}}<1,\frac{\sqrt[14]{c^2+a^2}}{\sqrt[7]{|c|}+\sqrt[7]{|a|}}<1$$

Adding them we get required inequality. So it is true.

107. Prove that for any real numbers x, y, z:

$$(x^2+2y^2+2z^2+3xy+5yz+3zx)^2\geq 8(x+y)(y+z)(z+x)(x+y+z)$$

Proposed by Nguyen Viet Hung –Hanoi- Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los números reales x, y, z***:***

$$(x^2+2y^2+2z^2+3xy+5yz+3zx)^2\geq 8(x+y)(y+z)(z+x)(x+y+z)$$

$$x^2+2y^2+2z^2+3xy+5yz+3zx=(x^2+2yx+xz)+(yx+2y^2+yz)+$$

$$+(2zx + 4yz + 2z^2)$$

$$\Rightarrow x^2 + 2y^2 + 2z^2 + 3xy + 5yz + 3zx = x(x + 2y + z) + y(x + 2y + z) +$$

$$+2z(x + 2y + z)$$

$$\Rightarrow x^2 + 2y^2 + 2z^2 + 3xy + 5yz + 3zx = (x + 2y + z)(x + y + 2z)$$

Realizamos los siguientes cambios de variables:

$$x + y = a,$$

$$y + z = b,$$

$$z + x = c,$$

$$2(x + y + z) = a + b + c$$

$$\Rightarrow \big((a + b)(b + c)\big)^2 \geq 4abc(a + b + c)$$

$$\Rightarrow \big(b(b + a) + c(b + a)\big)^2 \geq 4a^2bc + 4a^2bc + 4abc^2$$

$$\Rightarrow b^2(b^2 + a^2 + 2ab) + c^2(b^2 + a^2 + 2ab) + 2bc(b^2 + a^2 + 2ab) \geq$$

$$\geq 4a^2bc + 4b^2ac + 4abc^2$$

$$\Rightarrow a^2(b^2 + c^2 - 2bc) + b^2(b^2 + c^2 + 2bc) + 2ab(b^2 - c^2) \geq 0$$

$\Rightarrow \big(a(b - c) + b(b + c)\big)^2 \geq 0 \Leftrightarrow$ ***La igualdad se alcanza cuando:***

$$(x, y, z) \to (0, 0, 0)$$

108. If $a, b, c, d > 0$ then:

$$\frac{ac + bd + |ad - bc|}{\sqrt{(a^2 + b^2)(c^2 + d^2)}} + \frac{(a^2 + b^2)(c^2 + d^2)}{(ac + bd)|ad - bc|} \geq 2 + \sqrt{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a, b, c, d > 0 \Rightarrow$$

$$\frac{ac+bd+|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}}+\frac{(a^2+b^2)(c^2+d^2)}{(ac+bd)|ad-bc|}\underbrace{\geq}_{(1)}2+\sqrt{2}$$

Let $\sqrt{ac+bd}=x$, $\sqrt{|ad-bc|}=y$, $\sqrt{(a^2+b^2)(c^2+d^2)}=z$

$$LHS=\frac{x^2+y^2}{z}+\frac{z^2}{x^2y^2}$$

Now, $x^4+y^4=(ac+bd)^2+(ad-bc)^2$

$$=(a^2+b^2)(c^2+d^2)=z^2$$

$$\left.\begin{matrix}x^2=z\sin\theta\\ y^2=z\cos\theta\end{matrix}\right\}\ 0<\theta<\frac{\pi}{2}$$

$$\therefore LHS=\frac{z(\cos\theta+\sin\theta)}{z}+\frac{z^2}{z^2\sin\theta\cos\theta}$$

$$=\cos\theta+\sin\theta+\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$

$$=\cos\theta+\sin\theta+\tan\theta+\cot\theta=f(\theta)$$

$$f'(\theta)=\cos\theta-\sin\theta+\sec^2\theta-csc^2\theta=\cos\theta-\sin\theta+\frac{1}{\cos^2\theta}-\frac{1}{\sin^2\theta}=$$

$$=\cos\theta-\sin\theta-\frac{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}{\cos^2\theta\sin^2\theta}=(\cos\theta-\sin\theta)\left(1-\frac{\cos\theta+\sin\theta}{\cos^2\theta\sin^2\theta}\right)$$

$$f'(\theta)\Rightarrow(\cos\theta-\sin\theta)(\cos^2\theta\sin^2\theta-\cos\theta-\sin\theta)=0$$

If $\cos^2\theta\sin^2\theta=\cos\theta+\sin\theta$, *then*

$$\cos^4\theta\sin^4\theta=1+2\cos\theta\sin\theta$$

$\Rightarrow t^4-2t-1=0$ *(where* $t=\cos\theta\sin\theta$*)*$\Rightarrow t^4=2t+1$

Now, $\because 0<\theta<\frac{\pi}{2},\therefore t>0\Rightarrow 2t+1>1$

$\therefore RHS>1$. *Now,* $LHS=\left(\frac{1}{2}\sin^2\theta\right)^4<\frac{1}{16}$

So, $RHS>1$ *and* $LHS<\frac{1}{16}\Rightarrow t^4-2t-1=0$ *has no real root*

$$\Rightarrow \cos^2\theta\sin^2\theta - \cos\theta - \sin\theta \neq 0$$

$$\therefore f'(\theta) = 0 \Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$$

$$f''(\theta) = -\sin\theta - \cos\theta + 2\sec^2\theta\tan\theta + 2\csc^2\theta\cot\theta$$

At $\theta = \frac{\pi}{4}, f''(\theta) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 4 + 4 = 8 - \sqrt{2}$

$> 0 \Rightarrow$ ***at*** $\theta = \frac{\pi}{4}, f(\theta)$ ***attains a minimal and*** $\because f(\theta)$ ***never attains a maxima,***

$$\therefore f(\theta) \geq f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 + 1 = 2 + \sqrt{2}$$

$\therefore LHS$ ***of (1)*** $= f(\theta) \geq 2 + \sqrt{2}$

(Proved)

Solution 2 by Mithun Chakraborty – Kolkata – India

Lemma 1: Define $f: (0, \infty] \to \mathbb{R}$ ***as*** $f(x) = 2\sqrt{x} + \frac{1}{x}$

Then, $f(x)$ ***has a unique minimum at*** $x = 1$***.***

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2};\ f''(x) = -\frac{1}{2}x^{-\frac{3}{2}} + 2x^{-3}$$

$f'(x) = 0$ ***has unique root in*** $(0, \infty]$ ***at*** $x = 1; f''(1) = \frac{3}{2} > 0$

Let $ac + bd = p; |ad - bc| = q > 0$

Since $a, b, c, d > 0$***, we have*** $p > 0$

Now, $(a^2 + b^2)(c^2 + d^2) = (a^2c^2 + b^2d^2) + (a^2d^2 + b^2c^2) =$

$$= (ac + bd)^2 - 2abcd + |ad - bc|^2 + 2abcd = p^2 + q^2$$

$$\therefore LHS = \frac{p + q}{\sqrt{p^2 + q^2}} + \frac{p^2 + q^2}{pq}$$

$\geq 2\frac{\sqrt{pq}}{\sqrt{p^2+q^2}} + \frac{p^2+q^2}{pq}$ ***by AM-GM inequality since*** $p, q > 0$

$$= 2\sqrt{y} + \frac{1}{y} \text{ where } y = \frac{pq}{p^2+q^2} = f(y)$$

Now, $(p-q)^2 \geq 0 \Rightarrow p^2 + q^2 \geq 2pq \Rightarrow \frac{pq}{p^2+q^2} \leq \frac{1}{2}$

Since $f(x)$ ***is strictly decreasing on*** $(0,1)$,

$$LHS \geq f(y) \geq f\left(\frac{1}{2}\right) = 2\sqrt{\frac{1}{2}} + \frac{1}{\frac{1}{2}} = 2 + \sqrt{2} = RHS$$

(q.e.d.)

109. If $0 < x \leq y \leq z \leq t$ then:

$$\left(4\sqrt[4]{xyzt} - 3\sqrt[3]{xyz}\right)\left(3\sqrt[3]{xyz} - 2\sqrt{xy}\right) \leq zt$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

Now, let $\frac{x}{t} = x_1, \frac{y}{t} = y_1, \frac{z}{t} = z_1$ ***where*** $0 < x_1, y_1, z_1 \leq 1$

Now, $4(xyzt)^{\frac{1}{4}} - 3(xyz)^{\frac{1}{3}} = t\left[4(x_1y_1z_1)^{\frac{1}{4}} - 3(x_1y_1z_1)^{\frac{1}{3}}\right] \leq t$

[as $0 < x_1y_1z_1 \leq 1$***]***

Similarly, $3(xyz)^{\frac{1}{3}} - 2(xy)^{\frac{1}{2}} \leq z$

Thus, $\left[4(xyzt)^{\frac{1}{4}} - 3(xyz)^{\frac{1}{3}}\right] \cdot \left[3(xyz)^{\frac{1}{3}} - 2(xy)^{\frac{1}{2}}\right] \leq zt$

110. If $x, y, z \in [0, \infty)$ then:

$$\sqrt{x^2 - xy\sqrt{3} + y^2} + \sqrt{y^2 - yz\sqrt{2} + z^2} \geq \sqrt{x^2 - xz + z^2}$$

Proposed by Daniel Sitaru – Romania

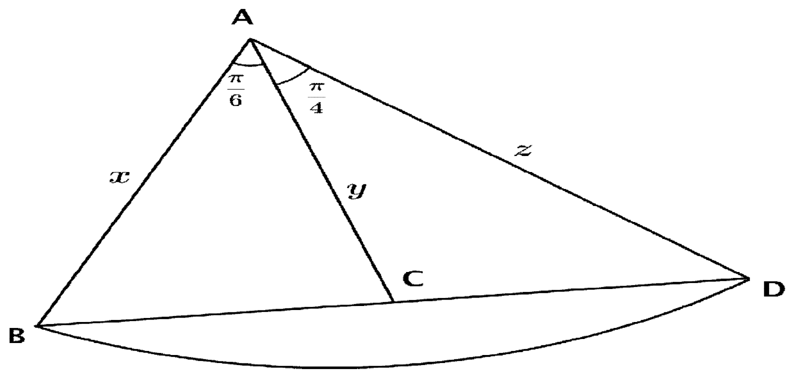
Solution 1 by Ravi Prakash - New Delhi – India

Consider two Δ^s ABC ***and*** ACD

such that

$$AB = x, AC = y, AD = z$$

$$\angle BAC = \frac{\pi}{6}, \angle CAD = \frac{\pi}{4}$$



$$\textit{Then } BC = \sqrt{x^2 - xy\sqrt{3} + y^2},\ CD = \sqrt{y^2 - \sqrt{2}yz + z^2}$$

$$\textit{Also, } LBAD = 75^\circ,\ BD = \sqrt{x^2 - 2\cos 75^\circ xz + z^2}$$

$$\textit{As } \cos 75^\circ < \cos 60^\circ - 2\cos 75^\circ > -2\cos 60^\circ = -1$$

$$\Rightarrow x^2 - 2\cos 75^\circ xz + z^2 > x^2 - xz + z^2$$

$$\textit{Now, } BC + CD \geq BD > \sqrt{x^2 - xz + z^2}$$

$$\Rightarrow \sqrt{x^2 - \sqrt{3}xy + y^2} + \sqrt{y^2 - \sqrt{2}yz + z^2} > \sqrt{x^2 - xz + z^2}$$

Solution 2 by Nguyen Minh Triet - Quang Ngai – Vietnam

$$\textit{If } x, y, z \in [0; \infty) \textit{ then:}$$

$$\sqrt{x^2 - xy\sqrt{3} + y^2} + \sqrt{y^2 - yz\sqrt{2} + z^2} \geq \sqrt{x^2 - xz + z^2}$$

By Minkowski inequlity, we have

$$LHS = \sqrt{\left(y - \frac{\sqrt{3}}{2}x\right)^2 + \left(\frac{x}{2}\right)^2} + \sqrt{\left(\frac{z}{\sqrt{2}} - y\right)^2 + \left(\frac{z}{\sqrt{2}}\right)^2} \geq$$

$$\geq \sqrt{\left(\frac{z}{\sqrt{2}} - \frac{\sqrt{3}}{2}x\right)^2 + \left(\frac{x}{2} + \frac{z}{\sqrt{2}}\right)^2} = \sqrt{x^2 + z^2 + \frac{1-\sqrt{3}}{\sqrt{2}}xz} \geq \sqrt{x^2 + z^2 - xz}$$

(q.e.d.)

$$\textit{(since} \begin{cases} \frac{1-\sqrt{3}}{\sqrt{2}} > -1 \\ xz \geq 0 \end{cases}$$

Solution 3 by Soumitra Mandal – Kolkata – India

$$\sqrt{x^2 - xy\sqrt{3} + y^2} + \sqrt{y^2 - yz\sqrt{2} + z^2} =$$

$$= \sqrt{\left(\frac{\sqrt{3}}{2}x - y\right)^2 + \left(\frac{x}{2}\right)^2} + \sqrt{\left(y - \frac{z}{\sqrt{2}}\right)^2 + \left(\frac{z}{\sqrt{2}}\right)^2} \geq$$

$$\geq \sqrt{\left(\frac{\sqrt{3}}{2}x - \frac{z}{\sqrt{2}}\right)^2 + \left(\frac{x}{2} + \frac{z}{\sqrt{2}}\right)^2} = \sqrt{x^2 + z^2 - \frac{\sqrt{3}-1}{\sqrt{2}}xz} \geq$$

$$\geq \sqrt{x^2 - xz + z^2} \text{ since } \sqrt{2} + 1 > \sqrt{3}$$

$$\therefore \sqrt{x^2 - xy\sqrt{3} + y^2} + \sqrt{y^2 - yz\sqrt{2} + z^2} \geq \sqrt{x^2 - xz + z^2}$$

(proved)

111. If $x, y, z \in (0, \infty), xyz = 1$ then:

$$x(x - 3(y + z))^2 + (3x - (y + z))^2(y + z) \geq 27$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ear Bunpheng - Phnom Penh – Cambodia

$$S = x(x - 3(y + z))^2 + (3x - (y + z))^2(y + z) \geq 27$$

Since $x, y, z \in (0, +\infty)$

By Cauchy: $x + y + z \geq 3\sqrt[3]{xyz}$

$$x+y+z\geq 3+(xyz-1)$$

$$\Rightarrow S\geq x\left(x-3(3-x)\right)^2+\left(3x-(3-x)\right)^2(y+z)$$

$$\geq x(4x-9)^2+(4x-3)^2(3-x)$$

expand all the factor.

$$\geq 81x-72x^2+16x^3+27-72x+4x^2-9x+24x^2-16x^3\geq 27$$

(True)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$x(x^2-6x\cdot(y+z)+9(y+z)^2)+$$

$$+(y+z)\cdot(9x^2-6x\cdot(y+z)+(y+z)^2)=$$

$$=x^3+3x^2\cdot(y+z)+3x(y+z)^2+(y+z)^3=$$

$$=(x+y+z)^3\geq\left(3\sqrt[3]{xyz}\right)^3=27$$

112. If $a,b,c\in\mathbb{R}, a+b\geq 0, a\leq c\leq b$ then:

$$2\sinh\left(\frac{a+b}{2}\right)\leq\sinh(c)+\sinh(a+b-c)\leq\sinh(a)+\sinh(b)$$

Proposed by Abdallah El Farissi – Bechar – Algerie

Solution 1 by Soumava Chakraborty – Kolkata – India

$$a,b,c\in\mathbb{R}, a+b\geq 0, a\leq c\leq b$$

$$\Rightarrow 2\sinh\left(\frac{a+b}{2}\right)\overset{(1)}{\leq}\sinh(c)+\sinh(a+b-c)\overset{(2)}{\leq}\sinh(a)+\sinh(b)$$

$$\sinh(c)+\sinh(a+b-c)=2\sinh\left(\frac{a+b}{2}\right)\cosh\left(c-\frac{a+b}{2}\right)$$

$$\therefore (1)\Leftrightarrow\underbrace{2\sinh\left(\frac{a+b}{2}\right)\left\{\cosh\left(c-\frac{a+b}{2}\right)-1\right\}\geq 0}_{(a_1)}$$

Now, $\sinh(\lambda)=\frac{e^\lambda-e^{-\lambda}}{2}=\frac{e^\lambda-\frac{1}{e^\lambda}}{2}=\frac{e^{2\lambda}-1}{2e^\lambda}$

If $= \frac{a+b}{2}$***, then*** $\sinh\left(\frac{a+b}{2}\right) = \frac{e^{a+b}-1}{2e^{\frac{a+b}{2}}} \underbrace{\geq}_{(b)} 0$

$$\left(\because e^{a+b} \geq 1, as, a+b \geq 0\right)$$

Again, $\cosh(\lambda) = \frac{e^{\lambda}+e^{-\lambda}}{2} \overset{A-G}{\geq} \frac{2\sqrt{e^{\lambda}\cdot e^{-\lambda}}}{2} = 1$

$$\Rightarrow \cosh\left(c - \frac{a+b}{2}\right) \geq 1 \Rightarrow \underbrace{\cosh\left(c - \frac{a+b}{2}\right) - 1 \geq 0}_{(c)}$$

(b) × (c) ⇒ (a₁) is true ⇒ (1) is true

Again, $\sinh(a) + \sinh(b) = 2\sinh\left(\frac{a+b}{2}\right)\cosh\left(\frac{a-b}{2}\right)$

$\therefore$ ***(2)*** $\Leftrightarrow 2\sinh\left(\frac{a+b}{2}\right)\left\{\cosh\left(\frac{a-b}{2}\right) - \cosh\left(c - \frac{a+b}{2}\right)\right\}$

$\Leftrightarrow 2^2\sinh\left(\frac{a+b}{2}\right)\sinh\left(\frac{c-b}{2}\right)\sinh\left(\frac{a-c}{2}\right) \geq 0$ ***(a₂)***

$$\sinh\left(\frac{c-b}{2}\right) = \frac{e^{c-b}-1}{2e^{\frac{c-b}{2}}} \underbrace{\leq}_{(d)} 0 \left(\because e^{c-b} \leq 1, as\ \ c \leq b\right)$$

$$\sinh\left(\frac{a-c}{2}\right) = \frac{e^{a-c}-1}{2e^{\frac{a-c}{2}}} \underbrace{\leq}_{(e)} 0 \left(\because e^{a-c} \leq 1, as\ a \leq c\right)$$

(b) × (d) × (e) ⇒ a_2 is true ⇒ (2) is true

(Proved)

Solution 2 by Abdallah El Farissi – Bechar – Algerie

Let a, b, c ***be real numbers such that*** $a + b \geq 0$ ***and*** $a \leq c \leq b$***. Prove that***

$$2sh\left(\frac{a+b}{2}\right) \leq sh\,(c) + sh(a+b-c) \leq sh(a) + sh(b)$$

Solution:

Let $f(x) = sh(x) + sh(a+b-c), x \in [a,b]$ ***we have***

$$f''(x)=sh\,(x)+sh\,(a+b-x)=sh\left(\frac{a+b}{2}\right)ch\left(x-\frac{a+b}{2}\right)$$ ***then f is a concex function, for all $c\in[a,b]$ there is $\lambda\in[0,1]$ such that***

$$c=\lambda a+(1-\lambda)b,$$ ***now***

$$2sh\left(\frac{a+b}{2}\right)=sh\left(\frac{a+b-c+c}{2}\right)$$
$$\le sh(c)+sh(a+b-c)=f(c)=f(\lambda+(1-\lambda)b)$$
$$\le \lambda f(a)+(1-\lambda)f(b)=sh(a)+sh(b)$$

113. If $a,b,c,d\ge 0$ then:

$$\sqrt{a^2+b^2-ab\sqrt{2}}+\sqrt{b^2+c^2-bc\sqrt{3}}+\sqrt{c^2+d^2-\frac{cd(\sqrt{6}+\sqrt{2})}{2}}\ge\sqrt{a^2+d^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$a^2+b^2-ab\sqrt{2}=a^2+b^2-2ab\cos\left(\frac{\pi}{4}\right)=\left|a-be^{\frac{i\pi}{4}}\right|^2=\left|ae^{-\frac{i\pi}{4}}-b\right|^2$$

Also, $b^2+c^2-bc\sqrt{3}=\left|b-ce^{\frac{\pi i}{6}}\right|^2$

$$c^2+d^2-\frac{cd}{2}(\sqrt{6}+\sqrt{2})=c^2+d^2-2cd\left(\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}+\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\right)=$$

$$=c^2+d^2-2cd\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=c^2+d^2-2cd\cos\left(\frac{\pi}{12}\right)=\left|c-de^{\frac{i\pi}{12}}\right|^2$$

$$\therefore LHS=\left|ae^{-\frac{i\pi}{4}}-b\right|+\left|b-ce^{\frac{\pi i}{6}}\right|+\left|c-de^{\frac{i\pi}{12}}\right|$$

$$\ge\left|ae^{-\frac{i\pi}{4}}-b+b-ce^{\frac{\pi i}{6}}\right|+\left|c-de^{\frac{i\pi}{12}}\right|=\left|e^{\frac{\pi i}{6}}\right|\left|ae^{-\frac{5i\pi}{2}}-c\right|+\left|c-de^{\frac{\pi}{12}}\right|$$

$$\geq \left|ae^{-\frac{5i\pi}{12}} - de^{\frac{\pi i}{12}}\right| = \left|a - de^{\frac{\pi i}{2}}\right| = \sqrt{a^2 + d^2}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

A result $\sqrt{a^2 + b^2} + \sqrt{x^2 + y^2} \geq \sqrt{(a + x)^2 + (b + y)^2}$ ***where***

$$a, b, x, y \geq 0$$

now $a^2 + b^2 - \sqrt{2}ab = \left(\frac{a}{\sqrt{2}} - b\right)^2 + \frac{a^2}{2}$ ***and*** $b^2 + c^2 - \sqrt{3}bc = \left(b - \frac{\sqrt{3}}{2}\right)^2 + \frac{c^2}{4}$

$$\therefore \sqrt{a^2 + b^2 - \sqrt{2}ab} + \sqrt{b^2 + c^2 - \sqrt{3}bc} =$$

$$\sqrt{\left(\frac{a}{\sqrt{2}} - b\right)^2 + \frac{a^2}{2}} + \sqrt{\left(b - \frac{\sqrt{3}}{2}c\right)^2 + \frac{c^2}{4}}$$

$$\geq \sqrt{\left(\frac{a}{\sqrt{2}} - \frac{\sqrt{3}}{2}c\right)^2 + \left(\frac{a}{\sqrt{2}} + \frac{c}{2}\right)^2} = \sqrt{a^2 + c^2 - \frac{\sqrt{3}-1}{\sqrt{2}}ac} =$$

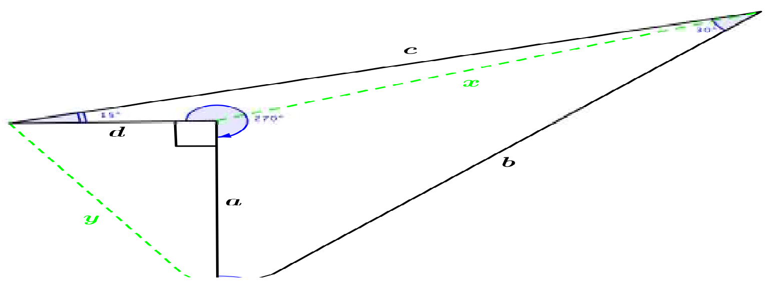
$$= \sqrt{\left(a \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} - c\right)^2 + \left(a \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}}\right)^2}$$

Again, $\sqrt{c^2 + d^2 - \frac{\sqrt{6}+\sqrt{2}}{2}cd} = \sqrt{\left(c - d \cdot \frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2 + \left(d \cdot \frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2}$

$$\therefore \sqrt{a^2 + c^2 - \frac{\sqrt{3} - 1}{\sqrt{2}}ac} + \sqrt{c^2 + d^2 - \frac{\sqrt{6} + \sqrt{2}}{2}cd}$$

$$\geq \sqrt{\left(a \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} - d \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}}\right)^2 + \left(a \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} + d \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}}\right)^2} = \sqrt{a^2 + d^2}$$

Solution 3 by Soumava Chakraborty-Kolkata-India



Let us consider a quadrilateral with angle between 'a' and 'b' = $45°$, angle between 'b' and 'c' = $30°$, angle between 'c' and 'd' = $15°$ and angle between 'a' and 'd' = $270°$

$$\textit{Then } x = \sqrt{a^2 + b^2 - 2ab\cos 45°} = \sqrt{c^2 + d^2 - 2cd\cos 15°}$$

$$\left(\cos 15° = \frac{\sqrt{6} + \sqrt{2}}{4}\right)$$

$$y = \sqrt{b^2 + c^2 - 2bc\cos 30°} = \sqrt{a^2 + d^2}$$

$$\therefore x + y + x > y$$

$$\Rightarrow \sqrt{a^2 + b^2 - ab\sqrt{2}} + \sqrt{b^2 + c^2 - bc\sqrt{3}} + \sqrt{c^2 + d^2 - \frac{cd(\sqrt{6} + \sqrt{2})}{4}} > \sqrt{a^2 + d^2}$$

114. If $x > y > z > 0$ then:

$$\frac{1}{2}\left(\sqrt{\frac{y}{x-y}} + \sqrt{\frac{z}{y-z}} + \sqrt{\frac{x}{x-z}}\right) > \frac{y}{x} + \frac{z}{y} + \frac{x}{2x-z}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sqrt{\frac{y}{x-y}}=\sqrt{1\cdot\left(\frac{y}{x-y}\right)}\geq\frac{2}{1+\frac{x-y}{y}}=\frac{2y}{x}$$

$$\therefore\frac{1}{2}\sqrt{\frac{y}{x-y}}\geq\frac{y}{x}$$

Equality at $y=x-y\Rightarrow 2y-x.$

Similarly, $\sqrt{\frac{z}{y-z}}\geq\frac{2z}{y}$

$$\therefore\frac{1}{2}\sqrt{\frac{z}{y-z}}\geq\frac{z}{y}$$

Equality at $2z=y$

Again, $\frac{1}{2}\sqrt{\frac{x}{x-z}}\geq\frac{1}{1+\frac{x-z}{x}}=\frac{x}{2x-z}$

Equality at $x=x-z\Rightarrow z=0$

(A contradiction)

Hence, Equality will not occur.

$$\frac{1}{2}\left(\sqrt{\frac{y}{x-y}}+\sqrt{\frac{z}{y-x}}+\sqrt{\frac{x}{x-z}}\right)>\frac{y}{x}+\frac{z}{y}+\frac{x}{2x-z}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{1}{2}\left(\sum\sqrt{\frac{y}{x-y}}\right)=\frac{1}{2}\left(\sum\frac{y}{\sqrt{(x-y)y}}\right)=$$

$$=\sum\frac{y}{2\sqrt{(x-y)y}}=\frac{y}{2\sqrt{(x-y)y}}+\frac{z}{2\sqrt{(y-z)z}}+\frac{x}{\sqrt{(x-z)x}}\overset{Cauchy}{\geq}$$

$$\geq \frac{y}{x-y+y}+\frac{z}{y-z+z}+\frac{x}{x-z+x}=\frac{y}{x}+\frac{z}{y}+\frac{x}{2x-z}$$

TRUE

115. Let $a,b,c\in(0,\infty)$ then $9(a^2+b^2+c^2)^2\geq 8(a+b+c)(a^3+b^3+c^3)$

Proposed by Richdad Phuc – Hanoi – Vietnam

Solution by Soumitra Mandal - Chandar Nagore – India

Let $a+b+c=1$, $ab+bc+ca=\frac{1-q^2}{3}$ and $abc=r$

$$\sum_{cyc} ab(a^2+b^2)=\frac{(1+2q^2)(1-q^2)}{9}-r,$$

$$\sum_{cyc} a^4=\frac{-1+8q^2+2q^4}{9}+4r$$

and

$$\sum_{cyc} a^2b^2=\frac{(1-q^2)^2}{9}-2r$$

$VQBC$ *inequality relation,* $r\leq\frac{(1+2q)(1-q)^2}{27}$

$$\therefore 9\left(\sum_{cyc} a^2\right)^2\geq 8\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right)\Rightarrow \sum_{cyc} a^4+18\left(\sum_{cyc} a^2b^2\right)\geq$$

$$\geq 8\sum_{cyc} ab(a^2+b^2)$$

$$\Leftrightarrow \frac{-1+8q^2+2q^4}{9}+4r+18\left(\frac{1-2q^2+q^4}{9}-2r\right)\geq 8\left(\frac{1+q^2-2q^4}{9}-r\right)$$

$$\Leftrightarrow \frac{-1+8q^2+2q^4+18-36q^2+18q^4-8-8q^2+16q^4}{9}\geq 24r$$

$$\Leftrightarrow \frac{36q^4 - 36q^2 + 9}{9} \ge 24 \Leftrightarrow 4q^4 - 4q^2 + 1 \ge \frac{8}{9}(1 - 3q^2 + 2q^3)$$

$$\Leftrightarrow 1 - 12q^2 + 36q^4 - 16q^3 \ge 0.$$ ***Let*** $f(q) = 36q^4 - 16q^3 - 12q^2 + 1$

for all $1 > q \ge 0$

$$f'(q) = 144q^3 - 48q^2 - 24q \Rightarrow f''(q) = 24(18q^2 - 4q - 1) < 0$$

for all $1 > q \ge 0$

$\therefore$ f ***is concave. Hence,*** $f(q) \ge f(1) = 9 > 0$. $\therefore 1 - 12q^2 + 36q^4 - 16q^3 > 0$

$$\therefore 9\left(\sum_{cyc} a^2\right)^2 \ge 8\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right)$$

(proved)

116. If $3 \le a \le b$ then

$$\frac{1}{\sqrt{3}}\arctan\frac{(a+b+1)\sqrt{3}}{1-a-b-2ab} - \frac{1}{2\sqrt{5}}\ln\frac{(2a-3-\sqrt{5})(2b-3+\sqrt{5})}{(2a-3+\sqrt{5})(2b-3-\sqrt{5})} \ge$$

$$\ge \int_a^b \frac{x\,dx}{(x^2+x+1)(x^2-3x+1)}$$

Proposed by Mihály Bencze – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\int_a^b \frac{x\,dx}{(x^2+x+1)(x^2-3x+1)} = \frac{1}{4}\int_a^b \frac{dx}{x^2+x+1} - \frac{1}{4}\int_a^b \frac{dx}{x^2-3x+1}$$

$$= \frac{1}{4}\left[\ln\frac{2x-3-\sqrt{5}}{2x-3+\sqrt{5}}\right]_{x=a}^{x=b} - \frac{1}{4}\left[\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]_{x=a}^{x=b}$$

$$=\frac{1}{4\sqrt{5}}\ln\frac{(2b-3-\sqrt{5})(2a-3+\sqrt{5})}{(2b-3+\sqrt{5})(2a-3-\sqrt{5})}-\frac{1}{2\sqrt{3}}\tan^{-1}\frac{(a+b+1)\sqrt{3}}{1-a-b-2ab}$$

we need to show,

$$\frac{1}{\sqrt{3}}\tan^{-1}\frac{(a+b+1)\sqrt{3}}{1-a-b-2ab}-\frac{1}{2\sqrt{5}}\ln\frac{(2a-3-\sqrt{5})(2b-3+\sqrt{5})}{(2a-3+\sqrt{5})(2b-3-\sqrt{5})}$$

$$\geq\frac{1}{4\sqrt{5}}\ln\frac{(2b-3-\sqrt{5})(2a-3+\sqrt{5})}{(2b-3+\sqrt{5})(2a-3-\sqrt{5})}-\frac{1}{2\sqrt{3}}\tan^{-1}\frac{(a+b+1)\sqrt{3}}{1-a-b-2ab}$$

$$\Leftrightarrow\frac{3}{4}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{(a+b+1)\sqrt{3}}{1-a-b-2ab}\geq\frac{1}{4\sqrt{5}}\ln\frac{(2b-3+\sqrt{5})(2a-3-\sqrt{5})}{(2b-3-\sqrt{5})(2a-3+\sqrt{5})}$$

$$\Leftrightarrow\frac{3}{4}\int_a^b\frac{dx}{x^2+x+1}+\frac{1}{4}\int_a^b\frac{dx}{x^2-3x+1}\geq 0\Leftrightarrow\frac{3}{x^2+x+1}+\frac{1}{x^2-3x+1}\geq 0$$

$\Leftrightarrow (x-2)(2x-1)\geq 0$, ***which is true*** $[\because x\in[a,b]\,and\ b\geq a\geq 3]$

hence proved

117. 1) If a,b,c,k are nonnegative real numbers such that $a+b+c>0$, then

$$\frac{ab}{b+2kc+k^2a}+\frac{bc}{c+2ka+k^2b}+\frac{ca}{a+2kb+k^2c}\leq\frac{a+b+c}{(1+k)^2}.$$

2) If x,y,z are nonnegative real numbers and a,b,c are positive real numbers such that $4ab\geq c^2$, then

$$\frac{xy}{ax+by+cz}+\frac{yz}{ay+bz+cx}+\frac{zx}{az+bx+cy}\leq\frac{x+y+z}{a+b+c}.$$

Proposed by Le Khansy Sy-Long An-Vietnam

Solution by Le Khansy Sy-Long An-Vietnam

1) Using the Cauchy – Schwarz inequality, we have

$$\sum_{cyc}\frac{ab(1+k)^2}{b+2kc+k^2a}\le\sum_{cyc}\left(\frac{ab}{b+kc}+\frac{abk}{c+ka}\right)=$$

$$=\sum_{cyc}\left(\frac{ab}{b+kc}+\frac{ack}{b+kc}\right)=\sum_{cyc}\left[\frac{a(b+kc)}{b+kc}\right]=a+b+c$$

Or $\frac{ab}{b+2kc+k^2a}+\frac{bc}{c+2ka+k^2b}+\frac{ca}{a+2kb+k^2c}\le\frac{a+b+c}{(1+k)^2}$.

The equality holds for $a=b=c$***, and for*** $a=0$ ***and*** $c=kb$ ***(or any cyc permuation)***

2) Case 1 $4ab+c^2$ ***We have a previous case.***

Case 2 $4ab>c^2$***: Using the identity***

$$\frac{xy}{ax+by+cz}=\frac{4bxy}{(4ab-c^2)x+c(cx+2bz)+2b(cz+2by)}$$

Using the Cauchy – Schwarz inequality gives

$$\frac{xy}{ax+by+cz}\le\frac{4bxy}{(4ab-c^2+c^2+2bc+2bc+4b^2)^2}\left[\frac{(4ab-c^2)^2}{(4ab-c^2)x}+\frac{(c^2+2bc)^2}{c(cx+2bz)}+\frac{(2bc+4b^2)^2}{2b(cz+2by)}\right]$$

$$=\frac{1}{4b(a+b+c)^2}\left[y(4ab-c^2)+(c+2b)^2\left(\frac{cxy}{cx+2bz}+\frac{2bxy}{cz+2by}\right)\right],$$

hence

$$\sum_{cyc}\frac{xy}{ax+by+cz}\le\sum_{cyc}\left\{\frac{1}{4b(a+b+c)^2}\left[y(4ab-c^2)+(c+2b)^2\left(\frac{cxy}{cx+2bz}+\frac{2bxy}{cz+2by}\right)\right]\right\}$$

$$=\sum_{cyc}\left\{\frac{1}{4b(a+b+c)^2}\left[y(4ab-c^2)+(c+2b^2)\left(\frac{cxy}{cx+2bz}+\frac{2byz}{cx+2bz}\right)\right]\right\}$$

$$=\sum_{cyc}\frac{y}{a+b+c}=\frac{x+y+z}{a+b+c}$$

118. Let a, b, c be positive real numbers such that $a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Prove that:

$$3(a^3b + b^3c + c^3a) \geq (a + b + c)^2.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c *números* $\mathbb{R}^+$ *de tal manera que:* $a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

Probar que: $3(a^3b + b^3c + c^3a) \geq (a + b + c)^2$

Siendo: $a, b, c > 0$. *Probar la desigualdad de Holder:*

$$(a^3b + b^3c + c^3a)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{a}\right)(1 + 1 + 1) \geq (a + b + c)^3$$

Luego:

$$\Rightarrow 3(a^3b + b^3c + c^3a) \geq \frac{(a+b+c)^3}{\frac{1}{b}+\frac{1}{c}+\frac{1}{a}} \geq \frac{(a+b+c)^3}{a+b+c} = (a + b + c)^2 \text{ ... (LQQD)}$$

119. Prove that if $a, b, c \in [1, 2]$

$$(3a + 4b + 5c)\left(\frac{3}{a} + \frac{4}{b} + \frac{5}{c}\right) \leq 162$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$(3a + 4b + 5c)\left(\frac{3}{a} + \frac{4}{b} + \frac{5}{c}\right) \leq 162$$

$$\Leftrightarrow (3^2 + 4^2 + 5^2) + \left(\frac{12a}{b} + \frac{15a}{c}\right) + \left(\frac{12b}{a} + \frac{20b}{c}\right) + \left(\frac{15c}{a} + \frac{20c}{b}\right) \leq 162$$

$$\Leftrightarrow \left(\frac{12a}{b} + \frac{15a}{c}\right) + \left(\frac{12b}{a} + \frac{20b}{c}\right) + \left(\frac{15c}{a} + \frac{20c}{b}\right) - 112 \leq 0$$

WLOG let $a = \min\{a, b, c\}$

$$\text{Let } f(a)=\frac{12a}{b}+\frac{15a}{c}+\frac{12b}{a}+\frac{15c}{a}+\frac{20b}{c}+\frac{20c}{b}-112 \text{ for all } a\in[1,2]$$

$$f'(a)=(12b+15c)\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}\right)\geq 0 \therefore f(a) \text{ is increasing}$$

$$f(2)\geq f(a)\Rightarrow \frac{24}{b}+\frac{30}{c}+6b+\frac{15c}{2}+\frac{20b}{c}+\frac{20c}{b}-112\geq f(a)$$

$$\text{Let } \varphi(b)=\frac{24}{b}+\frac{30}{c}+6b+\frac{15c}{2}+\frac{20b}{c}+\frac{20c}{b}-112 \text{ for all } b\in[1,2]$$

$$\varphi'(b)=6+\frac{20}{c}-\frac{24}{b^2}-\frac{20c}{b^2}\Rightarrow \varphi''(b)=\frac{2(24+20c)}{b^3}>0 \text{ for all } b\in[1,2]$$

$$\therefore \max\{\varphi(1),\varphi(2)\}\geq\varphi(b). \text{ So, } \varphi(1)\geq\varphi(b)$$

$$\Rightarrow \frac{50}{c}+\frac{55c}{2}-82\geq\varphi(b). \text{ Let } h(c)=\frac{50}{c}+\frac{55c}{2}-82 \text{ for all } c\in[1,2]$$

$$h'(c)=\frac{55}{2}-\frac{50}{c^2}=\frac{5}{2c^2}(11c^2-10)\geq 0 \text{ for all } c\in[1,2]$$

$$\therefore h \text{ is an increasing } f \text{ function} \therefore h(2)\geq h(c)\Rightarrow 0>-2>h(c)$$

$$\therefore \varphi(2)\geq\varphi(b)\Rightarrow \frac{35c}{2}+\frac{70}{c}-88\geq\varphi(b)$$

$$\text{Let } F(c)=\frac{35c}{2}+\frac{70}{c}-88 \text{ for all } c\in[1,2]. \therefore F'(c)=\frac{35}{2c^2}(c^2-4)\leq 0 \text{ for all } c\in[1,2]$$

$$\therefore F(c) \text{ is decreasing. So, } F(1)\geq F(c)\Rightarrow 0>-\frac{1}{2}\geq F(c)$$

$$\therefore 0>\max\{\varphi(1),\varphi(2)\}\geq\varphi(b).$$

$$\therefore (3a+4b+5c)\left(\frac{3}{a}+\frac{4}{b}+\frac{5}{c}\right)\leq 162 \text{ (proved)}$$

120. If $a,b,c>0$ then:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{c+a}{c+b}+\frac{a+b}{a+c}+\frac{b+c}{b+a}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow$

$$\frac{ab^2+bc^2+ca^2}{abc}\geq\frac{(a+b)^2(b+c)+(b+c)^2(c+a)+(c+a)^2(a+b)}{(a+b)(b+c)(c+a)}$$

$$\Leftrightarrow (ab^2+bc^2+ca^2)(a+b)(b+c)(c+a)\geq$$

$$abc[(a+b)\{(a+b)(b+c)\}+(b+c)\{(b+c)(c+a)\}+(c+a)\{(c+a)(a+b)\}]$$

$$\Leftrightarrow\left(\sum ab^2\right)\left(2abc+\sum ab^2+\sum a^2b\right)\geq$$

$$\geq abc\left\{(a+b)\left(\sum ab+b^2\right)+(b+c)\left(\sum ab+c^2\right)+(c+a)\left(\sum ab+a^2\right)\right\}$$

$$\Leftrightarrow 2abc\left(\sum ab^2\right)+\left(\sum ab^2\right)^2+\left(\sum ab^2\right)\left(\sum a^2b\right)\geq$$

$$\geq abc\left\{2(a+b+c)(ab+bc+ca)+\sum ab^2+\sum a^3\right\}$$

$$=abc\left\{2\left(\sum a^2b+\sum ab^2\right)+6abc+\sum ab^2+\sum a^3\right\}$$

$$=2abc\left(\sum ab^2\right)+2abc\left(\sum a^2b\right)+6a^2b^2c^2+abc\left(\sum ab^2\right)+abc(a^3+b^3+c^3)$$

$$\Leftrightarrow a^2b^4+b^2c^4+c^2a^4+2abc\left(\sum a^2b\right)+\sum a^3b^3+3a^2b^2c^2+abc\left(\sum a^3\right)\geq$$

$$\geq 2abc\left(\sum a^2b\right)+6a^2b^2c^2+abc\left(\sum ab^2\right)+abc\left(\sum a^3\right)$$

$$\Leftrightarrow a^2b^4+b^2c^4+c^2a^4+\sum a^3b^3\geq 3a^2b^2c^2+abc(ab^2+bc^2+ca^2)\quad \textbf{(i)}$$

Now,

$$\left.\begin{array}{l} a^3b^3+a^3b^3+b^3c^3\overset{A-G}{\geq}3b^3a^2c\\ b^3c^3+b^3c^3+c^3a^3\overset{A-G}{\geq}3c^3b^2a\\ c^3a^3+c^3a^3+a^3b^3\overset{A-G}{\geq}3a^3c^2b\end{array}\right\}\Rightarrow Adding,\ 3\sum a^3b^3\geq 3abc(ab^2+bc^2+ca^2)\Rightarrow$$

$$\Rightarrow\sum a^3b^3\geq abc(ab^2+bc^2+ca^2)\ \textbf{(1)}$$

Also, $a^2b^4+b^2c^4+c^2a^4\overset{A-G}{\geq}3a^2b^2c^2$ ***(2)***

(1) + (2) $\Rightarrow$ ***(i) is true (Proved)***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{c+a}{c+b}=\frac{c\cdot\left(1+\frac{a}{c}\right)}{c\cdot\left(1+\frac{b}{c}\right)}=\frac{1+\frac{a}{c}}{1+\frac{b}{c}}$$

$$\frac{a}{b}=x;\frac{b}{c}=y;\frac{c}{a}=z\Rightarrow \begin{matrix} xyz=1 \\ x+y+z\geq 3\end{matrix}$$

$$\frac{b}{a}=\frac{1}{x};\frac{c}{b}=\frac{1}{y};\frac{a}{c}=\frac{1}{z}$$

$$\frac{b}{a}=\frac{1}{x}=\frac{xyz}{x}=yz;\frac{c}{b}=zx;\frac{a}{c}=xy$$

$$x+y+z\geq\frac{1+xy}{1+y}+\frac{1+yz}{1+z}+\frac{1+zx}{1+x}$$

$$\underbrace{(x+y+z)(1+y)(1+z)(1+x)}_{LHS}\geq$$

$$\geq\underbrace{(1+xy)(1+z)(1+x)+(1+yz)(1+x)(1+y)+(1+zx)(1+y)(1+z)}_{RHS}$$

1) $LHS=2\cdot(x+y+z)+(x+y+z)^2+(x+y+z)(xy+yz+zx)$

2) $RHS\Rightarrow \left.\begin{matrix}(1+xy)(1+z)(1+z)=2+z+2x+xz+yx+x^2y \\ (1+yz)(1+x)(1+y)=2+x+2y+xy+yz+y^2z \\ (1+zx)(1+y)(1+z)=2+y+2z+zx+zy+z^2x\end{matrix}\right\}$

$$xyz=1$$

$$RHS=6+3(x+y+z)+2(xy+yz+zx)+x^2y+y^2z+z^2x$$

$$LHS=2(x+y+z)+x^2+y^2+z^2+2(xy+yz+zx)+3xyz+$$

$$+(x^2y+y^2z+z^2x)+(xy^2+yz^2+zx^2)\geq$$

$$\geq 6+3(x+y+z)+2(xy+yz+zx)+(x^2y+y^2z+z^2x)$$

$$x^2+y^2+z^2+(xy^2+yz^2+zx^2)\geq 6-3xyz+(x+y+z)$$

$$x^2+y^2+z^2+(xy^2+yz^2+zx^2)\geq 3+(x+y+z)$$

$$\textbf{a)}\ xy^2+yz^2+zx^2 \underset{Cauchy}{\geq} 3\cdot\sqrt[3]{(xyz)^3}=3xyz=3$$

$$\textbf{b)}\ x^2+y^2+z^2 \underset{Chebyshev}{\geq} \frac{1}{3}\cdot(x+y+z)\cdot(x+y+z) \overset{Cauchy}{\geq}$$

$$\geq \frac{1}{3}\cdot 3\sqrt[3]{xyz}\cdot(x+y+z)=x+y+z$$

$$x^2+y^2+z^2+(xy^2+yz^2+zx^2)\geq x+y+z+3$$

121. **Prove that if $a,b,c\geq 0$ then:**

$$\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6\leq 27(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: $a,b,c\geq 0$. ***Probar que:***

$$\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6\leq 27(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)$$

Tener en cuenta lo siguiente:

$$\textbf{1)}: a^2+ab+b^2\geq\frac{3}{4}(a+b)^2 \Leftrightarrow (a-b)^2\geq 0$$

$$\textbf{2) Si: } a,b,c>0\to\frac{9}{8}(a+b)(b+c)(c+a)\geq(a+b+c)(ab+bc+ca)$$

$$\textbf{3) Si: } a,b,c\in R\to(a+b+c)^2\geq 3(ab+bc+ca)$$

Por lo tanto:

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ac+c^2)\geq$$

$$\geq 27\left(\frac{3}{4}\right)^3(a+b)^2(b+c)^2(c+a)^2\geq 9(a+b+c)^2(ab+bc+ca)^2$$

Ahora bien: $9(a+b+c)^2(ab+bc+ca)^2\geq 27(ab+bc+ca)^3$

Es suficiente demostrar lo siguiente:

$$27(ab+bc+ca)^3\geq\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6$$

Por la desigualdad de Cauchy: $27(ab+bc+ca)^3 \geq 27\left(\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{3}\right)^3 \geq$

$\geq \left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6$ ***... (LQQD)***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $a, b, c \geq 0$ ***then***

$$27\prod_{cyc}(a^2+ab+b^2) \geq \left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6$$

We know, $a^2+ab+b^2 = \frac{3}{4}(a+b)^2+\frac{(a-b)^2}{4} \geq \frac{3}{4}(a+b)^2$

Similarly, $b^2+bc+c^2 \geq \frac{3}{4}(b+c)^2$ ***and*** $c^2+ca+a^2 \geq \frac{3}{4}(c+a)^2$

$$\therefore 27\prod_{cyc}(a^2+ab+b^2) \geq 27\cdot\left(\frac{3}{4}\right)^3\prod_{cyc}(a+b)^2$$

$$\geq 27\cdot\left(\frac{3}{4}\right)^3\cdot\frac{64}{81}(a+b+c)^2(ab+bc+ca)^2\left[\therefore 9\prod_{cyc}(a+b) \geq 8\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right]$$

$\geq 27(ab+bc+ca)^3$ ***[since,*** $(a+b+c)^2 \geq 3(ab+bc+ca)$***]***

$$\geq \left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6\left[\because \frac{ab+bc+ca}{3} \geq \left(\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{3}\right)^2\right]$$

(proved)

Solution 3 by Soumava Chakraborty-Kolkata-India

$$27(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq \left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6$$

Case 1: Exactly 1 of $a, b, c = 0$***. If*** $a = 0$***, given inequality***

$$\Leftrightarrow 27b^2c^2(b^2+bc+c^2) \geq b^3c^3$$

$\Leftrightarrow 27b^2+26bc+27c^2 \geq 0$***, which is true.***

Proceeding similarly, it can be shown that the inequality holds true for

$b = 0$ ***or*** $c = 0$

***Case 2:** Exactly 2 of $a, b, c = 0$. Inequality becomes $0 \geq 0$, which is true.*

***Case 3:** All of $a, b, c = 0$. Inequality becomes $0 \geq 0$, which is true.*

(Solution by Soumava C)

***Case 4:** $a, b, c > 0$. Using Wu's inequality $-1, LHS \geq 27(ab+bc+ca)^3$*

$\therefore$ ***it suffices to prove:*** $27(ab+bc+ca)^8 \geq \left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6$

$$\Leftrightarrow 3(ab+bc+ca) \geq \left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2$$

$\Leftrightarrow 3\sum x^2 = (\sum x)^2$, ***where*** $x=\sqrt{ab}, y=\sqrt{bc}, z=\sqrt{ca}$, ***which is true***

(Hence proved)

122. If $a, b, c > 0, a+b+c=3$ then:

$$\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}} \geq \frac{a^2+b^2+c^2}{2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Hoang Le Nhat Tung – Hanoi – Vietnam

$$\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}} \geq \frac{a^2+b^2+c^2}{2} \quad \textit{(1)}$$

**** Since inequality Buniakovski we have:***

$$\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}} = \frac{\left(\frac{a^2}{b^2}\right)^2}{\sqrt{2c(a^3+1)}}+\frac{\left(\frac{b^2}{c^2}\right)^2}{\sqrt{2a(b^3+1)}}+\frac{\left(\frac{c^2}{a^2}\right)^2}{\sqrt{2b(c^3+1)}}$$

$$\geq \frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2}{\sqrt{2c(a^3+1)}+\sqrt{2a(b^3+1)}+\sqrt{2b(c^3+1)}} \quad \textit{(2)}$$

- Other, since AM-GM for 3 positive real numbers:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} = \frac{\frac{a^2}{b^2}+\frac{a^2}{b^2}+\frac{b^2}{c^2}}{3}+\frac{\frac{b^2}{c^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}{3}+\frac{\frac{c^2}{a^2}+\frac{c^2}{a^2}+\frac{a^2}{b^2}}{3} \geq$$

$$\geq \frac{3\sqrt[3]{\frac{a^2}{b^2}\cdot\frac{a^2}{b^2}\cdot\frac{b^2}{c^2}}}{3}+\frac{3\sqrt[3]{\frac{b^2}{c^2}\cdot\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}}}{3}+\frac{3\sqrt[3]{\frac{c^2}{a^2}\cdot\frac{c^2}{a^2}\cdot\frac{a^2}{b^2}}}{3}=\sqrt[3]{\frac{a^4}{b^2c^2}}+\sqrt[3]{\frac{b^4}{c^2a^2}}+\sqrt[3]{\frac{c^4}{a^2b^2}}$$

$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{a^2+b^2+c^2}{\sqrt[3]{a^2b^2c^2}}$. ***Because:*** $3=a+b+c\geq 3\cdot\sqrt[3]{abc}\Rightarrow$

$$\sqrt[3]{abc}\leq 1\Leftrightarrow\sqrt[3]{a^2b^2c^2}\leq 1$$

$$\Rightarrow\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{a^2+b^2+c^2}{\sqrt[3]{a^2b^2c^2}}\geq\frac{a^2+b^2+c^2}{1}=a^2+b^2+c^2\Leftrightarrow\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq a^2+b^2+c^2\quad (3)$$

+ Since (2), (3):

$$\Rightarrow\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}}\geq\frac{(a^2+b^2+c^2)^2}{\sqrt{2c(a^3+1)}+\sqrt{2a(b^3+1)}+\sqrt{2b(c^3+1)}}$$

- Since AM-GM for 2 positive real numbers

$$\sqrt{2c(a^3+1)}+\sqrt{2a(b^3+1)}+\sqrt{2b(c^3+1)}$$

$$=\sqrt{(ca+c)(2a^2-2a+2)}+\sqrt{(ab+a)(2b^2-2b+2)}+\sqrt{(bc+b)(2c^2-2c+2)}\leq$$

$$\leq\frac{(ca+c)+(2a^2-2a+2)}{2}+\frac{(ab+a)+(2b^2-2b+2)}{2}+\frac{(bc+b)+(2c^2-2c+2)}{2}$$

$$\Rightarrow\sqrt{2c(a^3+1)}+\sqrt{2a(b^3+1)}+\sqrt{2b(c^3+1)}\leq a^2+b^2+c^2+\frac{ab+bc+ca}{2}-\frac{a+b+c}{2}+3\quad (5)$$

- Since (4), (5):

$$\Rightarrow\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}}\geq\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\frac{ab+bc+ca}{2}-\frac{a+b+c}{2}+3}\quad (6)$$

We will prove that: $\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\frac{ab+bc+ca}{2}-\frac{a+b+c}{2}+3}\geq\frac{a^2+b^2+c^2}{2}$ ***(7)***

$$\Leftrightarrow 2(a^2+b^2+c^2)\geq a^2+b^2+c^2+\frac{ab+bc+ca}{2}-\frac{a+b+c}{2}+3$$

$\Leftrightarrow a^2+b^2+c^2+\frac{a+b+c}{2}\geq\frac{ab+bc+ca}{2}+3\Leftrightarrow a^2+b^2+c^2+\frac{3}{2}\geq\frac{ab+bc+ca}{2}+3$ ***(Do*** $a+b+c=3$***)***

$$\Leftrightarrow a^2+b^2+c^2\geq\frac{ab+bc+ca}{2}+\frac{3}{2}\Leftrightarrow 2(a^2+b^2+c^2)\geq ab+bc+ca+3\quad (8)$$

- Other, such that: $a+b+c=3$. ***We have:***

$$2(a^2+b^2+c^2)+3=\frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2}+(a^2+1)+(b^2+1)+(c^2+1)\geq$$

$$\ge \frac{2ab}{2}+\frac{2bc}{2}+\frac{2ca}{2}+2\sqrt{a^2}+2\sqrt{b^2}+2\sqrt{c^2}=ab+bc+ca+2(a+b+c)=ab+bc+ca+6$$

$\Rightarrow 2(a^2+b^2+c^2)\ge ab+bc+ca+3 \Rightarrow$ ***Inequality (8) True*** $\Rightarrow$ ***(7) True.***

- Since (6), (7): $\Rightarrow \frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}}\ge \frac{a^2+b^2+c^2}{2}$

$\Rightarrow$ ***Inequality (1) true and we get the result:***

+ The occurs if: $\begin{cases} a,b,c>0; a+b+c=3 \\ \dfrac{\frac{a^2}{b^2}}{\sqrt{2c(a^3+1)}}=\dfrac{\frac{b^2}{c^2}}{\sqrt{2a(b^3+1)}}=\dfrac{\frac{c^2}{a^2}}{\sqrt{2b(c^3+1)}} \\ \frac{a^2}{b^2}=\frac{b^2}{c^2}=\frac{c^2}{a^2}; a=b=c=1 \\ ca+c=2a^2-2a+2 \\ ab+a=2b^2-2b+2 \\ bc+b=2c^2-2c+2 \end{cases} \Leftrightarrow a=b=c=1.$

Solution 2 by Pham Quy-Vietnam

Given $a,b,c>0, a+b+c=3$***. Prove that***

$$\sum \frac{a^4}{b^4\sqrt{2c(a^3+1)}}\ge \frac{\sum a^2}{2}$$

Lemma 1: $\forall\, a,b,c>0: \sum \frac{a}{b}\ge \frac{\sqrt{3(a^2+b^2+c^2)}}{\sqrt[3]{abc}}$ ***(1)***

Proof:

WLOG, suppose that $abc=1$***, (1) becomes***

$$\sum \frac{a}{b}\ge \sqrt{3(a^2+b^2+c^2)}$$

$$\Leftrightarrow \left(\sum \frac{a}{b}\right)^2\ge 3(a^2+b^2+c^2)$$

$$\Leftrightarrow \sum \frac{a^2}{b^2}+2\sum \frac{a}{c}\ge 3(a^2+b^2+c^2)$$

$$\Leftrightarrow a^4c^2+b^4a^2+c^4b^2+2a^2b+2b^2c+2c^2a\ge 3(a^2+b^2+c^2)$$

By AM-GM inequality $\begin{cases} a^4c^2 + a^2b + a^2b \ge 3\sqrt[3]{a^6 \cdot a^2b^2c^2} = 3a^2 \\ b^4a^2 + b^2c + b^2c \ge 3b^2 \\ c^4b^2 + c^2a + c^2a \ge 3c^2 \end{cases}$

$\Rightarrow$ ***Proved***

By Cauchy – Schwarz's inequality, we have

$$\sum \frac{a^4}{b^4\sqrt{2c(a^3+1)}} \ge \frac{\left(\sum \frac{a^2}{b^2}\right)^2}{\sum \sqrt{2c(a^3+1)}} \ge \frac{\sum \frac{a^2}{b^2} \cdot \frac{\sqrt{3(a^4+b^4+c^4)}}{\sqrt[3]{a^2b^2c^2}}}{\sum \sqrt{2c(a^3+1)}}$$

Form $a+b+c=3 \ge 3\sqrt[3]{abc}$ ***(AM-GM)*** $\Rightarrow abc \le 1$

$$\ge \frac{\sum \frac{a^2}{b^2} \cdot \sqrt{3(a^4+b^4+c^4)}}{\sum \sqrt{2c(a^3+1)}} \ge \frac{2\sum \frac{a^2}{b^2} \cdot \left(\sum a^2\right)}{2\sum \sqrt{2c(a^3+1)}}$$

$$\ge \frac{2\sum \frac{a^2}{b^2} \cdot \left(\sum a^2\right)}{\sum a^3 + 2\sum a + 3} \quad \textit{(AM-GM)}$$

On the other hand, we have

$$4\sum \frac{a^2}{b^2} \ge \frac{\sqrt{3(a^4+b^4+c^4)}}{\sqrt[3]{a^2b^2c^2}} + 3\sum \frac{a^2}{b^2} \overset{AM-GM \text{ and } abc\le 1}{\ge} \sqrt{(a+b+c)(a^4+b^4+c^4)} + 1$$

$\ge a^3+b^3+c^3+2\sum a+3$ ***(Cauchy – Schwarz and*** $a+b+c=3$***)***

$$\Rightarrow \sum \frac{a^4}{b^4\sqrt{2c(a^3+1)}} \ge \frac{\left(\sum a^3 + 2\sum a + 3\right)\left(\sum a^2\right)}{2\left(\sum a^3 + 2\sum a + 3\right)} = \frac{\sum a^2}{3}$$

(q.e.d.)

Solution 3 by Khung Long Xanh-Da Nang-Vietnam

$a, b, c > 0, a+b+c=3$***. Prove that***

$$\sum \frac{a^4}{b^4\sqrt{2c(a^3+1)}} \ge \frac{a^2+b^2+c^2}{2}$$

Lemma: $a^3b + b^3c + c^3a \le \frac{(a^2+b^2+c^2)^2}{3}$ ***(Which is Vasc's famous inequality)***

By AM-GM inequality we have:

$$a^2b^2 + b^2c^2 + c^2a^2 \le \frac{(a^2 + b^2 + c^2)^2}{3}$$

$$\sum \frac{a^4}{b^4\sqrt{2c(a^3+1)}} = \sum \frac{a^4}{b^2\cdot\sqrt{b}\cdot\sqrt{2ca^3b^3+2cb^3}} \ge \frac{a^4}{b^2\cdot\sqrt{b}\cdot\sqrt{2a^2b^2+2b^3c}}$$

Using Cauchy - Schwarz inequality we have

$$\sum \frac{a^4}{b^2\cdot\sqrt{b}\cdot\sqrt{2a^2b^2+2b^3c}} \cdot \left(\sum \sqrt{b}\cdot\sqrt{2a^2b^2 + 2b^3c}\right) \ge \left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)^2 \quad (*)$$

We have that:

1. $\sum \sqrt{b}\cdot\sqrt{2a^2b^2 + 2b^3c} \le \sqrt{2(a + b + c)(\sum a^2b^2 + \sum b^3c)} \le \sqrt{6\cdot\frac{2(a^2+b^2+c^2)^2}{3}}$

$$= 2(a^2 + b^2 + c^2)$$

2. $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge \frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a}$

By Cauchy Schwarz:

$$a^2b + b^2c + c^2a \le \sqrt{(ab + bc + ca)(a^3b + b^3c + c^3a)} \le \sqrt{3\cdot\frac{(a^2+b^2+c^2)^2}{3}} =$$

$$= a^2 + b^2 + c^2$$

Thus we have: $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge \frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a} \ge a^2 + b^2 + c^2$

So from (*). We have:

$$2(a^2 + b^2 + c^2)\cdot\sum \frac{a^4}{b^4\sqrt{2c(a^3 + 1)}} \ge (a^2 + b^2 + c^2)^2$$

$$\Leftrightarrow \sum \frac{a^4}{b^4\sqrt{2c(a^3 + 1)}} \ge \frac{a^2 + b^2 + c^2}{2}$$

123. If $a, b, c > 0, a^4 + b^4 + c^4 = 1$ then:

$$\frac{a^7 + b^7}{ab(a + b)} + \frac{b^7 + c^7}{bc(b + c)} + \frac{c^7 + a^7}{ca(c + a)} \ge 3(a^2b^2 + b^2c^2 + c^2a^2) - 2$$

Proposed by Marin Chirciu – Romania

Solution 1 by Shahlar Maharramov – Jebrail – Azerbaidian

$$\frac{a^7 + b^7}{ab(a + b)} = \frac{(a + b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6)}{ab(a + b)}$$

$$= \frac{a^6 + a^4b^2 + a^2b^4 + b^6}{ab} - \underbrace{\frac{a^5b + a^3b^3 + ab^4}{ab}}_{a^4 + a^3b^2 + b^4}$$

Then we have so prove

$$2 + \sum_{cycl} \frac{a^6 + a^4b^2 + a^2b^4 + b^6}{ab} \ge \sum_{cycl} (a^4 + a^2b^2 + b^4) + \sum 3a^2b^2$$

Since $a^4 + b^4 + c^4 = 1$

$$2\sum_{cycl} \frac{a^6 + a^4b^2 + a^2b^4 + b^6}{ab} \ge 2 + \sum_{cycl} 4\frac{\sqrt[4]{a^{12}b^{12}}}{ab} = 2 + \sum_{cycl} 4a^2b^2$$

But RHS

$$\sum_{cycl} (a^4 + a^2b^2 + b^4) + \sum 3a^2b^2$$

$$a^4 + b^4 + c^4 = 1$$

$$= 2 + \sum_{cycl} a^2b^2 + \sum 3a^2b^2 = 2 + \sum_{cycl} 4a^2b^2$$

It means LHS

$$= 2 + \sum_{cycl} 4a^2b^2, \ RHS = 2 + \sum_{cycl} 4a^2b^2$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $a,b,c>0$ ***and*** $\sum_{cyc}a^4=1$ ***then***

$$\sum_{cyc}\frac{a^7+b^7}{ab(a+b)}\geq 3\left(\sum_{cyc}a^2b^2\right)-2$$

We know, $a^3+b^3\geq ab(a+b), b^3+c^3\geq bc(b+c)$ ***and***

$c^3+a^3\geq ca(c+a).\therefore \sum_{cycl}\frac{a^7+b^7}{ab(a+b)}\geq\sum_{cycl}\frac{a^7+b^7}{a^3+b^3}=\sum_{cyc}\frac{(a^4+b^4)(a^3+b^3)-(ab)^3(a+b)}{a^3+b^3}$

$=\sum_{cyc}(a^4+b^4)-\sum_{cyc}(ab)^2\cdot\frac{ab(a+b)}{a^3+b^3}=2\sum_{cyc}a^4-\sum_{cyc}(ab)^2=4\sum_{cyc}a^4-$

$\sum_{cyc}(ab)^2-2\geq 3\sum_{cyc}(ab)^2-2$ ***(proved)***

$$\left[\because\sum_{cyc}a^4\geq\sum_{cyc}(ab)^2\right]$$

equality at $x=y=z=\frac{1}{\sqrt[4]{3}}$

Solution 3 by Soumava Chakraborty-Kolkata-India

$a,b,c>0,\sum a^4=1\Rightarrow\sum\frac{a^7+b^7}{ab(a+b)}\geq 3\sum a^2b^2-1\rightarrow$ ***(1)***

(1) $\Leftrightarrow\sum\frac{(a^7+b^7)(a+b)}{ab(a+b)^2}\overset{(2)}{\geq}3x-2,$ ***where*** $x=\sum a^2b^2$

Now, $(a^7+b^7)(a+b)\geq(a^4+b^4)^2\rightarrow$ ***(3) [***$\because a^7b+b^7a\geq 2a^4b^4$ ***by AM-GM]***

$\therefore\sum\frac{(a^7+b^7)(a+b)}{ab(a+b)^2}\geq\sum\frac{(a^4+b^4)^2}{ab(a+b)^2}$ ***(using (3))*** $\geq\frac{(\sum(a^4+b^4))^2}{\sum ab(a+b)^2}$ ***(Bergstrom)***

$\geq\frac{4}{\sum\frac{(a+b)^4}{4}}\left(\because ab\leq\frac{(a+b)^2}{4}\right)=\frac{16}{\sum(a+b)^4}\geq\frac{16}{8\sum(a^4+b^4)}$

$(\because a^4+b^4\geq\frac{1}{8}(a+b)^4$ ***by Chebyshev)*** $=1(\because\sum a^4=1)\Rightarrow\sum\frac{(a^7+b^7)(a+b)}{ab(a+b)^2}\geq 1\rightarrow$ ***(4)***

$$x=\sum a^2b^2\leq\sum a^4=1\Rightarrow 1\geq x\Rightarrow 3\geq 3x=1\overset{(5)}{\geq}3x-2$$

(4), (5) ⇒ (2) is true (Hence proved)

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{a^7+b^7}{ab(a+b)} \ge 3\cdot\sum a^2b^2 - 2$$

$$3\sum a^2b^2 - 2(a^4+b^4+c^4) \le 3\cdot\sum a^2b^2 - 2\cdot\sum a^2b^2 = \sum a^2b^2$$

$$\sum\left(\frac{a^7+b^7}{ab(a+b)} - a^2b^2\right) \ge 0 \textit{ (ASSURE)}$$

$$\sum\left(\frac{(a+b)\cdot(a^6-a^3b+\dots+b^6)}{(a+b)\cdot ab} - a^2b^2\right) =$$

$$= \sum\left(\frac{a^5(a-b)+a^3b^2(a-b)-a^2b^3(a-b)-b^5(a-b)}{ab}\right) =$$

$$= \sum\frac{(a-b)\cdot(a^3+a^3b^2-a^2b^3-b^3)}{ab} =$$

$$= \sum\frac{(a-b)^2\cdot(a^4+a^3b+2a^2b^2+ab^3+b^4)}{ab} \ge 0$$

TRUE

Solution 5 by Kunihiko Chikaya-Tokyo-Japan

$$\frac{a^7+b^7}{ab(a+b)} = \frac{a^7+b^7}{2}\cdot\frac{2}{ab(a+b)}$$

$$\underbrace{\ge}_{Jensen}\left(\frac{a+b}{2}\right)^7\cdot\frac{2}{ab(a+b)} = \frac{1}{2^6}\cdot\frac{\{(a+b)^2\}^3}{ab} \ge \frac{1}{2^6}\cdot\frac{(4ab)^3}{ab}$$

$$\left.\begin{array}{l}\therefore \frac{a^7+b^7}{ab(a+b)} + \frac{b^7+c^7}{bc(b+c)} + \frac{c^7+a^7}{ca(c+a)} \ge a^2b^2+b^2c^2+c^2a^2 \\ a^2b^2+b^2c^2+c^2a^2 \ge 3(a^2b^2+b^2c^2+c^2a^2)-2\end{array}\right\} \Rightarrow \textit{Goal!}$$

$$\Leftrightarrow a^2b^2+b^2c^2+c^2a^2 \le a^4+b^4+c^4 \Leftrightarrow \frac{1}{2}\{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2\} \ge 0$$

Solution 6 by Imad Zak-Saida-Lebanon

$$\textbf{Chebyshev's} \Rightarrow a^7+b^7 \ge \frac{1}{2}(a^4+b^4)(a^3+b^3) \ge \frac{1}{2}(a^4+b^4)(a+b)ab$$

$$LHS \ge \sum \frac{1}{2}(a^4+b^4) = \sum a^4 \overset{??}{\ge} 3\sum a^2b^2 - 2 \Leftrightarrow \sum a^4 \overset{??}{\ge} 3\sum a^2b^2 - 2\sum a^4 \Leftrightarrow$$

$$3\sum a^4 \overset{??}{\ge} 3\sum a^2b^2 \Leftrightarrow \sum a^4 \ge \sum a^2b^2.$$

True from $\sum x^2 \ge \sum xy$

Q.E.D

Equality at $(a,b,c) = \left(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$

Solution 7 by Horia Nicolcea-Romania

$$\sum_{cyc}\frac{a^7+b^7}{ab(a+b)} \ge (1)\sum_{cyc}\frac{(a^3+b^3)(a^4+b^4)}{2ab(a+b)} \ge (2)\sum_{cyc}\frac{a^4+b^4}{2} = 1 \ge (3)3a^2b^2+3b^2c^2+3c^2a^2-2$$

Where: (1) is $a^7+b^7 \ge \frac{(a^4+b^4)(a^3+b^3)}{2} \Leftrightarrow (a^4-b^4)(a^3-b^3) \ge 0;$

(2) is $a^3+b^3 \ge ab(a+b) \Leftrightarrow ab(a+b) \Leftrightarrow (a+b)(a-b)^2 \ge 0;$

(3) is $a^4+b^4+c^4 \ge a^2b^2+b^2c^2+c^2a^2 \Leftrightarrow \sum_{cyc}(a^2-b^2)^2 \ge 0;$

124. Let a,b,c be positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{2}{3}\left[\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}\right] \ge 2.$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo:** a,b,c **números** R^+**. Probar que:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{2}{3}\left[\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}\right] \ge 2 \ \ldots \ (A)$$

Desde que:** $a,b,c > 0$**. Por: MA** $\ge$ **MG

$$\frac{2}{3}\left[\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}\right] \ge \frac{2}{3} \times \sqrt[3]{\frac{\prod ab\prod(a+b)}{\prod(a+b)^3}} \ge \frac{2\sqrt[3]{8(abc)^3}}{\prod(a+b)} = \frac{4abc}{\prod(a+b)}$$

Por consiguiente, tenemos en ... (A)

$$\sum\frac{a}{b+c}+\frac{2}{3}\sum\frac{ab}{(a+b)^2}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\geq 2$$

Es suficiente demostrar:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\geq 2$$

$$\Leftrightarrow a(a+b)(a+c)+b(b+a)(b+c)+c(c+a)(c+b)+4abc\geq$$

$$\geq 2(a+b)(b+c)(c+a)$$

$$\Leftrightarrow a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)+7abc\geq$$

$$\geq 2ab(a+b)+2bc(b+c)+2ca(c+a)+4abc$$

$$\Leftrightarrow a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)\geq 0$$

$\Leftrightarrow a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b)\geq 0$ ... ***(Válido por desigualdad de Schur)***

Por transtividad:

$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{2}{3}\left[\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}\right]\geq 2$ ... ***(LQQD)***

125. In $\Delta\ ABC$ the following relationship holds:

$$\sqrt{\sqrt{\frac{a}{c}}+\sqrt{\frac{b}{c}}}+\sqrt{\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{a}}}+\sqrt{\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{b}}}\geq\frac{6r\sqrt{2}}{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS\overset{A-G}{\underset{(1)}{\geq}}3\sqrt[3]{\sqrt{\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}}\sqrt{\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}}\sqrt{\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}}}=3\sqrt[3]{\sqrt{\frac{(\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})}{\sqrt{abc}}}}=$$

$$= 3\sqrt[3]{\sqrt{\frac{(x+y)(y+z)(z+x)}{xyz}}} \text{ where } x = \sqrt{a} \text{ etc}$$

$$\text{Now, } (x+y)(y+z)(z+x) \overset{A-G}{\geq} 2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{zx}$$

$$\Rightarrow \frac{(x+y)(y+z)(z+x)}{xyz} \geq 8 \quad (2)$$

$$\therefore LHS \geq 3\sqrt[3]{\sqrt{8}} \text{ (from (1) and (2))} = 3\sqrt[3]{2\sqrt{2}} = 3\sqrt[3]{\left(\sqrt{2}\right)^3} = 3\sqrt{2}$$

$$\therefore \text{it suffices to prove: } 3\sqrt{2} \geq \frac{6r\sqrt{2}}{R} \Leftrightarrow R \geq 2r \to \text{true (Euler) (Proved)}$$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$RHS = \frac{6r\sqrt{2}}{R} \overset{Euler}{\leq} 3\sqrt{2};\ \frac{a}{c} = x^2, \frac{b}{c} = y^2, \frac{b}{a} = z^2$$

$$\sqrt{x+y} + \sqrt{z+\frac{1}{x}} + \sqrt{\frac{1}{y}+\frac{1}{z}} \overset{A-G}{\geq} 3\sqrt[6]{(x+y)\left(z+\frac{1}{x}\right)\left(\frac{1}{y}+\frac{1}{z}\right)} \geq$$

$$\overset{A-G}{\geq} 3\sqrt[6]{2\sqrt{xy}\cdot\frac{2\sqrt{z}}{x}\cdot\frac{2}{\sqrt{yz}}} = 3\sqrt{2}$$

126. If $A \in M_2(\mathbb{R})$ then:

$$det\left(A^5 + I_2\right)^2 \leq 3\left((det\, A)^{10} + Tr^2\left(A^5\right) + 1\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

Let $pA(x) = x^2 - tr\, Ax + \det A$ **with own values** λ_1, λ_2

$$\begin{cases} x_1 + \lambda_2 = tr\, A \in \mathbb{R} \\ \lambda_1\lambda_2 = \det A \in \mathbb{R} \end{cases}$$

If $p(x) = x^5 + 1 \Rightarrow \det p(A) = \det(A^5 + I_2) = (\lambda_1^4 + 1)(\lambda_2^5 + 1) \Rightarrow$

inequality becomes:

$$\left((\lambda_1^5 + 1)(\lambda_2^5 + 1)\right)^2 \le 3\left((\lambda_1\lambda_2)^{10} + (\lambda_1^5 + \lambda_2^5)^2 + 1\right) \Leftrightarrow$$

$$\Leftrightarrow \left((\lambda_1^5 + 1)(\lambda_2^5 + 1)\right)^2 \le 3\left((\lambda_1^5\lambda_2^5)^2 + (\lambda_1^5 + \lambda_2^5)^2 + 1\right) \quad \textit{(1)}$$

Let $x_1^5 = x_1, x_2^5 = x_2$ ***inequality (1) becomes:***

$$\left((x_1 + 1)(x_2 + 1)\right)^2 \le 3((x_1x_2)^2 + (x_1 + x_2)^2 + 1)^2 \Leftrightarrow$$

$$(x_1x_2 + x_1 + x_2 + 1)^2 \le 3((x_1x_2)^2 + (x_1 + x_2)^2 + 1) \quad \textit{(2)}$$

Let $x_1 + x_2 = S, x_1x_2 = p, S, p \in \mathbb{R}$***. Inequality (2) becomes:***

$(p + S + 1)^2 \le 3(p^2 + S^2 + 1)$ ***true because is Cauchy inequality.***

127. Let be: $a, b, c > 0$. Prove that the following relationship holds:

$$\frac{abc}{7\sqrt{7}} \le \frac{(a^2 - ab + b^2)(b^2 - bc + b^2)(c^2 - ca + a^2)}{\sqrt{(a^2 + 5ab + b^2)(b^2 + 5bc + c^2)(c^2 + 5ca + a^2)}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: $a, b, c > 0$***. Probar la siguiente desigualdad:***

$$\frac{abc}{7\sqrt{7}} \le \frac{(a^2 - ab + b^2)(b^2 - bc + b^2)(c^2 - ca + a^2)}{\sqrt{(a^2 + 5ab + b^2)(b^2 + 5bc + c^2)(c^2 + 5ca + a^2)}}$$

Se puede observar claramente que:

$$a^2 - ab + b^2 = \frac{3}{4}(a - b)^2 + \frac{1}{4}(a + b)^2 \ge \frac{1}{4}(a + b)^2$$

$$a^2 + 5ab + b^2 = -\frac{3}{4}(a - b)^2 + \frac{7}{4}(a + b)^2 \le \frac{7}{4}(a + b)^2$$

Desde que: $a, b, c > 0 \rightarrow (a + b)(b + c)(c + a) \ge 8abc$

Por lo tanto la desigualdad es equivalente:

$$\frac{(a^2-ab+b^2)(b^2-bc+b^2)(c^2-ca+a^2)}{\sqrt{(a^2+5ab+b^2)(b^2+5bc+c^2)(c^2+5ca+a^2)}} \geq \frac{\left(\frac{1}{4}\right)^3(a+b)^2(b+c)^2(c+a)^2}{\left(\frac{\sqrt{7}}{2}\right)^3(a+b)(b+c)(c+a)} =$$

$$= \frac{1}{56\sqrt{7}}(a+b)(b+c)(c+a) \geq \frac{abc}{7\sqrt{7}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$a^2 - ab + b^2 = \frac{3}{4}(a-b)^2 + \frac{1}{4}(a+b)^2 \overset{(1)}{\geq} \frac{(a+b)^2}{4}$$

Similarly, $b^2 - bc + c^2 \overset{(2)}{\geq} \frac{(b+c)^2}{4}$***, and*** $c^2 - ca + a^2 \overset{(3)}{\geq} \frac{(c+a)^2}{4}$

(1) $\times$ ***(2)*** $\times$ ***(3)*** $\Rightarrow \prod(a^2 - ab + b^2) \geq \frac{(a+b)^2(b+c)^2(c+a)^2}{64}$

$$= \frac{\big((a+b)(b+c)(c+a)\big)\{(a+b)(b+c)(c+a)\}}{64}$$

$$\overset{AM-GM}{\geq} \frac{(8abc)}{64}(a+b)(b+c)(c+a) = \frac{(a+b)(b+c)(c+a)abc}{8}$$

$\therefore$ ***it suffices to prove:*** $\frac{\{\prod(a+b)\}abc}{8} \geq \frac{abc}{7\sqrt{7}}\prod\sqrt{a^2 + 5ab + b^2}$

$$\Leftrightarrow \prod\left\{\frac{\sqrt{7}(a+b)}{2}\right\} \geq \prod\sqrt{a^2 + 5ab + b^2} \quad \textit{(a)}$$

Now, $\frac{\sqrt{7}(a+b)}{2} \geq \sqrt{a^2 + 5ab + b^2} \Leftrightarrow 7(a^2 + b^2 + 2ab) \geq 4(a^2 + 5ab + b^2)$

$$\Leftrightarrow 3(a-b)^2 \geq 0 \rightarrow \textbf{\textit{true}} \Rightarrow \frac{\sqrt{7}(a+b)}{2} \overset{(4)}{\geq} \sqrt{a^2 + 5ab + b^2}$$

Similarly, $\frac{\sqrt{7}(b+c)}{2} \overset{(5)}{\geq} \sqrt{b^2 + 5bc + c^2}$ ***and*** $\frac{\sqrt{7}(c+a)}{2} \overset{(6)}{\geq} \sqrt{c^2 + 5ca + a^2}$

(4) $\times$ ***(5)*** $\times$ ***(6)*** $\Rightarrow \prod\left\{\frac{\sqrt{7}(a+b)}{2}\right\} \geq \prod(a^2 + 5ab + b^2)$

$\Rightarrow$ ***(a) is true***

(Proved)

Solution 3 by Soumitra Mandal-Chandar Nagore-India

We need to prove,

$$\prod_{cyc}\frac{a^2-ab+b^2}{\sqrt{a^2+5ab+b^2}}\ge\frac{abc}{7\sqrt{7}}$$

$$\Leftrightarrow 7^3\prod_{cyc}(a^2-ab+b^2)^2\ge(ab)^2\prod_{cyc}(a^2+5ab+b^2)$$

Now, $7(a^2-ab+b^2)^2-ab(a^2+5ab+b^2)$

$$=7(a^2+b^2)^2-15ab(a^2+b^2)+2(ab)^2$$

$=(a-b)^2(7a^2+7b^2-ab)\ge 0$, ***so*** $7(a^2-ab+b^2)^2\ge ab(a^2+5ab+b^2)$

Similarly, $7(b^2-bc+c^2)^2\ge bc(b^2+5bc+c^2)$ ***and***

$$7(c^2-ca+a^2)^2\ge ca(c^2+5ca+a^2)$$

$$\therefore 7^3\prod_{cyc}(a^2-ab+b^2)^2\ge(abc)^2\prod_{cyc}(a^2+5ab+b^2)$$

$$\therefore\frac{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)}{\sqrt{(a^2+5ab+b^2)(b^2+5bc+c^2)(c^2+5ca+a^2)}}\ge\frac{abc}{7\sqrt{7}}$$

(Proved)

Solution 4 by Ravi Prakash-New Delhi-India

For $a,b,>0$. ***Consider***

$$7(a^2-ab+b^2)^2-ab(a^2+5ab+b^2)$$

$$=7(a^2-ab+b^2)^2-ab(a^2-ab+b^2)-6a^2b^2$$

$$=7(a^2-ab+b^2)-7ab(a^2-ab+b^2)+6ab(a^2-ab+b^2)-6a^2b^2$$

$$=7(a^2-ab+b^2)(a^2-ab+b^2-ab)+6ab(a^2-ab+b^2-ab)$$

$$=(a-b)^2[7a^2-7ab+7b^2+6ab]$$

$$=(a-b)^2[(a^2-ab+b^2)+6(a^2+b^2)]\ge 0$$

$$\therefore \frac{a^2 - ab + b^2}{\sqrt{a^2 + 5ab + b^2}} \geq \frac{\sqrt{ab}}{\sqrt{7}}$$

Similarly $\frac{b^2-bc+c^2}{\sqrt{b^2+5bc+c^2}} \geq \frac{\sqrt{bc}}{\sqrt{7}}$ ***and*** $\frac{c^2-ca+a^2}{\sqrt{c^2+5ca+a^2}} \geq \frac{\sqrt{ca}}{\sqrt{7}}$

Multiplying above inequalities we get desired inequality.

Solution 5 by Kunihiko Chikaya-Tokyo-Japan

$$f(a, b) := \frac{a^2-ab+b^2}{\sqrt{a^2+5ab+b^2}} = \frac{(a+b)^2+3(a-b)^2}{2\sqrt{7(a+b)^2-3(a-b)^2}} \quad (*)$$

$$\geq \frac{a + b}{2\sqrt{7}}$$

when $a = b$ ***each numerator, denominator of (*) is minimal,***

maximal respectively.

$$f(a, b)f(b, c)f(c, a) \geq \frac{(a + b)(b + c)(c + a)}{8 \cdot 7\sqrt{7}} = \frac{abc}{7\sqrt{7}}$$

$$\because (a + b)(b + c)(c + a) \geq 8abc, a, b, c \in R^+$$

128. Prove that the following inequality holds for all positive real numbers a, b, c

$$a\sqrt{b^2 + 7bc + c^2} + b\sqrt{c^2 + 7ca + a^2} + c\sqrt{a^2 + 7ab + b^2} \geq 2(ab + bc + ca) + \frac{9abc}{a + b + c}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar la siguiente desigualdad para todos los numerous R^+ a, b, c***:***

$$a\sqrt{b^2 + 7bc + c^2} + b\sqrt{c^2 + 7ca + a^2} + c\sqrt{a^2 + 7ab + b^2} \geq 2(ab + bc + ca) + \frac{9abc}{a+b+c} \ldots (A)$$

1) Desde que: $a, b, c > 0$***, se cumple lo siguiente:***

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{9}{2(a+b+c)} \ldots \textit{(Desigualdad de Cauchy)}$$

Para ello demostraremos previamente:

$$\sqrt{b^2+7bc+c^2}\geq b+c+\frac{2bc}{b+c}\Rightarrow b^2+7bc+c^2\geq(b+c)^2+\frac{4b^2c^2}{(b+c)^2}+4bc$$

$$\sqrt{b^2+7bc+c^2}\geq b+c\Rightarrow bc\geq\frac{4b^2c^2}{(b+c)^2}\Leftrightarrow bc\frac{(b-c)^2}{(b+c)^2}\geq 0,\text{ ya que: } bc>0$$

Luego, desigualdad es equivalente en ... (A):

$$\sum a\sqrt{b^2+7bc+c^2}\geq\sum a(b+c)+\sum\frac{2abc}{b+c}\geq 2(ab+bc+ca)+\frac{9abc}{a+b+c}\ldots$$

(LQQD)

129. If $a,b,c\in(1,\infty)$ then:

$$\sum\frac{\ln(ab)}{\ln(abe)}\leq\sum\frac{\ln a^2}{\ln(ae)}$$

Proposed by Daniel Sitaru – Romania

Solution by Mihalcea Andrei Ștefan – Romania

$$\ln a=x>0;\ \ln b=y>0;\ \ln c=z>0$$

Inequality $\Leftrightarrow\sum\frac{x+y}{x+y+1}\leq 2\sum\frac{x}{x+1}$. ***We'll prove:*** $\frac{x+y}{x+y+1}\leq\frac{x}{x+1}+\frac{y}{y+1}$

But $\frac{x}{x+1}+\frac{y}{y+1}\overset{Bergström}{\geq}\frac{(x+y)^2}{x^2+y^2+x+y}\geq\frac{x+y}{x+y+1}$

$$\Leftrightarrow x^2+xy+x+xy+y^2+y\geq x^2+y^2+x+y\Leftrightarrow xy\geq 0\ \textit{true}$$

130. If $a,b,c>0,a^6+b^6+c^6=9$ then:

$$2\left(\frac{a+b}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2}+\frac{b+c}{\left(b^3\sqrt{c}+c^3\sqrt{b}\right)^2}+\frac{c+a}{\left(c^3\sqrt{a}+a^3\sqrt{c}\right)^2}\right)\geq 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si $a,b,c>0,a^6+b^6+c^6=9$***. Probar que:***

$$\frac{2(a+b)}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2}+\frac{2(b+c)}{\left(b^3\sqrt{c}+c^3\sqrt{b}\right)^2}+\frac{2(c+a)}{\left(c^3\sqrt{a}+a^3\sqrt{c}\right)^2}=1 \quad \textbf{(A)}$$

Previamente probaremos que:

$$\frac{a^6+b^6}{2}\geq\left(\frac{a^3\sqrt{b}+b^3\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)^2\Leftrightarrow\frac{a^6+b^6}{2}\geq\left(\frac{a^3+b^3}{2}\right)^2\geq\left(\frac{a^3\sqrt{b}+b^3\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)^2\Leftrightarrow$$

$$\Leftrightarrow (a^3+b^3)\left(\sqrt{a}+\sqrt{b}\right)\geq 2\left(a^3\sqrt{b}+b^3\sqrt{a}\right)$$

Lo que es equivalente:

$$(a+b)(a^2+b^2-ab)\left(\sqrt{a}+\sqrt{b}\right)\geq 2\sqrt{ab}\left(a^2\sqrt{a}+b^2\sqrt{b}\right)$$

Es suficiente probar que:

$$(a^2+b^2-ab)\left(\sqrt{a}+\sqrt{b}\right)\geq\left(a^2\sqrt{a}+b^2\sqrt{b}\right)\Leftrightarrow a^2\sqrt{b}+b^2\sqrt{a}\geq ab\left(\sqrt{a}+\sqrt{b}\right)$$

$\Leftrightarrow \frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}}\geq\sqrt{a}+\sqrt{b}$***, aplicando desigualdad de Cauchy***

$$\Leftrightarrow\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}}\geq\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{\sqrt{a}+\sqrt{b}}=\sqrt{a}+\sqrt{b}$$

Como: $a,b,c>0$***, aplicamos nuevamente la desigualdad de Cauchy en (A):***

$$\sum\frac{2(a+b)}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2}\geq\sum\left(\frac{\sqrt{a}+\sqrt{b}}{a^3\sqrt{b}+b^3\sqrt{a}}\right)^2\geq$$

$$\geq 2\sum\frac{1}{a^6+b^6}\geq\frac{9}{a^6+b^6+c^6}=1\ \ldots\ \textbf{(LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$a,b,c>0,\ a^6+b^6+c^6=9$$

$$\Rightarrow 2\left\{\frac{a+b}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2}+\frac{b+c}{\left(b^3\sqrt{c}+c^3\sqrt{b}\right)^2}+\frac{c+a}{\left(c^3\sqrt{a}+a^3\sqrt{c}\right)^2}\right\}\geq 1$$

$$\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2\leq 2(a^6b+b^6a)\left(\because (x+y)^2\leq 2(x^2+y^2)\right)$$

$$\Rightarrow \frac{a+b}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2} \ge \frac{a+b}{2ab(a^5+b^5)} \underset{(1)}{\ge} \frac{2}{(a+b)(a^5+b^5)}$$

(∵ $\frac{2ab}{a+b} \le \frac{a+b}{2}$*, by HM ≤ HM)*

$$\therefore\ 2\sum \frac{a+b}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2} \ge 4\sum \frac{1}{(a+b)(a^5+b^5)}$$ *(by (1))*

$$\underset{(2)}{\overset{Bergstrom}{\ge}} 4\frac{(1+1+1)^2}{\sum(a+b)(a^5+b^5)} = \frac{36}{\sum\{a^6+b^6+ab(a^4+b^4)\}}$$

$$= \frac{36}{2\sum a^6 + \sum ab\,(a^4+b^4)}$$

Now, $\sum\{ab(a^4+b^4)\} \overset{G\le A}{\le} \sum\left(\frac{a^2+b^2}{2}\right)(a^4+b^4)$

$$= \frac{1}{2}\left\{\sum(a^6+b^6) + \left(\sum a^4b^2 + \sum a^2b^4\right)\right\}$$

$$\overset{(3)}{=} \sum a^6 + \frac{1}{2}\left(\sum a^4b^2 + \sum a^2b^4\right)$$

Now,

$$\begin{Bmatrix} a^6+a^6+b^6 \overset{A-G}{\ge} 3a^4b^2, & b^6+b^6+a^6 \overset{A-G}{\ge} 3a^2b^4 \\ b^6+b^6+c^6 \overset{A-G}{\ge} 3b^4c^2, & c^6+c^6+b^6 \overset{A-G}{\ge} 3b^2c^4 \\ c^6+c^6+a^6 \overset{A-G}{\ge} 3c^4a^2, & a^6+a^6+c^6 \overset{A-G}{\ge} 3c^2a^4 \end{Bmatrix}$$

Adding, $6\sum a^6 \ge 3\sum a^4b^2 + 3\sum a^2b^4$

$$\Rightarrow \frac{\sum a^4b^2 + \sum a^2b^4}{2} \overset{(4)}{\ge} \frac{2\sum a^6}{2} = \sum a^6$$

(3), (4) $\Rightarrow \sum\{ab(a^4+b^4)\} \le 2\sum a^6$ *(5)*

(2), (5) $\Rightarrow LHS \ge \frac{36}{4\sum a^6} = \frac{9}{\sum a^6} = \frac{9}{9} = 1 = RHS$

(Proved)

Solution 3 by Soumitra Mandal-Chandar Nagore-India

***Let** $a,b,c>0$,*

$$\sum_{cyc} a^6 = 9$$

then

$$2\left(\sum_{cyc}\frac{a+b}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2}\right)\geq 1$$

$$2\left(\sum_{cyc}\frac{a+b}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2}\right)\overset{Cauchy-Schwarz}{\geq}2\left(\sum_{cyc}\frac{a+b}{(a+b)(a^6+b^6)}\right)$$

$$=2\sum_{cyc}\frac{1}{a^6+b^6}\geq 2\frac{9}{\sum_{cyc}(a^6+b^6)}\left[\because \frac{1}{3}\left(\sum_{cyc}\frac{1}{x}\right)\geq\frac{3}{x+y+z}\right]$$

$= 1$ ***(proved) equality at*** $a=b=c=\sqrt[6]{3}$

Solution 4 by Dhanh Tang Thung-Vietnam

Apply Cauchy – Schwarz inequality, we have:

$$\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2\leq (a^6+b^6)(b+a)$$

$$\Rightarrow 2LHS\geq 2\left(\frac{1}{a^6+b^6}+\frac{1}{b^6+c^6}+\frac{1}{c^6+a^6}\right)\geq\frac{18}{2(a^6+b^6+c^6)}=1$$

Equality when $a=b=c=\sqrt[6]{3}$***.***

Solution 5 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$2\cdot\sum_{cycl}\frac{a+b}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2}=$$

$$=2\cdot\sum\frac{a+b}{a^6b+2a^3b^3\sqrt{ab}+ab^6}\overset{Cauchy}{\geq}2a^3b^3\sqrt{ab}=2\sqrt{a^7b^7}\leq a^7+b^7$$

$$\ge 2\cdot\sum\frac{a+b}{a^6b+a\cdot b^6+a^7+b^7}=2\cdot\sum\frac{a+b}{a^6(a+b)+b^6(a+b)}=$$

$$=2\cdot\sum\frac{1}{a^6+b^6}\overset{Cauchy}{\ge}\frac{6}{\sqrt[3]{(a^6+b^6)(b^6+c^6)(c^6+a^6)}}\overset{Cauchy}{\ge}$$

$$\ge\frac{6}{2\cdot\frac{(a^6+b^6+c^6)}{3}}=\frac{18}{18}=1$$

131. If $a,b,c>0, a+b+c=1$ then:

$$5\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\le\sum\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)}\le 5$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Anas Adlany-El Zemmara-Morocco

We have by AM-GM inequality

$$\sum\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)}\le\sum\frac{(a+4b)+(2a+3b)+(3a+2b)+(4a+b)}{4}=5,$$

Also, by AM-GM inequality we have

$$\sum\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)}\ge 5\sum\sqrt[4]{\sqrt[5]{ab^4a^2b^2a^4b}}=5\sum\sqrt{ab}$$

and we are done.

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Si: $a,b,c>0$. Probar que:

$$5\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\le\sum\sqrt[4]{(a+4b)(3a+2b)(3b+2a)(4a+b)}\le 5$$

Siendo: $a,b,c>0$. Por: MA≥MG

$$(a+4b)(4a+b)(3a+2b)(2a+3b)=(4a^2+4b^2+17ab)(6a^2+6b^2+13ab)\ge$$

$$\ge(25ab)(25ab)=5^4(ab)^2$$

Por la tanto:

$$\sum \sqrt[4]{(a+4b)(3a+2b)(3b+2a)(4a+b)} \geq 4\sum \sqrt{ab}$$

Luego, una vez más aplicando: $MA \geq MG$

$$\sum \sqrt[4]{(a+4b)(3a+2b)(3b+2a)(4a+b)} \leq \sum \frac{(a+4b)+(3a+2b)+(3b+2a)+(4a+b)}{4} =$$

$$= \frac{10}{2}(a+b+c) = 5$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$(a+4b)(2a+3b)(3a+2b)(4a+b)$$

$$= \{(a+4b)(4a+b)\}\{(2a+3b)(3a+2b)\}$$

$$= (4a^2+4b^2+17ab)(6a^2+6b^2+13ab)$$

$$= (4a^2+4b^2-8ab+25ab)(6a^2+6b^2-12ab+25ab)$$

$$= \{4(a-b)^2+25ab\}\{6(a-b)^2+25ab\}$$

$$\geq 625a^2b^2 \Rightarrow \sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \geq 5\sqrt{ab}$$

$$\therefore \sum \sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \geq 5\left(\sum \sqrt{ab}\right)$$

Again, $\sum \sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \leq$

$$\overset{G-A}{\leq} \sum \frac{10a+10b}{4} = \sum \frac{5(a+b)}{2} = \frac{5}{2}\cdot 2\sum a = \sum a = 5$$

(Proved)

Solution 4 by Soumitra Mandal-Chandar Nagore-India

Let $a,b,c > 0, a+b+c = 1$ ***then***

$$5\sum_{cyc} \sqrt{ab} \leq \sum_{cyc} \sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \leq 5$$

$$\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \leq \frac{10(a+b)}{4} = \frac{5(a+b)}{2}$$

Similarly, $\sqrt[4]{(b+4c)(2b+3c)(3b+2c)(4b+c)} \leq \frac{5}{2}(b+c)$ ***and***

$$\sqrt[4]{(c+4a)(2c+3a)(3c+2a)(4c+a)} \le \frac{5(c+a)}{2}$$, so

$$\sum_{cyc} \sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \le 5(a+b+c) = 5$$

$\frac{2a+3b}{5} \overset{Weighted\ A.M. \ge G.M}{\ge} \sqrt[5]{a^2b^3}$. Similarly, $\frac{3a+2b}{5} \ge \sqrt[5]{a^3b^2}$, $\frac{a+4b}{5} \ge \sqrt[5]{ab^4}$ and $\frac{4a+b}{5} \ge \sqrt[5]{a^4b}$. So, $\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \ge 5\sqrt{ab}$

similarly, $\sqrt[4]{(b+4c)(2b+3c)(3b+2c)(4b+c)} \ge 4\sqrt{bc}$ and

$$\sqrt[4]{(c+4a)(2c+3a)(3c+2a)(4c+a)} \ge 5\sqrt{ca}$$

$$5\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right) \le \sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \le 5$$

(proved)

Solution 5 by Dang Thanh Tung-Vietnam

AM-GM:

$$a+4b \ge 5\sqrt[5]{ab^4}$$

$$2a+3b \ge 5\sqrt[5]{a^2b^3}$$

$$3a+2b \ge 5\sqrt[5]{a^3b^2}$$

$$4a+b \ge 5\sqrt[5]{a^4b}$$

$$\Rightarrow (a+4b)(2a+3b)(3a+2b)(4a+b) \ge 5^4a^2b^2$$

$$\Rightarrow \sum \sqrt{(a+4b)(2a+3b)(3a+2b)(4a+b)} \ge 5\sum \sqrt[5]{ab}$$

AM-GM:

$$\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \le \frac{10(a+b)}{4} = \frac{5}{2}(a+b)$$

$$\Rightarrow \sum \sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)} \le 5(a+b+c) = 5$$

$$\Rightarrow 5(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})\le\sum\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)}\le 5$$

Equality when $a=b=c=\frac{1}{3}$.

Solution 6 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\textbf{1)}\ \sum\sqrt{\sqrt{\left(\left(2\sqrt{a}\right)^2+\left(\sqrt{b}\right)^2\right)\cdot\left(\left(2\sqrt{b}\right)^2+\left(\sqrt{a}\right)^2\right)}}\cdot$$

$$\cdot\sqrt{\sqrt{\left(\left(\sqrt{2a}\right)^2+\left(\sqrt{3b}\right)^2\right)\cdot\left(\left(\sqrt{2b}\right)^2+\left(\sqrt{3a}\right)^2\right)}}\ge$$

$$\ge\sum\sqrt{4\sqrt{ab}+\sqrt{ab}}\cdot\sqrt{2\sqrt{ab}+3\sqrt{ab}}=\sum\sqrt{5\sqrt{ab}}\cdot\sqrt{5\sqrt{ab}}=$$

$$=\sum 5\cdot\sqrt{ab}=5\cdot(ab+bc+ca)$$

$$\textbf{2)}\ \sum\sqrt{\sqrt{(4a+b)(a+4b)}}\cdot\sqrt{\sqrt{(3a+2b)(2a+3b)}}\le$$

$$\le\sum\sqrt{\frac{5(a+b)}{2}}\cdot\sqrt{\frac{5(a+b)}{2}}=\sum\frac{5(a+b)}{2}=5$$

132. If $a,b,c\in(0;+\infty)$ and $k\in\mathbb{R}$, prove that.

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{(k-1)^2}{2}\ge\frac{(k^2+2k+13)(a^2+b^2+c^2)}{2(a+b+c)^2}$$

Proposed by Le Khanh Sy-Long An-Vietnam

Solution by Le Khanh Sy-Long An-Vietnam

The inequality becomes as follows.

$$\Leftrightarrow (a+b+c)^2\sum_{cyc}\frac{a}{b}+(k-1)^2\sum_{cyc}ab\ge 2(k+3)\sum_{cyc}a^2$$

$$\Leftrightarrow \sum_{cyc}\left[\frac{a^3}{b}+\frac{a^2b}{c}+ac+2a^2+\frac{2ab^2}{c}+2cb\right]+(k-1)^2\sum_{cyc}ab \geq 2(k+3)\sum_{cyc}a^2$$

$$\Leftrightarrow \sum_{cyc}\left[\frac{a^3}{b}+\frac{a^2b}{c}+\frac{2ab^2}{c}\right]+\left(k^2-2k+4\right)\sum_{cyc}ab \geq 2(k+2)\sum_{cyc}a^2$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{a^3}{b}+\frac{a^2b}{c}-\frac{2ab^2}{c}\right)+\sum_{cyc}\left[\frac{4a^2c}{b}-(4k-k^2)ab\right] \geq 2(k+2)\left[\sum_{cyc}a^2-\sum_{cyc}ab\right]$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{b^3}{c}+\frac{a^2b}{c}-\frac{2ab^2}{c}\right)+\sum_{cyc}\left(\frac{4a^2c}{b}-4kac+k^2cb\right) \geq 2(k+2)\left[\sum_{cyc}a^2-\sum_{cyc}ab\right]$$

Using the AM-GM inequality, we have.

$$\sum_{cyc}\left[\frac{b(a-b)^2}{c}+\frac{c(2a-kb)^2}{b}\right] \geq 2\sum_{cyc}[(a-b)(2a-kb)]$$

$$= 2(k+2)\left[\sum_{cyc}a^2-\sum_{cyc}ab\right]$$

We are done.

133. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$a\sqrt{4a+4b+c}+b\sqrt{4b+4c+a}+c\sqrt{4c+4a+b} \geq 9$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c números R^+ de tal manera que $a+b+c=3$. Probar la siguiente desigualdad

$$a\sqrt{4a+4b+c}+b\sqrt{4b+4c+a}+c\sqrt{4c+4a+b} \geq 9$$

Como $a, b, c > 0$, por la desigualdad de Holder

$$\left(a\sqrt{4a+4b+c}+b\sqrt{4b+4c+a}+c\sqrt{4c+4a+b}\right)^2\left(\frac{a}{4a+4b+c}+\frac{b}{4b+4c+a}+\frac{c}{4c+4a+b}\right)\geq$$

$$\geq (a+b+c)^3 \quad \textit{(M)}$$

Es suficiente probar que:

$$N=\frac{a}{4a+4b+c}+\frac{b}{4b+4c+a}+\frac{c}{4c+4a+b}\leq\frac{1}{a+b+c}$$

$$\frac{4a(a+b+c)}{4a+4b+c}+\frac{4b(b+c+a)}{4b+4c+a}+\frac{4c(c+a+b)}{4c+4a+b}\leq 4$$

$$\left(a+\frac{3ca}{4a+4b+c}\right)+\left(b+\frac{3ab}{4b+4c+a}\right)+\left(c+\frac{3bc}{4c+4a+b}\right)\leq 4$$

$$\frac{3ca}{4a+4b+c}+\frac{3ab}{4b+4c+a}+\frac{3bc}{4c+4a+b}\leq 1\Leftrightarrow$$

$$\Leftrightarrow\frac{9ca}{4a+4b+c}+\frac{9ab}{4b+4c+a}+\frac{9bc}{4c+4a+b}\leq 3$$

Partiendo de la desigualdad de Cauchy:

$$\frac{9ca}{4a+4b+c}=\frac{(2+1)^2ca}{2(2a+b)+2b+c}\leq\frac{2ca}{2a+b}+\frac{ca}{2b+c} \quad \textit{(A)}$$

Análogamente para los siguientes términos:

$$\frac{9ab}{4b+4c+a}=\frac{(2+1)^2ab}{2(2b+c)+2c+a}\leq\frac{2ab}{2b+c}+\frac{ab}{2c+a} \quad \textit{(B)}$$

$$\frac{9bc}{4c+4a+b}\leq\frac{(2+1)^2bc}{2(2c+a)+2a+b}\leq\frac{2bc}{2c+a}+\frac{bc}{2a+b} \quad \textit{(C)}$$

Sumando (A) + (B) + (C) se obtiene lo pedido

$$\frac{9ca}{4a+4b+c}+\frac{9ab}{4b+4c+a}+\frac{9bc}{4c+4a+b}\leq\frac{c(2a+b)}{2a+b}+\frac{a(2b+c)}{2b+c}+\frac{b(2c+a)}{2a+b}$$

$$\frac{9ca}{4a+4b+c}+\frac{9ab}{4b+4c+a}+\frac{9bc}{4c+4a+b}\leq a+b+c=3 \quad \textit{(LQQD)}$$

Luego, por transitividad tenemos en (M)

$$\left(a\sqrt{4a+4b+c}+b\sqrt{4b+4c+a}+c\sqrt{4c+4a+b}\right)^2\geq\frac{(a+b+c)^3}{N}\geq$$

$$\geq (a+b+c)^4 = 81$$

$$\Rightarrow a\sqrt{4a+4b+c} + b\sqrt{4b+4c+a} + c\sqrt{4c+4a+b} \geq 9$$

134. Prove that for all positive real numbers a, b, c the inequality holds:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{b+c}{2a+b+c} + \frac{c+a}{2b+c+a} + \frac{a+b}{2c+a+b}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los numerous R^+ la siguiente desigualdad:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{b+c}{2a+b+c} + \frac{c+a}{2b+c+a} + \frac{a+b}{2c+a+b}$$

$$\left(\frac{a}{b+c} + 1\right) + \left(\frac{b}{c+a} + 1\right) + \left(\frac{c}{a+b} + 1\right) \geq$$

$$\geq \left(\frac{b+c}{2a+b+c} + 1\right) + \left(\frac{c+a}{2b+c+a} + 1\right) + \left(\frac{a+b}{2c+a+b} + 1\right)$$

$$(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq$$

$$\geq (a+b+c)\left(\frac{2}{2a+b+c} + \frac{2}{2b+c+a} + \frac{2}{2c+a+b}\right)$$

$$\Leftrightarrow \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq$$

$\geq \frac{2}{2a+b+c} + \frac{2}{2b+c+a} + \frac{2}{2c+a+b}$ ***(lo cual demostraremos)***

Como $a, b, c > 0$, pro la desigualdad de Cauchy:

$\frac{1}{a+b} + \frac{1}{c+a} \geq \frac{4}{2a+b+c}$ ***(A),***

$\frac{1}{b+c} + \frac{1}{b+a} \geq \frac{4}{2b+c+a}$ ***(B),***

$\frac{1}{c+a} + \frac{1}{c+b} \geq \frac{4}{2c+a+b}$ ***(C).***

Sumando (A) + (B) + (C) se obtiene

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\ge\frac{2}{2a+b+c}+\frac{2}{2b+c+a}+\frac{2}{2c+a+b}\ \textit{(LQQD)}$$

135. Prove the inequality holds for all positive real numbers a, b, c

$$\sum_{cyc}\frac{b^2+c^2}{a^2+1}+\sum_{cyc}\frac{2}{ab+1}\ge 6$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar la desigualdad para todos los numerous R^+ a, b, c

$$\sum\frac{b^2+c^2}{a^2+1}+\sum\frac{2}{ab+1}\ge 6$$

$$\Leftrightarrow\frac{b^2+c^2}{a^2+1}+\frac{c^2+a^2}{b^2+1}+\frac{a^2+b^2}{c^2+1}+\sum\frac{2}{ab+1}+\sum\frac{2ab}{ab+1}\ge 6+\sum\frac{2ab}{ab+1}$$

$$\Leftrightarrow\left(\frac{b^2}{a^2+1}+\frac{a^2}{b^2+1}\right)+\left(\frac{c^2}{b^2+1}+\frac{b^2}{c^2+1}\right)+\left(\frac{a^2}{c^2+1}+\frac{c^2}{a^2+1}\right)\ge$$

$$\ge\frac{2ab}{ab+1}+\frac{2bc}{bc+1}+\frac{2ca}{ca+1}$$

Es suficiente domostrar lo siguiente:

$$\frac{b^2}{a^2+1}+\frac{a^2}{b^2+1}\ge\frac{2ab}{ab+1}$$

$$\Leftrightarrow\left((b^4+a^4)+(a^2+b^2)\right)[ab+1]\ge 2ab(a^2+1)(b^2+1)$$

$$\Leftrightarrow\left((a+b)^2(a-b)^2+(a-b)^2+2ab(ab+1)\right)[ab+1]\ge 2ab(a^2+1)(b^2+1)$$

$$\Leftrightarrow(a-b)^2(a+b)^2(ab+1)+(a-b)^2(ab+1)+2ab\left[(ab+1)^2-(a^2+1)(b^2+1)\right]\ge 0$$

$$\Leftrightarrow(a-b)^2(a+b)^2(ab+1)+(a-b)^2(ab+1)-2ab(a-b)^2\ge 0$$

$$\Leftrightarrow(a-b)^2[(a+b)^2(ab+1)+(ab+1)-2ab]=$$

$$= (a-b)^2(a^3b + b^3a + 2a^2b^2 + a^2 + b^2 + ab + 1) \geq 0$$

Por lo tanto:

$$\left(\frac{b^2}{a^2+1} + \frac{a^2}{b^2+1}\right) + \left(\frac{c^2}{b^2+1} + \frac{b^2}{c^2+1}\right) + \left(\frac{a^2}{c^2+1} + \frac{c^2}{a^2+1}\right) \geq$$

$$\geq \frac{2ab}{ab+1} + \frac{2bc}{bc+1} + \frac{2ca}{ca+1} \quad \textit{(LQQD)}$$

136. If $a, b, c > 0$ then:

$$\sum \frac{a^5}{(2a+3b)^3} + \sum \frac{a^5}{(2a+3c)^3} \geq \frac{2(a^2+b^2+c^2)}{125}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} \frac{a^5}{(2a+3b)^3} + \sum_{cyc} \frac{a^5}{(2a+3c)^3} \geq \frac{2}{125}(a^2+b^2+c^2)$$

$$\sum_{cyc} \frac{a^5}{(2a+3b)^3} + \sum_{cyc} \frac{a^5}{(2a+3c)^3} = \sum_{cyc} \frac{a^8}{(2a^2+3ab)^3} + \sum_{cyc} \frac{a^8}{(2a^2+3ac)^3}$$

$$\overset{RADON'S\ INEQUALITY}{\geq} \frac{(a^2+b^2+c^2)^4}{(2a^2+2b^2+2c^2+3ab+3bc+3ca)^3}$$

$$+ \frac{(a^2+b^2+c^2)^4}{(2a^2+2b^2+2c^2+3ab+3bc+3ca)^3}$$

$$\geq \frac{2(a^2+b^2+c^2)^4}{125(a^2+b^2+c^2)^3} = \frac{2}{125}(a^2+b^2+c^2)$$

(Proved)

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Si:** $a, b, c > 0$**, probar que

$$\sum\frac{a^5}{(2a+3b)^3}+\sum\frac{a^5}{(2a+3c)^3}\geq\frac{2(a^2+b^2+c^2)}{125}$$

La desigualdad es equivalente:

$$\left(\frac{a^5}{(2a+3b)^3}+\frac{b^5}{(2b+3a)^3}\right)+\left(\frac{b^5}{(2b+3c)^3}+\frac{c^5}{(2c+3b)^3}\right)+$$

$$+\left(\frac{c^5}{(2c+3a)^3}+\frac{a^5}{(2a+3c)^3}\right)\geq\frac{2(a^2+b^2+c^2)}{125}\quad \textbf{(A)}$$

Como $a,b,c>0$, por la desigualdad de Holder:

$$\left(\frac{a^5}{(2a+3b)^3}+\frac{b^5}{(2b+3a)^3}\right)\big(a(2a+3b)+b(2b+3a)\big)\cdot$$

$$\cdot\big(a(2a+3b)+b(2b+3a)\big)\big(a(2a+3b)+b(2a+3b)\big)\geq(a^2+b^2)^4$$

Ahora bien

$$a(2a+3b)+b(2b+3a)=2(a^2+b^2)+6ab\leq5(a^2+b^2)$$

Por lo tanto:

$$\left(\frac{a^5}{(2a+3b)^3}+\frac{b^5}{(2b+3a)^3}\right)\geq\frac{(a^2+b^2)^4}{\big(a(2a+3b)+b(2b+3a)\big)^3}\geq$$

$$\geq\frac{(a^2+b^2)^4}{125(a^2+b^2)^3}=\frac{a^2+b^2}{125}$$

Finalmente aplicando para (A) se obtiene

$$\left(\frac{a^5}{(2a+3b)^3}+\frac{b^5}{(2b+3a)^3}\right)+\left(\frac{b^5}{(2b+3c)^3}+\frac{c^5}{(2c+3b)^3}\right)+$$

$$+\left(\frac{c^5}{(2c+3a)^3}+\frac{a^5}{(2a+3c)^3}\right)\geq\frac{a^2+b^2}{125}+\frac{b^2+c^2}{125}+\frac{c^2+a^2}{125}=$$

$$=\frac{2(a^2+b^2+c^2)}{125}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$a,b,c>0\Rightarrow\sum\frac{a^5}{(2a+3b)^3}+\sum\frac{a^5}{(2a+3c)^3}\underset{(1)}{\geq}\frac{2\sum a^2}{125}$$

$$\textbf{(1)}\Leftrightarrow\left[\frac{a^5}{(2a+3b)^3}+\frac{b^5}{(2b+3a)^3}\right]+\left[\frac{b^5}{(2b+3c)^3}+\frac{c^5}{(2c+3b)^3}\right]+\left[\frac{c^5}{(2c+3a)^3}+\frac{a^5}{(2a+3c)^3}\right]$$

$$\textbf{(i)}\leftarrow\geq\frac{a^2+b^2}{125}+\frac{b^2+c^2}{125}+\frac{c^2+a^2}{125}$$

$$\frac{a^5}{(2a+3b)^3}+\frac{b^5}{(2b+3a)^3}=a^2\left(\frac{a}{2a+3b}\right)^3+b^2\left(\frac{b}{2b+3a}\right)^3$$

WLOG, we may assume $a\geq b\geq c\quad\therefore\frac{a}{2a+3b}\geq\frac{b}{2b+3a}$

$$\therefore a^2\left(\frac{a}{2a+3b}\right)^3+b^2\left(\frac{b}{2b+3a}\right)^3\overset{Chebyshev}{\geq}\frac{1}{2}(a^2+b^2)\left[\left(\frac{a}{2a+3b}\right)^3+\left(\frac{b}{2b+3a}\right)^3\right]$$

$$\underset{(a)}{\overset{Chebyshev}{\geq}}\frac{1}{8}(a^2+b^2)\left(\frac{a}{2a+3b}+\frac{b}{2b+3a}\right)^3$$

$$\frac{a}{2a+3b}+\frac{b}{2b+3a}=\frac{3a^2+3b^2+4ab}{(2a+3b)(2b+3a)}$$

$$=\frac{2(a^2+ab+b^2)+(a+b)^2}{\left\{\sqrt{(2a+3b)(2b+3a)}\right\}^2}\overset{G\leq A}{\geq}\frac{2(a^2+ab+b^2)+(a+b)^2}{(a+b)^2}\cdot\frac{4}{25}\geq$$

$$\geq\frac{2\cdot\frac{3}{4}(a+b)^2+(a+b)^2}{(a+b)^2}\cdot\frac{4}{25}\left(\because a^2+ab+b^2=\frac{3}{4}(a+b)^2+\frac{1}{4}(a-b)^2\right)$$

$$=\frac{5}{2}\cdot\frac{4}{25}=\frac{2}{5}\Rightarrow\left(\frac{a}{2a+3b}+\frac{b}{2b+3a}\right)^3\geq\frac{8}{125}$$

$$\Rightarrow\frac{1}{8}(a^2+b^2)\left(\frac{a}{2a+3b}+\frac{b}{2b+3a}\right)^3\geq\frac{a^2+b^2}{125}\ \textbf{(b)}$$

$$\textbf{(a), (b)}\Rightarrow\frac{a^5}{(2a+3b)^3}+\frac{b^5}{(2b+3a)^3}\overset{(c)}{\geq}\frac{a^2+b^2}{125}$$

Similarly, $\frac{b^5}{(2b+3c)^3}+\frac{c^5}{(2c+3b)^5}\overset{(d)}{\geq}\frac{b^2+c^2}{125}$ ***and,***

$$\frac{c^5}{(2c+3a)^3}+\frac{a^5}{(2a+3c)^3}\overset{(e)}{\geq}\frac{c^2+a^2}{125}$$

(c) + (d) + (e) ⇒ (i) is true (Proved)

137. If $a,b,c>0, a\neq b\neq c\neq a$ then:

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}>\frac{81}{4(a^2+b^2+c^2)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si $a,b,c>0, a\neq b\neq c$, probar

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}>\frac{81}{4(a^2+b^2+c^2)}$$

Como: $a-b\neq 0, b-c\neq 0, c-a\neq 0 \wedge a,b,c>0$

Solo basta demostrar lo siguiente:

$$\frac{1}{a^2+b^2-2ab}+\frac{1}{ab}>\frac{9}{2(a^2+b^2)}\Leftrightarrow\frac{a^2+b^2}{a^2+b^2-2ab}+\frac{a^2+b^2}{ab}>\frac{9}{2}$$

$$\Leftrightarrow 1+\frac{a^2+b^2}{ab}>\frac{9}{2}-\frac{ab}{a^2+b^2-2ab}\Leftrightarrow\frac{a^2+b^2}{ab}+\frac{2ab}{a^2+b^2-2ab}>\frac{7}{2}$$

$$\Leftrightarrow 2(a^2+b^2)^2-4ab(a^2+b^2)+4a^2b^2>7ab(a^2+b^2)-14a^2b^2$$

$$\Leftrightarrow 2(a^2+b^2)^2-11ab(a^2+b^2)+18a^2b^2=2\left((a^2+b^2)-3ab\right)+ab(a^2+b^2)>0$$

Luego por desigualdad de Cauchy:

$$\sum\frac{1}{(a-b)^2}+\sum\frac{1}{ab}>\sum\frac{9}{2(a^2+b^2)}\geq\frac{81}{4(a^2+b^2+c^2)}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{16}{a^2+b^2+c^2} > \frac{9}{a^2+b^2+c^2} \quad \textbf{\textit{(TRUE)}}$$

$$\frac{4\cdot 9}{a^2+b^2+c^2} > \frac{81}{4\cdot(a^2+b^2+c^2)}$$

$$\frac{4\cdot 9}{a^2+b^2+c^2} = 4\cdot\frac{27}{3(a^2+b^2+c^2)} =$$

$$= 4\cdot\frac{27}{3\cdot(a^2+b^2+c^2)} \le 4\cdot\frac{27}{2\cdot(a^2+b^2+c^2)+ab+bc+ca} \le$$

$$\le 4\cdot\left(\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\right) =$$

$$= \frac{4}{a^2+ab+b^2}+\frac{4}{b^2+bc+c^2}+\frac{4}{c^2+ca+a^2} <$$

$$< \left(\frac{4}{a^2-ab+b^2}\right)+\left(\frac{4}{b^2-bc+c^2}\right)+\left(\frac{4}{c^2-ca+a^2}\right) \le$$

$$\le \left(\frac{1}{(a-b)^2}+\frac{1^2}{ab}\right)+\left(\frac{1^2}{(b-c)^2}+\frac{1^2}{bc}\right)+\left(\frac{1^2}{(c-a)^2}+\frac{1^2}{ca}\right) =$$

$$= \frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS = \frac{1^3}{|a-b|^2}+\frac{1^3}{|b-c|^2}+\frac{1^3}{|c-a|^2}+\frac{1^3}{\left(\sqrt{ab}\right)^2}+\frac{1^3}{\left(\sqrt{bc}\right)^2}+\frac{1^3}{\left(\sqrt{ca}\right)^2}$$

$$\underset{(1)}{\overset{Radon}{>}} \frac{(1+1+1+1+1+1)^3}{\left(\sum|a-b|+\sum\sqrt{ab}\right)^2} = \frac{216}{\left(\sum|a-b|+\sum\sqrt{ab}\right)^2}$$

$$\left(\sum|a-b|\right)^2 < 3\left(\sum|a-b|^2\right)\left(\because \left(\sum x\right)^2 < 3\sum x^2\right)$$

$$\Rightarrow \sum|a-b| \overset{(a)}{<} \sqrt{3}\sqrt{2\sum a^2 - 2\sum ab} = \sqrt{3}\sqrt{2u-2v},$$

where $u = \sum a^2$, $v = \sum ab$

Again, $\left(\sum \sqrt{ab}\right)^2 < 3\left(\sum ab\right) = 3v \Rightarrow \sum \sqrt{ab} < \sqrt{3}\sqrt{v}$ ***(b)***

(a) + (b) $\Rightarrow \sum|a-b| + \sum\sqrt{ab} < \sqrt{3}\left(\sqrt{2u-2v} + \sqrt{v}\right)$

$$\Rightarrow \left(\sum|a-b| + \sum\sqrt{ab}\right)^2 < 3\left(2u - v + 2\sqrt{2uv - 2v^2}\right)$$

$$\Rightarrow \frac{216}{\left(\sum|a-b|+\sum\sqrt{ab}\right)^2} > \frac{72}{2u-v+2\sqrt{2uv-2v^2}} \quad \textbf{(2)}$$

(1), (2) $\Rightarrow LHS > \frac{72}{2u-v+2\sqrt{2uv-2v^2}}$

$\therefore$ ***it suffices to prove:*** $\frac{72}{2u-v+2\sqrt{2uv-2v^2}} > \frac{81}{4u}$

$$\Leftrightarrow 32u > 18u - 9v + 18\sqrt{2uv - 2v^2}$$

$$\Leftrightarrow (14u + 9v)^2 > 324(2uv - 2v^2) \Leftrightarrow 196u^2 + 729v^2 - 396uv > 0$$

$$\Leftrightarrow 196u^2 + 729v^2 - 2(14u)(27v) + 360uv > 0$$

$$\Leftrightarrow (14u - 27v)^2 + 360uv > 0 \rightarrow \textbf{true}$$

$$\left(\because u = \sum a^2 > 0, v = \sum ab > 0 \text{ as } a, b, c > 0\right)$$

(Proved)

Solution 4 by Soumitra Mandal-Chandar Nagore-India

Let $a, b, c > 0$ ***and*** $a \neq b \neq c \neq a$ ***then***

$$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{81}{4(a^2+b^2+c^2)}$$

$$\sum_{cyc}\frac{1}{(a-b)^2} + \sum_{cyc}\frac{1}{ab}$$

$$= \sum_{cyc}\left(\frac{1}{(a-b)^2} + \frac{4}{4ab}\right) \overset{BERGSTORM\ INEQUALITY}{\geq} \sum_{cyc}\frac{(1+2)^2}{(a-b)^2 + 4ab}$$

[equality doesn't hold $\because a \neq b \neq c \neq a$***]***

$$= \sum_{cyc} \frac{9}{(a+b)^2} > \frac{9}{2} \sum_{cyc} \frac{1}{a^2+b^2} \left[\because 2(a^2+b^2) > (a+b)^2 \text{ and } a \neq b\right]$$

$$> \frac{81}{4(a^2+b^2+c^2)} \textit{ (proved)}$$

$$\left[\because \frac{1}{3} \sum_{cyc} \frac{1}{a^2+b^2} > \frac{3}{\sum_{cyc}(a^2+b^2)} \text{ and } a \neq b \neq c \neq a\right]$$

138. **If** $a, b, c \in \mathbb{R}, a \neq b \neq c \neq a$

$\omega = \min(|a+b|, |b+c|, |c+a|), \Omega = \max(|a|, |b|, |c|)$ **then:**

$$\omega < \frac{1}{3}\left(\frac{a|a| - b|b|}{a-b} + \frac{b|b| - c|c|}{b-c} + \frac{c|c| - a|a|}{c-a}\right) < 2\Omega$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\omega < \underbrace{\frac{1}{3}\left(\frac{a|a| - b|b|}{a-b} + \frac{b|b| - c|c|}{b-c} + \frac{c|c| - a|a|}{c-a}\right)}_{E} < 2\Omega$$

WLOG, we may assume $\omega = |a+b|$ ***and*** $\Omega = |a|$

We shall first prove: $\frac{a|a|-b|b|}{a-b} \overset{(1)}{\geq} |a+b|, \forall a, b \in \mathbb{R}$

Case 1: $a \geq 0, b \geq 0$

(1) $\Leftrightarrow \frac{a^2-b^2}{a-b} \geq a+b \Leftrightarrow a+b \geq a+b \to$ ***true***

Case 2: $a \geq 0, b < 0$

(1) $\Leftrightarrow \frac{a^2+b^2}{a-b} \geq |a+b| \Leftrightarrow a^2+b^2 \geq (a-b)|a+b| (\because a > b)$

$$\Leftrightarrow (a^2+b^2)^2 \geq (a-b)^2(a+b)^2 = (a^2-b^2)^2$$

$\Leftrightarrow a^4 + b^4 + 2a^2b^2 \geq a^4 + b^4 - 2a^2b^2 \Leftrightarrow a^2b^2 \geq 0 \to$ ***true***

Case (3) $a < 0, b \geq 0$

(1) $\Leftrightarrow \frac{-a^2-b^2}{a-b} \geq |a+b| \Leftrightarrow \frac{a^2+b^2}{b-a} \geq |a+b|$

$\Leftrightarrow a^2 + b^2 \geq (b-a)|a+b|(\because b > a)$

$\Leftrightarrow a^4 + b^4 + 2a^2b^2 \geq (b^2-a^2)^2 = a^4 + b^4 - 2a^2b^2 \Leftrightarrow a^2b^2 \geq 0 \to$ ***true***

Case 4: $a < 0, b < 0$

(1) $\Leftrightarrow \frac{-(a^2-b^2)}{a-b} \geq -a-b(\because a+b<0) \Leftrightarrow -a-b \geq -a-b \to$ ***true***

Combining the 4 cases, $\forall a, b \in \mathbb{R}, \frac{a|a|-b|b|}{a-b} \geq |a+b|$ ***(1)***

Similarly, $\frac{b|b|-c|c|}{b-c} \geq |b+c| \overset{(2)}{\overbrace{>}} |a+b| \; (\because \omega = |a+b|)$,

$$\frac{c|c|-a|a|}{c-a} \geq |c+a| \overset{(3)}{\overbrace{>}} |a+b|(\because \omega = |a+b|)$$

(1) + (2) + (3) $\Rightarrow E > 3|a+b| = 3\omega$

$\Rightarrow \frac{E}{3} > \omega \Rightarrow$ ***left inequality is true***

We shall now prove: $\frac{a|a|-b|b|}{a-b} \overset{(i)}{\overbrace{\leq}} |a| + |b|, \forall a, b \in \mathbb{R}$

Case 1: $a \geq 0, b \geq 0$

(i) $\Leftrightarrow a + b \leq a + b \to$ ***true***

Case 2: $a \geq 0, b < 0$

(i) $\Leftrightarrow \frac{a^2+b^2}{a-b} \leq a - b \Leftrightarrow a^2 + b^2 \leq (a-b)^2(\because a > b)$

$\Leftrightarrow ab \leq 0 \to$ ***true*** $\because a \geq 0, b < 0$

Case 3: $a < 0, b \geq 0$

(i) $\Leftrightarrow \frac{-a^2-b^2}{a-b} \leq -a + b \Leftrightarrow \frac{a^2+b^2}{b-a} \leq b - a$

$$\Leftrightarrow a^2 + b^2 \leq (b-a)^2 (\because b > 0) \Leftrightarrow ab \leq 0 \rightarrow \textit{true}$$

$$\because a < 0, b \geq 0$$

Case 4: $a < 0, b < 0$

$$\textit{(i)} \Leftrightarrow -a - b \leq -a - b \rightarrow \textit{true}$$

Combining above 4 cases, $\frac{a|a|-b|b|}{a-b} \leq |a| + |b|$ ***(i)*** **,** $\forall a, b \in \mathbb{R}$

Similarly, $\frac{b|b|-c|c|}{b-c} \leq |b| + |c|$ ***(ii)*** **and,** $\frac{c|c|-a|a|}{c-a} \leq |c| + |a|$ ***(iii)***

$$\textit{(i)} + \textit{(ii)} + \textit{(iii)} \Rightarrow E \leq 2(|a| + |b| + |c|)$$

$$\Rightarrow \frac{E}{3} \leq \frac{2}{3}(|a| + |b| + |c|) < \frac{2}{3} \cdot 3|a| = 2|a| = 2\Omega$$

$$(\because |b| < |a|, |c| < a|, |a| = \Omega$$

i.e. $\frac{E}{3} < 2\Omega \Rightarrow$ ***right inequality is true (Done)***

Solution 2 by Ravi Prakash-New Delhi-India

Thus, for $x < 0 \leq y$

$$|x + y| \leq \frac{x|x| - y|y|}{-x + y} \leq |x| + |y|$$

For $x, y \geq 0, x \neq y$

$$\frac{x|x| - y|y|}{x - y} = \frac{x^2 - y^2}{x - y} = x + y \Rightarrow |x + y| = \frac{x|x| - y|y|}{x - y} = |x| + |y|$$

For $x, y < 0, x \neq y$

$$\frac{x|x| - y|y|}{x - y} = \frac{-x^2 + y^2}{x - y} = -x - y$$

$$\therefore |x + y| = \frac{x|x| - y|y|}{x - y} = |x| + |y|$$

Thus, for $x, y \in R\ x \neq y$

$$|x+y| \leq \frac{x|y|-y|y|}{x-y} \leq |x|+|y|$$

For $x < 0 \leq y$

$$\frac{x|x|-y|y|}{x-y} = \frac{-x^2-y^2}{x-y} = \frac{y^2+x^2}{y-x}$$

As $xy < 0, (y-x)^2 = y^2+x^2-2xy > x^2+y^2$

$$\Rightarrow \frac{x^2+y^2}{-x+y} < y-x = |y|+|x| \Rightarrow \frac{x^2+y^2}{y-x} < |x|+|y|$$

If $x+y \geq 0,$

$$(y-x)(y+x) = y^2-x^2 < y^2+x^2 \Rightarrow |y+x| = y+x \leq \frac{y^2+x^2}{y-x}$$

If $x+y < 0$**, then**

$$-(x+y)(y-x) = x^2-y^2 < x^2+y^2$$

$$\Rightarrow -(x+y) < \frac{x^2+y^2}{y-x} \Rightarrow |x+y| < \frac{x^2+y^2}{y-x}$$

Now, for $a,b,c \in R, a,b,c$ **distinct**

$$3\omega < |a+b|+|b+c|+|c+a| \leq$$

$$\leq \frac{a|a|-b|b|}{a-b} + \frac{b|b|-c|c|}{b-c} + \frac{c|c|-a|a|}{c-a} \leq$$

$$\leq (|a|+|b|)+(|b|+|c|)+(|c|+|a|) < 6\Omega$$

139. If $a,b,c > 0, a+b+c = 1$ **then:**

$$a^3+b^3+c^3+6abc \geq a^{a^2+2bc} \cdot b^{b^2+2ac} \cdot c^{c^2+2ab}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Stefan-Romania

AM-GM, because $\sum(a^2+2bc)=(\sum a)^2=1$

$$\prod_{cyc} a^{a^2+2bc} \le \sum a(a^2+2abc)=\sum a^3+6abc$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $a,b,c\ge 0$ **and** $a+b+c=1$ **then**

$$\sum_{cyc} a^3+6abc \ge a^{a^2+2bc}\cdot b^{b^2+2ca}\cdot c^{c^2+2ab}$$

Applying Weighted A.M ≥ G.M;

$$\frac{\sum_{cyc} a(a^2+2bc)}{\sum_{cyc}(a^2+2bc)} \ge \left(\prod_{cyc} a^{a^2+2bc}\right)^{\frac{1}{\sum_{cyc}(a^2+2bc)}}$$

$$\Rightarrow \frac{a^3+b^3+c^3+6abc}{(a+b+c)^2} \ge \left(\prod_{cyc} a^{a^2+2bc}\right)^{\frac{1}{(a+b+c)^2}}$$

$$\therefore \sum_{cyc} a^3+6abc \ge \prod_{cyc} a^{a^2+2bc}$$

(proved) equality at $a=b=c=\frac{1}{3}$

Solution 3 by Fotini Kaldi-Greece

$$a+b+c=1 \Rightarrow (a+b+c)^2=1$$

$$a^3+b^3+c^3+6abc=$$

Applying Weighted A.M ≥ G.M;

$$(a^2+2bc)a+(b^2+2ac)b+(c^2+2ba)c \ge a^{a^2+2bc}+b^{b^2+2ac}+c^{c^2+2ab}$$

140. If $a, b, c \in \mathbb{R}$ then:

$$\sum(a^2+b^2-c^2)^2+8\sum a^2b^2 \geq 27abc\sqrt[3]{abc}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum(a^2+b^2-c^2)^2 \geq \frac{1}{3}(a^2+b^2+c^2)^2 \left(\because \sum x^2 \geq \frac{1}{3}\left(\sum x\right)^2\right)$$

$$\geq \frac{1}{3}\left\{\frac{1}{3}(a+b+c)^2\right\}^2 = \frac{1}{27}(a+b+c)^4$$

$$= \frac{1}{27}(a+b+c)^3(a+b+c) \underset{(1)}{\overset{A-G}{\geq}} \frac{1}{27}\cdot 27abc\cdot 3\sqrt[3]{abc} = 3abc\sqrt[3]{abc}$$

$$8\sum a^2b^2 \geq 8abc(a+b+c)\left(\because \sum x^2 \geq \sum xy\right)$$

$$\Rightarrow 8\sum a^2b^2 \overset{A-G}{\geq} 24abc\sqrt[3]{abc} \ \textbf{(2)}$$

$$\textbf{(1)}+\textbf{(2)} \Rightarrow \sum(a^2+b^2-c^2)^2+8\sum a^2b^2 \geq 27abc\sqrt[3]{abc}$$

(Proved)

Solution 2 by Shahlar Maharramov-Jebrail-Azerbaidian

$$8\sum a^2b^2 = 8\cdot 3\sqrt[3]{a^4b^4c^4} = 24abc\sqrt[3]{abc} \ \ (*)$$

$$\begin{cases}(a^2+b^2-c^2)^2 = a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2\\(b^2+c^2-a^2)^2 = b^4+c^4+a^4+2b^2c^2-2b^2a^2-2c^2a^2\\(c^2+a^2-b^2)^2 = c^4+a^4+b^4+2c^2a^2-2a^2b^2-2b^2c^2\end{cases}$$

adding these

$$\Rightarrow \sum(a^2+b^2-c^2) = 2(a^4+b^4)+2(a^4+c^4)+2(b^4+c^4)+$$

$$+a^3+b^3+c^3-4a^2b^2-4a^2c^2-4b^2c^2 \overset{A-G}{\geq}$$

$$2\cdot 2a^2b^2+2\cdot 2b^2c^2+2\cdot 2a^2c^2+a^4+b^4+c^4-$$

$$-4a^2b^2-4a^2c^2-4b^2c^2=a^4+b^4+c^4\overset{A-G}{\geq}3abc\sqrt[3]{abc}\quad (**)$$

Adding (*) and (**) $\Rightarrow$

$$\Rightarrow LHS\geq 24abc\sqrt[3]{abc}+3abc\sqrt[3]{abc}=27abc\sqrt[3]{abc}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$(a^2+b^2+c^2)^2+\sum(a^2+b^2-c^2)^2=$$

$$=(a^2+b^2+c^2)^2+(a^2+b^2-c^2)^2+(a^2-b^2+c^2)^2+(-a^2+b^2+c^2)^2=$$

$$=2(a^2+b^2)^2+2c^4+2(a^2-b^2)^2+2c^4=4(a^4+b^4+c^4)$$

$$\therefore \sum(a^2+b^2-c^2)^2+8\sum a^2b^2$$

$$=4(a^4+b^4+c^4)+8\sum a^2b^2-(a^2+b^2+c^2)^2$$

$$=4(a^2+b^2+c^2)^2-(a^2+b^2+c^2)^2=3(a^2+b^2+c^2)^2$$

$$\geq 3\left[3|abc|^{\frac{2}{3}}\right]^2=27|abc|^{\frac{4}{3}}$$

$$\Rightarrow \sum(a^2+b^2-c^2)^2+8\sum a^2b^2\geq 27(abc)(abc)^{\frac{1}{3}}$$

141. If $x,y\geq 0$ then:

$$(x^3+y^3)^3(x^2-xy+y^2)\geq x^2y^2\sqrt{xy}(x^2+y^2)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

Chebyshev: $x^3+y^3\geq\frac{1}{2}(x+y)(x^2+y^2)$

$x^2-xy+y^2\geq xy$ ***(AM-GM)***

$$(x^3+y^3)^3(x^2-\overset{\geq xy}{xy}+y^2)\geq\frac{xy}{8}(x+y)^3(x^2+y^2)^3\overset{AM-GM}{\geq}RHS$$

$$(x+y)^3 \geq \left(2\sqrt{xy}\right)^3 \geq 8xy\sqrt{xy}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Si: $x, y \geq 0$***. Probar que***

$$(x^3+y^3)^3(x^2-xy+y^2) \geq x^2y^2\sqrt{xy}(x^2+y^2)^3$$

1) Si: $x, y \in R$ ***se cumple la siguiente desigualdad***

$$x^2-xy+y^2 \geq \frac{1}{2}(x^2+y^2) \geq \frac{1}{4}(x+y)^2, x^2+y^2 \geq 2xy$$

2) Si $x, y \geq 0$ ***se cumple la siguiente desigualdad***

$$x+y \geq 2\sqrt{xy}$$

La desigualdad es equilvalente

$$(x+y)^3(x^2-xy+y^2)^3(x^2-xy+y^2) \geq x^2y^2\sqrt{xy}(x^2+y^2)^3$$

Luego

$$(x+y)^3(x^2-xy+y^2) \geq \frac{1}{4}(x+y)^4(x+y) \geq 8x^2y^2\sqrt{xy} \wedge \left(x^2-xy+y^2\right)^3 \geq \frac{1}{8}\left(x^2+y^2\right)^3$$

Multiplicando las 2 desigualdades, se obtiene

$$(x+y)^3(x^2-xy+y^2)^3(x^2-xy+y^2) \geq x^2y^2\sqrt{xy}(x^2+y^2)^3 \quad \textbf{(LQQD)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$(x^3+y^3)^3(x^2-xy+y^2) \geq x^2y^2\sqrt{xy}(x^2+y^2)^3 | \cdot (x+y)$$

$$(x^3+y^3)^4 \geq x^2y^2\sqrt{xy} \cdot (x^2+y^2)^3 \cdot (x+y) \ \textbf{(ASSURE)}$$

$$\left(x^3+y^3\right)^4 \geq \left(\frac{1}{2} \cdot (x+y) \cdot \left(x^2+y^2\right)\right)^4 = \underbrace{\left(\frac{x+y}{2}\right)^2}_{\geq xy} \cdot \underbrace{\left(\frac{x^2+y^2}{2}\right)}_{\geq xy} \cdot \frac{(x+y)^2}{2} \cdot \left(x^2+y^2\right)^3 \geq$$

$$\geq (xy)^2 \cdot \underbrace{\frac{x+y}{2}}_{\geq\sqrt{xy}} \cdot (x+y)(x^2+y^2)^3 \geq x^2y^2 \cdot \sqrt{xy}(x^2+y^2)^3(x+y)$$

Solution 4 by Ravi Prakash-New Delhi-India

$$(x^3+y^3)^3(x^2-xy+y^2)\ge(xy)^{\frac{5}{2}}(x^2+y^2)^3 \quad \textbf{(1)}$$

If $x=0$ ***or*** $y=0$***, we have nothing to prove so we may assume that***

$$x>0, y>0.$$

Put $x=r\cos\theta$, $y=r\sin\theta$***, and now (1) reduces to***

$$r''(\cos^3\theta+\sin^3\theta)^3(\cos^2\theta-\cos\theta\sin\theta+\sin^2\theta)\ge r''(\cos\theta\sin\theta)^{\frac{5}{2}}$$

$$\textbf{or } (\cos^3\theta+\sin^3\theta)^3(\cos^2\theta-\cos\theta\sin\theta+\sin^2\theta)\ge(\cos\theta\sin\theta)^{\frac{5}{2}}$$

$$\textbf{or } (\cos\theta+\sin\theta)^3(1-\cos\theta\sin\theta)^4\ge(\cos\theta\sin\theta)^{\frac{5}{2}}$$

Solution 5 by Soumava Chakraborty-Kolkata-India

$$\textbf{If } y=0, x>0, LHS=x'' \textbf{ and } RHS=0\Rightarrow LHS>RHS$$

$$\textbf{If } x=0, y>0, LHS=y'' \textbf{ and } RHS=0\Rightarrow LHS>RHS$$

Let's now consider $x,y>0$

We have, $(x+y)(x^3+y^3)\ge(x^2+y^2)^2$

$$\Leftrightarrow x^3y+xy^3\ge2x^2y^2\Leftrightarrow x^2+y^2\ge2xy\rightarrow \textbf{true by A-G}$$

$$\therefore (x+y)(x^3+y^3)\ge(x^2+y^2)^2 \ \textbf{(1)}$$

$$\textbf{Again, } x^2-xy+y^2=\frac{1}{4}(x+y)^2+\frac{3}{4}(x-y)^2$$

$$\textbf{(2)}\ge\frac{(x+y)^2}{4}$$

$$LHS=(x+y)(x^2-xy+y^2)(x^3+y^3)^2(x^2-xy+y^2)$$

$$=\left((x+y)(x^3+y^3)\right)(x^3+y^3)(x^2-xy+y^2)^2$$

$$\ge(x^2+y^2)^2(x^3+y^3)(x^2-xy+y^2)^2 \ \textbf{(a)} \ \textbf{(using (1))}$$

(a) $\Rightarrow$ ***it suffices to prove:***

$$(x^3+y^3)(x^2-xy+y^2)^2\ge x^2y^2\sqrt{xy}(x^2+y^2) \ \textbf{(b)}$$

Now, $x^2 - xy + y^2 \geq \frac{(x+y)^2}{4}$ ***(from (2))***

$\therefore$ ***(b)*** $\Rightarrow$ ***it suffices to prove:***

$$(x^3 + y^3)(x^2 - xy + y^2)\frac{(x+y)^2}{4} \geq x^2y^2\sqrt{xy}(x^2 + y^2)$$

$$\Leftrightarrow \underbrace{(x^3 + y^3)(x + y)}_{\geq (x^2+y^2)^2}\underbrace{(x^2 - xy + y^2)}_{\geq xy}\underbrace{(x + y)}_{\geq 2\sqrt{xy}} \geq 4x^2y^2\sqrt{xy}(x^2 + y^2)$$

$\therefore$ ***it suffices to prove:***

$$(x^2 + y^2)^2(xy)\left(2\sqrt{xy}\right) \geq 4x^2y^2\sqrt{xy}(x^2 + y^2)$$

$\Leftrightarrow x^2 + y^2 \geq 2xy \rightarrow$ ***true by A-G (Proved)***

Solution 6 by Shahlar Maharramov-Jebrail-Azerbaidian

Take substitution $y = m^2x$ ***then***

$$(x^3 + m^6x^3)^3(x^2 - m^2x^2 + m^4x^2) \geq x^2x^2m^4 \cdot xm(x^2 + m^4x^2)^3$$

$$\Rightarrow x''(1 + m^6)^3(1 - m^2 + m^4) \geq x''(1 + m^4)^3m^5$$

Since $m^4 - m^2 + 1 \geq m^2$ ***then***

$(1 + m^6)(1 - m^2 + m^4) \geq (1 + m^6)m^2$ ***then we have to prove***

$$(1 + m^6)m^2 \geq m^5(1 + m^4)^3 \Rightarrow$$

$(1 + m^6)^3 \geq \left(m + m^5\right)^3$? ***- we have to prove***

$$\Rightarrow 1 + m^6 \geq m + m^5? \Rightarrow$$

$$\Rightarrow m^6 - m^5 + 1 - m^4 \geq 0 \Rightarrow 4m^5(m - 1) - (m - 1) \geq 0?$$

$(m - 1)\left(m^5 - 1\right) \geq 0$? ***(*)***

as if $m \geq 1$ ***then correct (*)***

b) if $0 < m \leq 1$ ***then*** $m - 1 \leq 0 \quad m^5 - 1 \leq 0$ ***then (*) correct***

142. If $a, b, c \in (0, \infty)$ then:

$$a^8b^8 + b^8c^8 + c^8a^8 \geq a^5b^5c^5\sqrt[4]{27(a^4 + b^4 + c^4)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} a^8b^8 \geq (abc)^4\left(\sum_{cyc} a^4\right) = (abc)^5\frac{\sum_{cyc} a^4}{abc}$$

Now,

$$\left(\sum_{cyc} a^4\right)^3 \geq 27(abc)^4 \Rightarrow \frac{\sum_{cyc} a^4}{abc} \geq \sqrt[4]{27\left(\sum_{cyc} a^4\right)}$$

So,

$$\sum_{cyc} a^8b^8 \geq (abc)^5\sqrt[4]{27\left(\sum_{cyc} a^4\right)}$$

(Proved)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$a^4 = x;\ b^4 = y;\ c^4 = z$$

$$x^2y^2 + y^2z^2 + z^2x^2 \geq xyz \cdot \sqrt[4]{27\, xyz \cdot (x + y + z)}$$

$$(x^2y^2 + y^2z^2 + z^2x^2)^4 \geq (xyz)^4 \cdot 27 \cdot xyz \cdot (x + y + z)$$

$$(x^2y^2 + y^2z^2 + z^2x^2) \cdot (x^2y^2 + y^2z^2 + z^2x^2)^3 \geq (xyz)^4 \cdot 27xyz \cdot (x + y + z) \quad \textbf{(*) (ASSURE)}$$

a) $x^2y^2 + y^2z^2 + z^2x^2 \geq (xy)(yz) + (yz)(zx) + (zx)(xy) =$

$$= xyz(x + y + z)$$

b) $(x^2y^2 + y^2z^2 + z^2x^2)^3 \overset{AM-GM}{\geq} \left(3 \cdot \sqrt[3]{(xyz)^4}\right)^3 = 27(xyz)^4$

$$\textbf{a); b)} \Rightarrow (x^2y^2+y^2z^2+z^2x^2)(x^2y^2+y^2z^2+z^2x^2)^3 \overset{a);b)}{\geq}$$

$$\geq (xyz)^4 \cdot 27xyz \cdot (x+y+z) \ \textbf{(*)}$$

143. If $a,b,c>0, a^2+b^2+c^2=26(a+b+c)$ then:

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}} \geq \frac{1}{\sqrt{a+b+c}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marjan Milanovic-Nis-Serbia

By Jensen, since $x^{\left(-\frac{1}{2}\right)}$ is convex,

$$\sum(a+b^2)^{\left(-\frac{1}{2}\right)} \geq 3\left(\frac{a+b+c+a^2+b^2+c^2}{3}\right)^{\left(-\frac{1}{2}\right)} =$$

$$=3\left(\frac{27(a+b+c)}{3}\right)^{\left(-\frac{1}{2}\right)} = (a+b+c)^{\left(-\frac{1}{2}\right)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS \overset{Bergstrom}{\geq} \frac{9}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}} \ \textbf{(1)}$$

$$\left(\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}\right)^2 \leq 3\left(\sum a+\sum a^2\right)\left(\because \left(\sum x\right)^2 \leq 3\sum x^2\right)$$

$$=\frac{8}{\left(\sum a\right)}\left(\because \sum a^2 = 26\sum a\right)$$

$$\Rightarrow \frac{1}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}} \geq \frac{1}{9\sqrt{\sum a}} \Rightarrow \frac{9}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}} \geq \frac{1}{\sqrt{\sum a}} \ \textbf{(2)}$$

$$\textbf{(1), (2)} \Rightarrow LHS \geq \frac{1}{\sqrt{\sum a}}$$

(Proved)

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{1}{\sqrt{a+b+c}}$$

$$\left.\begin{array}{r} a+b^2=x^2 \\ b+c^2=y^2 \\ c+a^2=z^2 \end{array}\right| \begin{array}{c} a+b+c+a^2+b^2+c^2=x^2+y^2+z^2 \\ 27(a+b+c)=x^2+y^2+z^2 \end{array}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{3\sqrt{3}}{\sqrt{x^2+y^2+z^2}}\geq\frac{9}{x+y+z}\Rightarrow x^2+y^2+z^2\geq\frac{(x+y+z)^2}{3}$$

$$(x+y+z)(xy+yz+xz)\geq 9xyz \quad \textit{(AM-GM)}$$

Solution 4 by Shivam Sharma-New Delhi-India

Applying A.M ≥ H.M, we get,

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{9}{\left(\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}\right)}$$

Applying Cauchy – Schwarz inequality and then put

$a^2+b^2+c^2=26(a+b+c)$***, we get,***

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{9}{\sqrt{(1+1+1)(a+b+c+a^2+b^2+c^2)}}$$

$$=\frac{9}{\sqrt{3\left(27(a+b+c)\right)}}=\frac{9}{\sqrt{9\times 9\,(a+b+c)}}=\frac{1}{\sqrt{a+b+c}}$$

Hence, $\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{1}{\sqrt{a+b+c}}$ ***(Proved)***

Solution 5 by Abdul Aziz-Semarang-Indonesia

$$a,b,c>0;\ a^2+b^2+c^2=26(a+b+c)$$

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{9}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}} \quad \textit{(AM-HM)}$$

$$\geq \frac{9}{\sqrt{(1^2+1^2+1^2)(a+b^2+b+c^2+c+a^2)}} \quad (cs)$$

$$= \frac{9}{\sqrt{3(a^2+b^2+c^2+a+b+c)}} = \frac{9}{\sqrt{3\big(27(a+b+c)\big)}} = \frac{1}{\sqrt{a+b+c}}$$

Inequality holds when $a = b = c = 2b$

Solution 6 by Uche Eliezer Okeke-Nigeria

Given $\sum a^2 = 2b\sum a\,; a, b, c > 0$

Show $\frac{1}{\sqrt{a+b^2}} + \frac{1}{\sqrt{b+c^2}} + \frac{1}{\sqrt{c+a^2}} \geq \frac{1}{\sqrt{a+b+c}}$

$$\sum\left(\frac{1}{\sqrt{a+b^2}}\right)\sum\left(\frac{1}{\sqrt{a+b^2}}\right)\sum(a+b^2) \overset{Hölder}{\geq} (1+1+1)^3$$

$$\Rightarrow \sum\frac{1}{\sqrt{a+b^2}} \geq \sqrt{\frac{27}{\sum(a+b^2)}} = \frac{1}{\sqrt{\frac{\sum(a+b^2)}{27}}}$$

It suffices to show $\sqrt{\frac{\sum(a+b^2)}{27}} \leq \sqrt{a+b+c} = \sqrt{\sum a}$

$$\sqrt{\frac{\sum(a+b^2)}{27}} = \sqrt{\frac{\sum a+\sum b^2}{27}} = \sqrt{\frac{\sum a+26\sum a}{27}} = \sqrt{\sum a} \textbf{ (Proved)}$$

144. If $a, b, c \in (0, \infty)$ then:

$$\sum\left(a^2b + ab\sqrt{ab} + ab^2\right) \leq \frac{3\sqrt{2}}{2}\sum\sqrt{a^6+b^6}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$a^2b + ab\sqrt{ab} + ab^2 \leq \frac{3\sqrt{2}}{2}(a^6+b^6)^{\frac{1}{2}}$$

$$ab\left(a+\sqrt{ab}+b\right)\le\frac{3\sqrt{2}}{2}(a^6+b^6)^{\frac{1}{2}}$$

$$a^6+b^6=(a^2+b^2)(a^4-a^2b^2+b^4)\overset{A-G}{\ge}$$

$$\overset{A-G}{\ge}a^2b^2(a^2+b^2)\overset{A-G}{\ge}\frac{a^2b^2(a+b)^2}{2}$$

$$\frac{3\sqrt{2}}{2}\cdot\frac{ab(a+b)}{\sqrt{2}}\ge ab\left(a+\sqrt{ab}+b\right)$$

$$3(a+b)\ge 2\left(a+\sqrt{ab}+b\right);\ a+b\overset{A-G}{\ge}2\sqrt{ab}$$ ***(Proved)***

Solution 2 by Ravi Prakash-New Delhi-India

Let $x,y>0$. ***Put*** $x=r\cos\theta,\ y=r\sin\theta,\ 0<\theta<\frac{\pi}{2}$. ***Now, consider***

$$a(x^6+y^6)-2\left(x^2y+xy\sqrt{xy}+xy^2\right)^2=r^6E$$ ***where***

$$E=a(\cos^6\theta+\sin^6\theta)-2\sin^2\theta\cos^2\theta\left(\cos\theta+\sqrt{\cos\theta\sin\theta}+\sin\theta\right)^2$$

$$=a[(\cos^2\theta+\sin^2\theta)^3-3\cos^2\theta\sin^2\theta(\cos^2\theta+\sin^2\theta)]-$$

$$-2\sin^2\theta\cos^2\theta\left\{\cos^2\theta+\sin^2\theta+3\cos\theta\sin\theta+2(\cos\theta+\sin\theta)\sqrt{\cos\theta\sin\theta}\right\}$$

$$=9\left[1-3\cos^2\theta\sin^2\theta\right]-2\sin^2\theta\cos^2\theta\left[1+3\cos\theta\sin\theta+2(\cos\theta\sin\theta)\sqrt{\cos\theta\sin\theta}\right]$$

$$=9-\frac{29}{4}\sin^2 2\theta-\frac{6}{8}\sin^3 2\theta-\frac{4}{4\sqrt{2}}(\sin 2\theta)^{\frac{5}{2}}\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$$

$$=\frac{29}{4}(1-\sin^2 2\theta)+\frac{3}{4}(1-\sin^3 2\theta)+\left(1-(\sin 2\theta)^{\frac{5}{2}}\sin\left(\theta+\frac{\pi}{4}\right)\right)\ge 0$$

$$\Rightarrow 3\sqrt{x^6+y^6}\ge\sqrt{2}\left(x^2y+xy\sqrt{xy}+xy^2\right)\ \ \forall x,y>0.$$

Equality when $x=y$. ***Put*** $x=a,\ y=b+0$ ***get***

$a^2b+ab\sqrt{ab}+ab^2\le\frac{3}{\sqrt{2}}\sqrt{a^6+b^6}$ ***. Similarly, for other expressions.***

$$\Rightarrow\sum\left(a^2b+ab\sqrt{ab}+ab^2\right)\le\frac{3}{2}\sqrt{2}\sum\sqrt{a^6+b^6}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\left(a^2b+ab\sqrt{ab}+ab^2\right)=\sum ab\left(a+\sqrt{ab}+b\right)\underset{Cauchy}{\leq}$$

$$\leq\sum\frac{a^2+b^2}{2}\cdot\left(a+\frac{a+b}{2}+b\right)=$$

$$=\sum\frac{a^2+b^2}{2}\cdot\frac{3(a+b)}{2}=\sum\frac{3}{4}(a^2+b^2)(a+b)\overset{Chebyshev}{\leq}$$

$$\leq\sum\frac{3}{4}\cdot 2(a^3+b^3)=\sum\frac{3}{2}\cdot\sqrt{(a^3+b^3)^2}=$$

$$=\sum\frac{3}{2}\cdot\sqrt{a^6+b^6+2a^3b^3}\underset{Cauchy}{\leq}\sum\frac{3}{2}\cdot\sqrt{2(a^6+b^6)}=$$

$$=\frac{3\sqrt{2}}{2}\cdot\sum\sqrt{a^6+b^6}$$

Solution 4 by Soumava Chakraborty-Kolkata-India

We shall first prove that:

$a^2b+ab\sqrt{ab}+ab^2\leq\frac{3}{\sqrt{2}}\sqrt{a^6+b^6}$, ***which of course proves given***

inequality $\Leftrightarrow ab\left(a+b+\sqrt{ab}\right)\leq\frac{3}{\sqrt{2}}\sqrt{a^6+b^6}$

$$\Leftrightarrow a^2b^2\left(a+b+\sqrt{ab}\right)^2\leq\frac{9(a^6+b^6)}{2}\quad \textbf{(1)}$$

Now, $\sqrt{ab}\leq\frac{a+b}{2}\Rightarrow\sqrt{ab}+a+b\leq\frac{3}{2}(a+b)$

$$\Rightarrow a^2b^2\left(a+b+\sqrt{ab}\right)^2\leq a^2b^2\cdot\frac{9}{4}(a+b)^2\quad \textbf{(2)}$$

(1), (2) ⇒ it suffices to prove:

$$\frac{a^2b^2(a+b)^2}{4}\leq\frac{a^6+b^6}{2}\Leftrightarrow a^6+b^6\geq\frac{a^2b^2(a+b)^2}{2}\quad \textbf{(3)}$$

Now, $a^6+b^6=(a^3)^2+(b^3)^2\geq\frac{\left(a^3+b^3\right)^2}{2}\geq\frac{\{ab(a+b)\}^2}{2}$

$$\left(\because a^3+b^3\ge ab(a+b)\right)$$

$$\Rightarrow a^6+b^6\ge \frac{a^2b^2(a+b)^2}{2}\Rightarrow \textit{(3) is true (Proved)}$$

Solution 5 by Soumava Chakraborty-Kolkata-India

We shall first prove: $a^2b+ab\sqrt{ab}+ab^2\overset{(a)}{\le}\frac{3}{\sqrt{2}}\sqrt{a^6+b^6}$***, which of course proves the given inequality***

$$a^2b+ab^2=ab(a+b)\overset{G\le A}{\le}\frac{a^2+b^2}{2}(a+b)\overset{Chebyshev}{\le}a^3+b^3$$

$$\textbf{(1)}\overset{C-B-S}{\le}\sqrt{2}\sqrt{a^6+b^6}$$

$$\textit{Again } ab\sqrt{ab}\overset{G\le A}{\le}\frac{a^2+b^2}{2}\cdot\frac{a+b}{2}\overset{Chebyshev}{\le}\frac{a^3+b^3}{2}\underset{(2)}{\overset{C-B-S}{\le}}\frac{1}{\sqrt{2}}\sqrt{a^6+b^6}$$

$$\textbf{(1)}+\textbf{(2)}\Rightarrow a^2b+ab^2+ab\sqrt{ab}\le\frac{3}{\sqrt{2}}\sqrt{a^6+b^6}$$

$$\Rightarrow \textit{(a) is true (Proved)}$$

145. If $x>0$ then:

$$\left(1+\frac{x}{5}\right)^{20}>(1+x)\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{3}\right)^3\left(1+\frac{x}{4}\right)^4$$

Proposed by Daniel Sitaru – Romania

Solution by Șerban George Florin – Romania

$$P=(1+x)\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{3}\right)^3\left(1+\frac{x}{4}\right)^4<\left(1+\frac{x}{5}\right)^{20}$$

$$\sqrt[10]{P}\le\frac{(1+x)+2\left(1+\frac{x}{2}\right)+3\left(1+\frac{x}{3}\right)+4\left(1+\frac{x}{4}\right)}{10}\quad (Mg\le M_a)$$

$$\sqrt[10]{P}\le\frac{10+4x}{10}=1+\frac{2x}{5}\Rightarrow P\le\left(1+\frac{2x}{5}\right)^{10}$$

$$\left(1+\frac{2x}{5}\right)^{10}<\left(1+\frac{x}{5}\right)^{20}\Rightarrow\left(1+\frac{2x}{5}\right)^{10}<\left(1+\frac{2x}{5}+\frac{x^2}{25}\right)^{10}$$

$$\Rightarrow 1+\frac{2x}{5}<1+\frac{2x}{5}+\frac{x^2}{25}\Rightarrow\frac{x^2}{25}>0\quad (A)$$

$$\Rightarrow P\le\left(1+\frac{2x}{5}\right)^{10}<\left(1+\frac{x}{5}\right)^{20}\Rightarrow P<\left(1+\frac{x}{5}\right)^{20}$$

146. Prove that if $x,y,z,t\in(0,\infty)$ then:

$$\left(\frac{x}{y}+\frac{z}{t}\right)\left(\frac{y}{x}+\frac{z}{t}\right)\left(\frac{x}{y}+\frac{t}{z}\right)\left(\frac{y}{x}+\frac{t}{z}\right)\ge 12+\left(\frac{xt}{yz}+\frac{yz}{xt}\right)\left(\frac{xz}{yt}+\frac{yt}{xz}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Manish Tayal – India

Let be $a=\frac{x}{y};b=\frac{z}{t}$

$$(a+b)\left(1+\frac{b}{a}\right)\left(a+\frac{1}{b}\right)\left(1+\frac{1}{ab}\right)\ge 12+\left(\frac{a}{b}+\frac{b}{a}\right)\left(ab+\frac{1}{ab}\right)$$

$$(a+b)\left(\frac{a+b}{a}\right)\left(\frac{ab+1}{b}\right)\left(\frac{ab+1}{ab}\right)\ge 12+\left(\frac{a}{b}+\frac{b}{a}\right)\left(ab+\frac{1}{ab}\right)$$

$$\frac{(a+b)^2(ab+1)^2}{(ab)^2}\ge 12+\left(\frac{a^2+b^2}{ab}\right)\left(\frac{a^2b^2+1}{ab}\right)$$

$$(a+b)^2(ab+1)^2\ge 12(ab)^2+(a^2+b^2)(a^2b^2+1)$$

$$(a^2+b^2+2ab)(a^2b^2+1+2ab)\ge 12(ab)^2+(a^2+b^2)(a^2b^2+1)$$

$$(a^2+b^2)(a^2b^2+1)+(a^2+b^2)2ab+(ab+1)^2 2ab\ge$$

$$\ge 12(ab)^2+(a^2+b^2)(a^2b^2+1)$$

$$(a^2+b^2)2ab+(ab+1)^2\cdot 2ab\ge 12(ab)^2$$

$$(a^2+b^2)+(ab+1)^2\ge 6ab$$

$$(a-b)^2+(ab-1)^2\ge 0$$

147. Prove that if $0 < a \leq b$ then:

$$\left(\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\right)\leq\frac{(a+b)^2}{ab}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar que si: $0 < a \leq b$***:***

$$\left(\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\right)\leq\frac{(a+b)^2}{ab}$$

$$\Rightarrow 1+1+\left(\frac{a+b}{2ab}\right)\left(\sqrt{\frac{a^2+b^2}{2}}\right)+\left(\frac{2ab}{a+b}\right)\left(\sqrt{\frac{2}{a^2+b^2}}\right)\leq 2+\frac{a}{b}+\frac{b}{a}$$

$$\Rightarrow \frac{1}{\sqrt{2(a^2+b^2)}}\left(\frac{a+b}{2ab}(a^2+b^2)+\frac{4ab}{a+b}\right)\leq\frac{1}{\sqrt{2(a^2+b^2)}}\left(\frac{a+b}{2ab}(a^2+b^2)+(a+b)\right)\leq$$

$$\leq\frac{1}{(a+b)}\left(\frac{a+b}{2ab}(a^2+b^2)+(a+b)\right)$$

$$\Rightarrow \frac{a^2+b^2}{2ab}+1\leq\frac{a}{b}+\frac{b}{a}\to\frac{a}{b}+\frac{b}{a}\geq 2 \ldots$$ ***(Válido por: MA ≥ MG)***

Solution 2 by Soumava Chakraborty – Kolkata – India

$$\frac{2ab}{a+b}\leq\frac{2}{\frac{1}{a}+\frac{1}{b}}\overset{HM\leq AM}{\leq}\frac{a+b}{2}\leq\sqrt{\frac{a^2+b^2}{2}}\quad \textit{(QM} \geq \textit{AM)}$$

$$\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\leq 2\sqrt{\frac{a^2+b^2}{2}}\quad (1)$$

AM ≤ QM, $\frac{a+b}{2}\left(\frac{1}{ab}\right)\leq\sqrt{\frac{a^2+b^2}{2}}\left(\frac{1}{ab}\right)$

$$\Rightarrow\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\leq\sqrt{\frac{a^2+b^2}{2}}\left(\frac{1}{ab}\right)+\sqrt{\frac{2}{a^2+b^2}}\quad (2)$$

$$(1)\ \times\ (2) \Rightarrow \left(\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\right) \le$$

$$\le 2\sqrt{\frac{a^2+b^2}{2}}\left(\sqrt{\frac{a^2+b^2}{2}}\left(\frac{1}{ab}\right)+\sqrt{\frac{2}{a^2+b^2}}\right) = \frac{a^2+b^2}{ab}+2 = \frac{(a+b)^2}{ab} \text{ (proved)}$$

Solution 3 by Daniel Sitaru – Romania

We prove that:

$$\begin{cases} \frac{2ab}{a+b} \le \frac{a+b}{2} \\ \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}} \end{cases} \Leftrightarrow \begin{cases} 4ab \le (a+b)^2 \\ (a+b)^2 \le 2(a^2+b^2) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0 \le a^2 - 2ab + b^2 \\ a^2 + 2ab + b^2 \le 2a^2 + 2b^2 \end{cases} \Leftrightarrow \begin{cases} 0 \le (a-b)^2 \\ 0 \le (a-b)^2 \end{cases}$$

It follows: $0 < a \le \frac{2ab}{a+b} \le \sqrt{\frac{a^2+b^2}{2}} \le b$

From Schweitzer inequality for $n = 2$***, if*** $x_1, x_2 \in [a, b]$ ***then:***

$$(x_1 + x_2)\left(\frac{1}{x_1}+\frac{1}{x_2}\right) \le \frac{(a+b)^2}{ab}$$

Let be $x_1 = \frac{2ab}{a+b}; x_2 = \sqrt{\frac{a^2+b^2}{2}}$***; It follows:***

$$\left(\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\right) \le \frac{(a+b)^2}{ab}$$

(Schweitzer's inequality:

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \le \frac{(m+M)^2 n^2}{4mM}$$

$x_1, x_2, \dots, x_n \in [m, M]; 0 < m \le x_k \le M; k \in \overline{1, n}; n \in \mathbb{N}^*$***)***

148. **If $A, B \in M_n(\mathbb{C})$ then:**

$$rank\ A + rank\ B + rank\ (I_n - A - B) \geq n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Dimitris Kastriotis-Greece

$$Rank\ (A) + Rank\ (B) + Rank\ (I_n - A - B) \geq Rank\ \big(A + B + (I_n - A - B)\big)$$

$$= Rank\ \big((A - A) + (B - B) + I_n\big) = Rank\ (O + O + I_n)$$

$$= Rank(I_n) = n$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $rank\ (A) = r, rank\ (B) = r_2$

If $r_1 + r_2 \geq n$**, we are done. Suppose** $r_1 + r_2 < n$.

We know $rank\ (A + B) \leq rank(A) + rank\ (B)$.

Now, $rank\ (A) + rank\ (B) + rank\ (I_n - A - B) \geq$

$$\geq rank\ (A + B) + rank\ (I_n - A - B)$$

$$\geq rank\ \{(A + B) + I_n - (A + B)\} = rank\ (I_n) = n.$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

Use, $rank\ (A + B) \Leftarrow rank\ (A) + rank\ (B)$

for some order matrix A **and** B.

Hence, $rank\ (A) + rank\ (B) + rank\ (I - A - B) \Rightarrow$

$$\Rightarrow rank\ (A + B) + rank\ (I - A - B) \Rightarrow rank\ (A + B + I - A - B) = n$$

149. **If $a, b, c, d > 0$ then:**

$$\sqrt{\frac{a}{b}} + \sqrt[4]{\frac{b}{c}} + \sqrt[6]{\frac{c}{d}} + \sqrt[8]{\frac{d}{a}} > 1$$

Proposed by Daniel Sitaru – Romania

Solution by Daniel Sitaru – Romania

$$\sqrt{\frac{a}{b}}+\sqrt[4]{\frac{b}{c}}+\sqrt[6]{\frac{c}{d}}+\sqrt[8]{\frac{d}{a}}=$$

$$=2\cdot\frac{1}{2}\sqrt{\frac{a}{b}}+4\cdot\frac{1}{4}\sqrt[4]{\frac{b}{c}}+6\cdot\frac{1}{6}\sqrt[6]{\frac{c}{d}}+8\cdot\frac{1}{8}\sqrt[8]{\frac{d}{a}}\overset{AM-GM}{\geq}$$

$$\geq 20\sqrt[20]{\left(\frac{1}{2}\sqrt{\frac{a}{b}}\right)^2\cdot\left(\frac{1}{4}\sqrt[4]{\frac{b}{c}}\right)^4\cdot\left(\frac{1}{6}\sqrt[6]{\frac{c}{d}}\right)^6\cdot\left(\frac{1}{8}\sqrt[8]{\frac{d}{a}}\right)^8}=$$

$$=20\sqrt[20]{\frac{1}{2^2\cdot 4^4\cdot 6^6\cdot 8^8}}=\frac{20}{\sqrt[20]{2^{40}\cdot 3^6}}=\frac{5}{\sqrt[10]{27}}>1\leftrightarrow 5^{10}>27$$

150. Prove that if $0<a\leq b$ then:

$$\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right)\leq 5+2\left(\frac{a}{b}+\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

$$LHS=3+\frac{2\sqrt{ab}}{a+b}+\frac{4ab}{(a+b)^2}+\frac{a+b}{2\sqrt{ab}}+\frac{2\sqrt{ab}}{a+b}+\frac{(a+b)^2}{4ab}+\frac{a+b}{2\sqrt{ab}}$$

$$=3+\frac{4\sqrt{ab}}{a+b}+\frac{a+b}{\sqrt{ab}}+\frac{4ab}{(a+b)^2}+\frac{(a+b)^2}{4ab}$$

$$\leq 3+2+\frac{a+b}{\sqrt{ab}}+1+\frac{(a+b)^2}{4ab}$$

$$\left(2\sqrt{ab}\leq a+b,\ 4ab\leq(a+b)^2\right)$$

$$= 6 + \frac{a + b}{\sqrt{ab}} + \frac{(a + b)^2}{4ab}$$

It suffices to show: $6 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab} \le 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right)$

$$\Leftrightarrow 1 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab} \le \frac{2(a^2+b^2)}{ab} \quad (1)$$

Now, $\sqrt{ab} \ge \frac{2ab}{a+b}$ ***(GM ≥ HM)***

$$\Rightarrow \frac{1}{\sqrt{ab}} \le \frac{a + b}{2ab} \Rightarrow \frac{a + b}{\sqrt{ab}} \le \frac{(a + b)^2}{2ab}$$

$$\left(\frac{a + b}{\sqrt{ab}}\right) + 1 + \frac{(a + b)^2}{4ab} \le \frac{(a + b)^2}{2ab} + 1 + \frac{(a + b)^2}{4ab}$$

$$= 1 + \frac{3(a+b)^2}{4ab} = \frac{3(a+b)^2+4ab}{4ab} \quad (2)$$

(1), (2) ⇒ it is sufficient to prove:

$$\frac{3(a + b)^2 + 4ab}{4ab} \le \frac{2(a^2 + b^2)}{ab}$$

$$\Leftrightarrow 3a^2 + 3b^2 + 10ab \le 8a^2 + 8b^2$$

$$\Leftrightarrow 5a^2 + 5b^2 - 10ab \ge 0 \Leftrightarrow 5(a - b)^2 \ge 0 \rightarrow \textit{ true (proved)}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Prove that if: $0 < a \le b$ ***then:***

$$\left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\right) \le 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right) \ldots (A)$$

Tener en cuenta lo siguiente:

$$1)\ \left(\sqrt{b} - \sqrt{a}\right)^2 \ge 0 \rightarrow a + b \ge 2\sqrt{ab} \rightarrow \frac{2\sqrt{ab}}{a + b} \le 1 \rightarrow \frac{4ab}{(a + b)^2} \le 1$$

Desarrollando en ... (A)

$$\Rightarrow \left(\frac{2ab}{a + b}\right)\left(\frac{a + b}{2ab}\right) + \frac{a + b}{2ab}\sqrt{ab} + \frac{(a + b)^2}{4ab} + \frac{2ab}{a + b} \cdot \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{ab}}\sqrt{ab} +$$

$$+\frac{a+b}{2}\cdot\frac{1}{\sqrt{ab}}+\frac{4ab}{(a+b)^2}+\frac{2\sqrt{ab}}{a+b}+\left(\frac{a+b}{2}\right)\left(\frac{2}{a+b}\right)\le$$

$$\le\left(1+\frac{(a+b)^2}{4ab}+\frac{(a+b)^2}{4ab}\right)+\left(1+1+\frac{a+b}{2ab}\sqrt{ab}\right)+(1+1+1)\le$$

$$\le\left(6+\frac{(a+b)^2}{2ab}+\frac{(a+b)^2}{4ab}\right)\le 5+2\left(\frac{a}{b}+\frac{b}{a}\right)$$

⇒ Esto es suficiente probar que:

$$\Rightarrow\left(6+\frac{(a+b)^2}{2ab}+\frac{(a+b)^2}{4ab}\right)\le 5+2\left(\frac{a}{b}+\frac{b}{a}\right)\to$$

$$\to 1+\frac{3(a+b)^2}{4ab}=\frac{5}{2}+\frac{3}{4}\left(\frac{a}{b}+\frac{b}{a}\right)\le 2\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$\Rightarrow\frac{5}{2}\le\frac{5}{4}\left(\frac{a}{b}+\frac{b}{a}\right)\to\frac{a}{b}+\frac{b}{a}\ge 2\to(b-a)^2\ge 0\ \ldots\ \textit{(LQQD)}$$

Solution 3 by Soumava Chakraborty – Kolkata – India

$$\frac{2ab}{a+b}\le\sqrt{ab}\ (HM\le GM)$$

Also, $\frac{a+b}{2}\ge\sqrt{ab}$ ***(AM ≥ GM)*** $\Rightarrow\frac{2}{a+b}\le\frac{1}{\sqrt{ab}}$

$$LHS\le\left(\sqrt{ab}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{ab}}\right)$$

$$=\left(2\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{2}{\sqrt{ab}}\right)$$

$$=\frac{a+b}{\sqrt{ab}}+4+\frac{(a+b)^2}{4ab}+\frac{a+b}{\sqrt{ab}}=4+\frac{(a+b)^2}{4ab}+\frac{2(a+b)}{\sqrt{ab}}$$

It suffices to show:

$$4+\frac{(a+b)^2}{4ab}+\frac{2(a+b)}{\sqrt{ab}}\le 5+\frac{2(a^2+b^2)}{ab}$$

$$\Leftrightarrow \frac{(a+b)^2}{4ab}+\frac{2(a+b)}{\sqrt{ab}}\le 1+\frac{2(a^2+b^2)}{ab}=\frac{2a^2+2b^2+ab}{ab}$$

Now, $\sqrt{ab}\ge\frac{2ab}{a+b}\Rightarrow\frac{1}{\sqrt{ab}}\le\frac{a+b}{2ab}\Rightarrow\frac{2(a+b)}{\sqrt{ab}}\le\frac{(a+b)^2}{ab}$

$$\Rightarrow \frac{(a+b)^2}{4ab}+\frac{2(a+b)}{\sqrt{ab}}\le\frac{(a+b)^2}{4ab}+\frac{(a+b)^2}{ab}=\frac{5(a+b)^2}{4ab}$$

it suffices to prove: $\frac{5(a+b)^2}{4ab}\le\frac{2a^2+2b^2+ab}{ab}$

$$\Leftrightarrow 5a^2+5b^2+10ab\le 8a^2+8b^2+4ab$$

$\Leftrightarrow 3a^2+3b^2-6ab\ge 0\Leftrightarrow 3(a-b)^2\ge 0\rightarrow$ ***true (proved)***

Solution 4 by Abdallah El Farisi – Bechar – Algerie

$$\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{2}{a+b}+\frac{1}{\sqrt{ab}}+\frac{a+b}{2ab}\right)=\frac{1}{ab}\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)^2\le$$

$$\le\frac{1}{ab}\left(\sqrt{ab}+a+b\right)^2=\frac{1}{ab}\left(ab+2\sqrt{ab}(a+b)+(a+b)^2\right)\le$$

$$\le\frac{1}{ab}\left(ab+2(a+b)^2\right)=\frac{1}{ab}\left(5a+2(a^2+b^2)\right)=5+2\left(\frac{a}{b}+\frac{b}{a}\right)$$

Solution 5 by Soumava Chakraborty – Kolkata – India

Let $x=\frac{2ab}{a+b}, y=\sqrt{ab}, z=\frac{a+b}{2}$; ***HM*** $\le$ ***GM*** $\le$ ***AM*** $\Rightarrow x\le y\le z$***. Also,*** $y^2=zx$

$$LHS=\left(\sum x\right)\left(\sum\frac{1}{x}\right)=\frac{(\sum x)(xy+yz+zx)}{xyz}$$

$$=\frac{(\sum x)(xy+yz+y^2)}{xyz}=\frac{(\sum x)^2}{zx}=\frac{(x+y+z)^2}{y^2}$$

$$RHS=\frac{2a^2+2b^2+5ab}{ab}=\frac{2(a+b)^2+ab}{ab}=1+\frac{8z^2}{y^2}$$

$$LHS\le RHS\Leftrightarrow\frac{(x+y+z)^2}{y^2}\le 1+\frac{8z^2}{y^2}$$

$$\Leftrightarrow (x+y+z)^2-y^2\le 8z^2$$

$$\Leftrightarrow (x+2y+z)(x+z) \le 8z^2$$

$$x+2y+z \overset{x\le y\le z}{\underset{=}{\le}} z+2z+z = 4z \text{ and } x+z \overset{x\le z}{\underset{=}{\le}} 2z$$

$$\Rightarrow (x+2y+z)(x+z) \le 8z^2$$

(Proved)

151. If $a,b,c>0, a+b+c=3, x=\sqrt{\frac{3-ab-bc-ca}{3}}$ then:

$$a^3+b^3+c^3+3abc \ge 6+8x^2-10x^3+\log(1+x^2-2x^3)$$

Proposed by Andrei Bâră – Romania

Solution 1 by Soumitra Mandal-Kolkata-India

Let $a+b+c=p, abc+bc+ca=q$ **and** $abc=r$

Then $q=3(1-x^2)$ ***and*** $0\le x<1$

$$\sum_{cyc} a^3+3r \ge 6+8x^2-10x^3+\log(1+x^2-2x^3)$$

$$\Leftrightarrow p^3-3pq+6r \ge 6+8x^2-10x^3+\log(1+x^2-2x^3)$$

$$\Leftrightarrow 27x^2+6r \ge 6+8x^2-10x^3+\log(1+x^2-2x^3)$$

$$\Leftrightarrow 6r \ge 6-19x^2-10x^3+\log(1+x^2-2x^3)$$

again from SCHUR'S INEQUALITY $p^3+9r\ge 4pq$

$\therefore 6r \ge 6-24x^2$ ***by putting values of*** p ***and*** q

we need to prove $6-24x^2 \ge 6-19x^2-10x^3+\log(1+x^2-2x^3)$

$$\Leftrightarrow 5(2x^3-x^2)-\log(1+x^2-2x^3) \ge 0 \ldots \textbf{(1)}$$

Let $f(x)=5(2x^3-x^2)-\log(1+x^2-2x^3)$ ***for all*** $0\le x<1$

Now, $f'(x)=\frac{(6x^2-2x)(6+5x^2-10x^3)}{1+x^2-2x^3}$

Again $f'(x) \leq 0$ ***for all*** $\frac{1}{2} \geq x \geq 0$ ***and*** $f'(x) \geq 0$ ***for all*** $1 > x \geq \frac{1}{2}$

$\therefore x = \frac{1}{2}$ ***is the global minimum and*** $f(x) \geq f\left(\frac{1}{2}\right) = 0$

$\therefore$ ***relation (1) is established hence***

$$\sum_{cyc} a^3 + 3abc \geq 6 + 8x^2 - 10x^3 + \log(1 + x^2 - 2x^3)$$

(proved)

Solution 2 by Andrei Bâră – Romania

$$a + b + c = 3 \Rightarrow 3(ab + bc + ca) \leq (a + b + c)^2 = 9 \Rightarrow ab + bc + ca \leq 3 \Rightarrow$$

$\Rightarrow ab + bc + ca = 3(1 - t^2)$, ***where*** $t \in [0, 1] \Rightarrow x = t$

So $a^2 + b^2 + c^2 = 3 + 6t^2 = 3(1 + 2t^2) \geq 3$

From Vo Quoc Ba Can's Theorem, it results that

$$(1 + t)^2(1 - 2t) \leq abc \leq (1 - t)^2(1 + 2t) \Leftrightarrow$$

$$\Leftrightarrow 1 - 3t^2 - 2t^3 \leq abc \leq 1 - 3t^2 + 2t^3$$

$$a^3 + b^3 + c^3 = (a + b + c)^3 - 3(a + b + c)(ab + bc + ca) + 3abc$$

$\Rightarrow a^3 + b^3 + c^3 = 27t^2 + 3abc$. ***Let*** $abc = r \Rightarrow a^3 + b^3 + c^3 = 3r + 27t^2$

$a^3 + b^3 + c^3 + 3abc \geq 6 + 8x^2 - 10x^3 + \ln(1 + x^2 - 2x^3)$ ***(1)***

(1) $\Leftrightarrow 6r + 27t^2 \geq 6 + 8t^2 - 10t^3 + \ln(1 + t^2 - 2t^3)$

Case I

$t \geq \frac{1}{2} \Rightarrow (1 + t)^2(1 - 2t) \leq 0$, ***but*** $a, b, c \geq 0 \Rightarrow r \geq 0 \Rightarrow 6r + 27t^2 \geq 27t^2$

$$27t^2 \geq 3t^2 + 24t^2 \geq 6 + 3t^2 \geq 6 + 8t^2 - 10t^3 + \ln(1 + t^2 - 2t^3) \Leftrightarrow$$

$$\Leftrightarrow 10t^3 \geq 5t^2 + \ln(1 + t^2 - 2t^3)$$

It remains to show that $10t^3 \geq 5t^2 + \ln(1 + t^2 - 2t^3)$

$$t \geq \frac{1}{2} \Rightarrow 1 + t^2 - 2t^3 \leq 1 \Rightarrow \ln(1 + t^2 - 2t^3) \leq 0 \Rightarrow$$

$$\Rightarrow 5t^2 + \ln(1 + t^2 - 2t^3) \le 5t^2 \le 10t^3 \Leftrightarrow t \ge \frac{1}{2}, \text{ true.}$$

Equality $\Leftrightarrow t = \frac{1}{2}, abc = 0 \Rightarrow a = 0, b = c = \frac{3}{2}$

Case II

$$t \in \left[0, \frac{1}{2}\right] \Rightarrow r \ge 1 - 3t^2 - 2t^3.$$

$$6r + 27t^2 \ge 6 + 9t^2 - 12t^3 \ge 6 + 8t^2 - 10t^3 + \ln(1 + t^2 - 2t^3) \Leftrightarrow$$

$\Leftrightarrow t^2 - 2t^3 \ge \ln(1 + t^2 - 2t^3)$. *Denote* $y = t^2 - 2t^3 \ge 0$. *Inequality becomes:*

$y \ge \ln(1 + y)$, *what is true for any* $y \ge 0$, *being a consequence of inequality:*

$e^y \ge y + 1$ *for any* y *real number, with equality* $\Leftrightarrow y = 0 \Rightarrow y \ge \ln(1 + y)$, *with equality* $\Leftrightarrow y = 0 \Leftrightarrow t = 0$, *or* $t = \frac{1}{2}$ *and* $r = 1 - 3t^2 - 2t^3 \Leftrightarrow a = b = c = 1$ *or* $a = 0, b = c = \frac{3}{2}$ *and its permutations.*

152. Let $x, y, z \ge 0$.

Prove that

$$\frac{x + y + z}{3} \ge \sqrt[3]{xyz} + k$$

$$k = \frac{3\left|\left(\sqrt[3]{x} - \sqrt[3]{y}\right)\left(\sqrt[3]{y} - \sqrt[3]{z}\right)\left(\sqrt[3]{z} - \sqrt[3]{x}\right)\right|}{4}$$

Proposed by Adil Abdullayev – Baku – Azerbaidjian

Solution by Rahim Shabazov-Baku-Azerbaidjian

$$\frac{x + y + z}{3} \ge \sqrt[3]{xyz} + k$$

$$k=\frac{3\left|\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\sqrt[3]{y}-\sqrt[3]{z}\right)\left(\sqrt[3]{z}-\sqrt[3]{x}\right)\right|}{4}$$

$$x=a^3,y=b^3,z=c^3,a,b,c>0$$

$$a^3+b^3+c^3\ge 3abc+\frac{9}{4}|(a-b)(b-c)(c-a)|\Rightarrow$$

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ac)\ge\frac{9}{4}\cdot|(a-b)(b-c)(c-a)|\Rightarrow$$

$$\Rightarrow(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\ge\frac{9}{2}\cdot|(a-b)(b-c)(c-a)|$$

$$\Rightarrow 2\cdot(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\ge 9\cdot|(a-b)(b-c)(c-a)|$$

$$2\cdot(a+b+c)\ge 0=a-b+b-c+c-a$$

$$[(a-b)+(b-c)+(c-a)]\cdot[(a-b)^2+(b-c)^2+(c-a)^2]\ge$$

$$\ge 9\cdot|(a-b)(b-c)(c-a)|$$

$$a-b=x,b-c=y,c-a=z$$

$$(x+y+z)(x^2+y^2+z^2)\ge 9\cdot|xyz|$$

153. Prove that if $a,b\in(0,1)$ then:

$$\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{a+2ab+b}}\le\left(\sqrt{ab}\right)^{\frac{1}{1+\sqrt{ab}}}\le\left(\frac{a+b}{2}\right)^{\frac{2}{a+b+2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Daniel Sitaru – Romania

$$\textit{Let be } f:(0,1)\to\mathbb{R};f(x)=\frac{\ln x}{x+1}$$

$$f'(x)=\frac{(\ln x)'(x+1)-\ln x\cdot(x+1)'}{(x+1)^2}=\frac{x+1-x\ln x}{(x+1)^2}$$

$$\textit{Let be } g:(0,1)\to\mathbb{R};g(x)=x+1-x\ln x$$

$$g'(x)=1-1-\ln x=-\ln x>0;(\forall)\,x\in(0,1)$$

g increasing on $(0,1)$

$$\inf g(x)=\lim_{\substack{x\to 0\\ x>0}} g(x)=\lim_{\substack{x\to 0\\ x>0}}(x+1-x\ln x)=$$

$$=1-\lim_{\substack{x\to 0\\ x>0}}\frac{\ln x}{\frac{1}{x}}=1+\lim_{\substack{x\to 0\\ x>0}} x=1>0$$

$f'(x)=\frac{g(x)}{(x+1)^2}>0\Rightarrow f$ ***increasing on*** $(0,1)$

$$0<a\le\frac{2}{\frac{1}{a}+\frac{1}{b}}\le\sqrt{ab}\le\frac{a+b}{2}\le b<1$$

from means inequality. It follows:

$$f\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right)\le f\left(\sqrt{ab}\right)\le f\left(\frac{a+b}{2}\right)$$

$$\frac{\ln\frac{2ab}{a+b}}{\frac{2ab}{a+b}+1}\le\frac{\ln\sqrt{ab}}{\sqrt{ab}+1}\le\frac{\ln\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}+1}$$

$$\ln\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab+a+b}}\le\ln\left(\sqrt{ab}\right)^{\frac{1}{1+\sqrt{ab}}}\le\ln\left(\frac{a+b}{2}\right)^{\frac{1}{\frac{a+b}{2}+1}}$$

$$\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{a+2ab+b}}\le\left(\sqrt{ab}\right)^{\frac{1}{1+\sqrt{ab}}}\le\left(\frac{a+b}{2}\right)^{\frac{2}{a+b+2}}$$

The equality holds if $a=b=c$.

Solution 2 by Soumitra Mandal – Kolkata – India

Let $A=\frac{a+b}{2}, B=\sqrt{ab}$ ***and*** $H=\frac{2}{\frac{1}{a}+\frac{1}{b}}$

Let $f(x)=x^{\frac{1}{x+1}}\Rightarrow\ln f(x)=\frac{\ln x}{x+1}$ ***for all*** $x\in(0,1)$

$$f'(x)=x^{\frac{1}{x+1}}\frac{x(1-\ln x)+1}{x(x+1)^2}>0 \text{ since } x\in(0,1)$$

so, f is continuous on $[0,1]$ and $f'(x)>0$ for all $x\in(0,1)$

so, f increasing hence $A,B,H\in[0,1]$

and $A\geq B\geq H\Rightarrow f(A)\geq f(B)\geq f(H)$

$$\therefore\left(\frac{a+b}{2}\right)^{\frac{2}{a+b+2}}\geq\left(\sqrt{ab}\right)^{\frac{1}{1+\sqrt{ab}}}\geq\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{a+2ab+b}} \text{ (proved)}$$

154. Let a,b,c be real number such that $a,b,c\geq\frac{1}{2}$ and $a+b+c=6$.

Prove that: $ab+bc+ca\geq 3\sqrt{abc+ab+bc+ca-4}$

Proposed at Hai Phong-Contest-Vietnam

Solution by Ngo Minh Ngoc Bao-Ho Chi Minh-Vietnam

We have lemma: Considering Polynomial

$$f(x,y,z)=\sum x^4+A\sum x^2y^2+Bxyz\sum x+C\sum x^3y+D\sum xy^3$$

(with A,B,C,D are the constant).

$$f(x,y,z)\geq 0\Leftrightarrow\begin{cases}1+A+B+C+D\geq 0\\3(1+A)\geq C^2+CD+D^2\end{cases},(\forall\, x,y,z\geq 0).$$

$$ab+bc+ca\geq 3\sqrt{abc+ab+bc+ca-4}\Leftrightarrow$$

$$\Leftrightarrow(ab+bc+ca)^2\geq 9abc+9(ab+bc+ca)-36$$

$$\Leftrightarrow\left(\sum ab\right)^2-\frac{3}{2}abc\sum a-\frac{1}{4}\left(\sum a\right)^2+\frac{(\sum a)^4}{36}\geq 0$$

$$\Leftrightarrow\frac{(\sum ab)^2}{2}-\frac{3}{2}abc\sum a-\frac{1}{4}\sum a^2\sum ab+\frac{\sum a^4}{36}+\frac{\sum a^3b}{9}+\frac{\sum ab^3}{9}+$$

$$+\frac{abc\sum a}{3}+\frac{\sum a^2b^2}{6}\geq 0$$

$$\Leftrightarrow \frac{\sum a^4}{36}+\frac{\sum a^2b^2}{2}-\frac{abc\sum a}{2}-\frac{\sum a^3b+\sum ab^3+abc\sum a}{4}+\frac{\sum a^3b}{9}+$$

$$+\frac{\sum ab^3}{9}+\frac{abc\sum a}{3}+\frac{\sum a^2b^2}{6}\geq 0$$

$$\Leftrightarrow \frac{\sum a^4}{36}+\frac{2\sum a^2b^2}{3}-\frac{5abc\sum a}{12}-\frac{5\sum a^3b}{36}-\frac{5\sum ab^3}{36}\geq 0 \quad (*)$$

Use lemma with $A=\frac{2}{3}, B=-\frac{5}{12}, C=D=-\frac{5}{36}$,

we have: $\begin{cases}1+A+B+C+D=1+\frac{2}{3}-\frac{5}{12}-\frac{5}{36}-\frac{5}{36}=\frac{35}{36}>0\\ 3(1+A)=5\geq C^2+CD+D^2=3\cdot\left(\frac{-5}{36}\right)^2 (true)\end{cases}\Rightarrow$

$\Rightarrow LHS(*)\geq RHS(*)$. *Equality occur when* $a=b=c=2$.

155. Let $a,b\in(0,\infty)$ and $+b=2$. Prove that:

$$(a+1)^a(b+1)^b+2ab\geq 6$$

Proposed by Richdad Phuc-Hanoi -Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

$$(a+1)^a(b+1)^b+2ab\geq 6\Rightarrow (a+1)^a(b+1)^b+2(1+a)(1+b)\geq 12$$

Let $f(x)=\frac{x+2}{3}\ln(1+x)$ *for all* $x\in(0,2)$ *then* $f''(x)=\frac{1}{3}\frac{x}{(1+x)^2}\geq 0$ *for all* $x\in(0,\infty)$. *Applying Jensen's Inequality,*

$$\frac{1}{2}\left\{\frac{a+2}{3}\ln(1+a)+\frac{b+2}{3}\ln(1+b)\right\}\geq\frac{\frac{a+b}{2}+2}{3}\ln\left(\frac{a+b}{2}+1\right)$$

$$\therefore (1+a)^{\frac{a+2}{3}}(1+b)^{\frac{b+2}{2}}\geq 4; \text{ applying A.M} \geq \text{G.M},$$

$$(1+a)^a(1+b)^b+2(1+a)(1+b)\geq 3(1+a)^{\frac{a+2}{3}}(1+b)^{\frac{b+2}{3}}\geq 12$$

hence, $(1+a)^a(1+b)^b+2ab\geq 6$ *(proved)*

156. Let $a, b, c > 0$ and $ab + bc + ca + abc = 4$ then

$$a^3 + b^3 + c^3 + abc \geq 4$$

Proposed by Richdad Phuc-Hanoi-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

Let $p = a + b + c, q = ab + bc + ca$ and $abc = r$

now, $q + r = 4 \Rightarrow \frac{p^2}{3} + \frac{p^3}{27} \geq 4 \Rightarrow p \geq 3$

$$\therefore a^3 + b^3 + c^3 + abc \geq 4 \Leftrightarrow p^3 - 3pq + 4r \geq 4$$

$$\Leftrightarrow p^3 - 3pq + 4(4 - q) \geq 4 \Leftrightarrow p^3 + 12 \geq q(3p + 4) \Leftrightarrow \frac{p^3 + 12}{3p + 4} \geq q$$

again, from Schur's Inequality, $p^3 + 9r \geq 4pq \Rightarrow p^3 + 9(4 - q) \geq 4pq$

$\Rightarrow \frac{p^3+36}{4p+9} \geq q$. Hence, we need to show that

$$\frac{p^3 + 12}{3p + 4} \geq \frac{p^3 + 36}{4p + 9} \Leftrightarrow 4p^4 + 9p^3 + 48p + 108 \geq 3p^4 + 4p^3 + 108p + 144$$

$$\Leftrightarrow p^4 + 5p^3 - 60p - 36 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow p^3(p - 3) + 8p^2(p - 3) + 24p(p - 3) + 12(p - 3) \geq 0$$

$\Leftrightarrow (p - 3)(p^3 + 8p^2 + 24p + 12) \geq 0$, which is true $\because p \geq 3$

$\therefore a^3 + b^3 + c^3 + abc \geq 4$ (proved)

157. If $0 < a_1 \leq 1 \leq a_2 \leq 2 \leq \cdots \leq 2015 \leq a_{2016} \leq 2016$ then:

$$\sum_{k=1}^{2016} \left(a_k + \frac{k^2}{a_k}\right) > 2016\left(2016 + \frac{1}{\sqrt[2016]{a_1 a_2 \cdot \ldots \cdot a_{2016}}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

For $k \geq 3$,

$a_k + \frac{k^2-1}{a_k} \geq 2\sqrt{k^2-1} > (2k-1)$. *Also,* $a_2 + \frac{2^2-1}{a_2} \geq 2\sqrt{3} > 1+3$. ***Now,***

$$\sum_{k=1}^{2016}\left(a_k + \frac{k^2}{a_k}\right) = \sum_{k=2}^{2016}\left(a_k + \frac{k^2-1}{a_k}\right) + \sum_{k=1}^{2016}\frac{1}{a_k}$$

$$> \sum_{k=1}^{2016}(2k-1) + 2010\left(\frac{1}{a_1a_2\dots a_k}\right)^{\frac{1}{2016}}$$

$$= 2016^2 + 2016\left(\frac{1}{a_1a_2\dots a_k}\right)^{\frac{1}{2016}}$$

158. 1. If $a, b > 0$ **then:** $b^b \cdot e^{a+\frac{1}{a}} \geq 2e^b$

2. If $a > 0, 0 < b \leq 1$ **then:** $b^b \cdot e^{1+\frac{1}{a}} \geq 2b \cdot e^b$

Proposed by Abdallah El Farissi-Bechar-Algerie

Solution by Soumitra Mandal - Chandar Nagore – India

Let $a, b > 0$ ***then*** $b^b \cdot e^{a+\frac{1}{a}} \geq 2e^b$

Now $b\ln b + a + \frac{1}{a} - \ln 2 - b = b\ln b + \left(a + \frac{1}{a} - 2\right) + 2 - \ln 2 - b$

$\geq b\ln b + \left(a + \frac{1}{a} - 2\right) + 2 + 1 - e^{\ln 2} - b$ ***since,*** $e^{\ln 2} \geq 1 + \ln 2$

$$\geq b(\ln b - 1) + \left(a + \frac{1}{a} - 2\right) + 1 \geq b\left(\frac{b-1}{b} - 1\right) + \left(a + \frac{1}{a} - 2\right) + 1$$

$$\because \ln(x+1) \geq \frac{x}{x+1} = a + \frac{1}{a} - 2 \geq 0$$

Hence, $b\ln b + a + \frac{1}{a} \geq \ln 2 + b \Rightarrow b^b \cdot e^{a+\frac{1}{a}} \geq 2e^b$ ***(proved)***

Let $a > 0, 0 < b \leq 1$ ***then*** $b^b \cdot e^{1+\frac{1}{a}} \geq (2e)^b$

Now, $b\ln b + a + \frac{1}{a} - b\ln 2 - b = b\ln b + 2 + \left(a + \frac{1}{a} - 2\right) - b\ln 2 - b$

$\geq b\ln b + 2 + \left(a + \frac{1}{a} - 2\right) + b\left(1 - e^{\ln 2}\right) - b$ *since,* $e^{\ln 2} \geq 1 + \ln 2$

$\geq b\left(\frac{b-1}{b}\right) + 2(1 - b) + \left(a + \frac{1}{a} - 2\right)$ *since,* $\ln(1 + x) \geq \frac{x}{x+1}$ *for all* $x \geq 0$

$= 1 - b + \left(a + \frac{1}{a} - 2\right) \geq 0 \because 0 < b \leq 1$ *and* $a + \frac{1}{a} \geq 2$

Hence, $b\ln b + a + \frac{1}{a} \geq b\ln 2 + b \Rightarrow b^b \cdot e^{a+\frac{1}{a}} \geq (2e)^b$ *(proved)*

159. If $a, b > 0$ then:

$$\sqrt{\frac{a^2 + b^2}{2}} + \frac{2}{\frac{1}{a} + \frac{1}{b}} \geq \frac{a + b}{2} + \sqrt{ab}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a, b > 0 \Rightarrow \sqrt{\frac{a^2 + b^2}{2}} + \frac{2}{\frac{1}{a} + \frac{1}{b}} \overset{(1)}{\geq} \frac{a + b}{2} + \sqrt{ab}$$

$$(1) \Leftrightarrow \sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab} \geq \frac{a+b}{2} - \frac{2ab}{a+b} \Leftrightarrow \frac{\frac{a^2+b^2}{2} - ab}{\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}} \geq \frac{(a+b)^2 - 4ab}{2(a+b)}$$

$$\Leftrightarrow \frac{(a-b)^2}{2\left(\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}\right)} - \frac{(a-b)^2}{2(a+b)} \geq 0 \Leftrightarrow (a-b)^2\left(\frac{1}{\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}} - \frac{1}{a+b}\right) \geq 0$$

$$\Leftrightarrow \frac{1}{\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}} - \frac{1}{a+b} \geq 0 (\because (a-b)^2 \geq 0) \Leftrightarrow a + b \geq \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}$$

$$\Leftrightarrow a^2+b^2+2ab \ge \frac{a^2+b^2}{2}+ab+2\sqrt{\frac{ab(a^2+b^2)}{2}}$$

$$\Leftrightarrow \frac{a^2+b^2}{2}+ab \ge 2\sqrt{\frac{ab(a^2+b^2)}{2}} \Leftrightarrow (a+b)^2 \ge 4\sqrt{\frac{ab(a^2+b^2)}{2}}$$

$$\Leftrightarrow (a+b)^4 \ge 8ab(a^2+b^2) \Leftrightarrow a^4+b^4+6a^2b^2 \ge 4a^3b+4ab^3$$

$$\Leftrightarrow (a^2+b^2)^2+(2ab)^2-2(a^2+b^2)(2ab) \ge 0$$

$$\Leftrightarrow (a^2+b^2-2ab)^2 \ge 0 \Leftrightarrow (a-b)^4 \ge 0 \to \textbf{\textit{true (Proved)}}$$

Solution 2 by Serban George-Florin-Romania

$$\sqrt{\frac{a^2+b^2}{2}}-\sqrt{a\cdot b} \ge \frac{a+b}{2}-\frac{2ab}{a+b}, \qquad \frac{\frac{a^2+b^2}{2}-a\cdot b}{\sqrt{\frac{a^2+b^2}{2}}+\sqrt{a\cdot b}} \ge \frac{(a+b)^2-4ab}{2\cdot(a+b)}$$

$$\frac{\frac{(a-b)^2}{2}}{\sqrt{\frac{a^2+b^2}{2}}+\sqrt{a\cdot b}} \ge \frac{(a-b)^2}{2\cdot(a+b)},$$

$$\frac{1}{\sqrt{\frac{a^2+b^2}{2}}+\sqrt{a\cdot b}} \ge \frac{1}{a+b}, \frac{1}{\sqrt{\frac{a^2+b^2}{2}}+\sqrt{a\cdot b}} \ge \frac{1}{a+b}$$

$$\sqrt{\frac{a^2+b^2}{2}}+\sqrt{a\cdot b} \le a+b, \frac{a^2+b^2}{2}+2\sqrt{\frac{a^2+b^2}{2}\cdot ab}+ab \le a^2+2ab+b^2,$$

$$2\sqrt{\frac{a^2+b^2}{2}\cdot ab} \le \frac{(a+b)^2}{2}, \qquad 16\frac{a^2+b^2}{2}\cdot ab \le (a+b)^4,$$

$$8ab\cdot(a^2+b^2) \le (a+b)^4, \quad 8ab\cdot(a^2+b^2) \le [(a^2+b^2)+2ab]^2,$$

$$8ab\cdot(a^2+b^2) \le (a^2+b^2)^2+4ab\cdot(a^2+b^2)+(2ab)^2,$$

$$0 \le (a^2+b^2)^2-4ab\cdot(a^2+b^2)+(2ab)^2,$$

$$[(a^2+b^2)-2ab]^2 \ge 0 \ \textbf{\textit{(A).}}$$

Solution 3 by Abdallah El Farissi-Bechar-Algerie

$$a, b > 0 \text{ By Cauchy – Schwarz we have } \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} \le a + b$$

$$\text{it follow that } \frac{\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}}{a+b} \le 1 \text{ then } \sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab} \ge \frac{\frac{a^2+b^2}{2}-ab}{a+b} = \frac{a+b}{2} - \frac{2}{\frac{1}{a}+\frac{1}{b}}$$

Solution 4 by Rovsen Pirguliev-Sumgait-Azerbaidjian

$$\text{Denote } \frac{y}{x} = k, \text{ we have: } \sqrt{\frac{1+k^2}{2}} + \frac{2k}{1+k} \ge \frac{1+k}{2} + \sqrt{k} \Leftrightarrow \sqrt{\frac{1+k^2}{2}} + \sqrt{k} \le 1 + k \Leftrightarrow$$

$$\Leftrightarrow \frac{1+k^2}{2} + 2\sqrt{k \cdot \frac{1+k^2}{2}} + k \le 1 + 2k + k^2 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{2k(1+k^2)} \le 1 + 2k + k^2 - k - \frac{1+k^2}{2} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{2k(1+k^2)} \le \frac{2k^2 + 2k + 2 - 1 - k^2}{2} \Leftrightarrow \sqrt{2k(1+k^2)} \le \frac{(k+1)^2}{2} \Leftrightarrow$$

$$\Leftrightarrow 2k(1+k^2) \le \frac{(k+1)^4}{4} \Leftrightarrow (k+1)^4 \ge 8k(1+k^2) \Leftrightarrow (k-1)^4 \ge 0$$

160. If $1 < x < y$ then:

$$x^{\frac{1}{x-1}} > y^{\frac{1}{y-1}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$\forall t > 1: \ln\left(\frac{1}{t}\right) + 1 \le \frac{1}{t} \Leftrightarrow \frac{1}{t}(t-1) - \ln t \le 0 \Leftrightarrow \frac{\frac{1}{t}(t-1) - \ln t}{(t-1)^2} \le 0 \Leftrightarrow$$

$$\frac{d\left(\frac{\ln t}{t-1}\right)}{dt} \le 0$$

$$\mathbf{1} < x < y \Rightarrow \int_1^x \frac{d\left(\frac{\ln t}{t-1}\right)}{dt}dt > \int_1^y \frac{d\left(\frac{\ln t}{t-1}\right)}{dt}dt \Leftrightarrow \frac{\ln x}{x-1} > \frac{\ln y}{y-1} \Leftrightarrow x^{\frac{1}{x-1}} > y^{\frac{1}{y-1}}$$

Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece

$$x^{\frac{1}{x-1}} < y^{\frac{1}{y-1}} \overset{\ln x\nearrow}{\Leftrightarrow} \frac{\ln x}{x-1} - \frac{\ln y}{y-1} > 0 \Leftrightarrow \left[\frac{\ln u}{u-1}\right]_y^x > 0 \Leftrightarrow \int_y^x \left(\frac{\ln u}{u-1}\right)' du > 0$$

$$\Leftrightarrow \int_y^x \frac{1-\frac{1}{u}-\ln u}{(y-1)^2} > 0 \overset{x<y}{\Leftrightarrow} \int_x^y \frac{1-\frac{1}{u}-\ln u}{(u-1)^2} du < 0$$

true, because $g(u) = 1 - \frac{1}{u} - \ln u;\ g'(u) = \frac{1}{u^2} - \frac{1}{u} = \frac{1-u}{y^2};\ g'(1) = 0$

u	0	1	$+\infty$
g'	$+ + + + + + + + +$	0	$- - - - - - - - -$
g	↗	0	↘

thus $g(u) \le 0$; "=" $u = 1$

Solution 3 by Ravi Prakash-New Delhi-India

Let $f(x) = \frac{\ln x}{x-1}, x > 1, f'(x) = \frac{1}{x(x-1)} - \frac{\ln x}{(x-1)^2} = \frac{x-1-x\ln x}{x(x-1)^2}$

Let $g(x) = x - 1 - x\ln x, x \ge 1, g'(x) = 1 - 1 - \ln x < 0\ \forall x > 1$

$\therefore g(x)$ ***decreases on*** $[1,\infty) \Rightarrow g(x) < g(1)\ \forall x > 1 \Rightarrow$

$\Rightarrow x - 1 - x\ln x < x\ \ \forall x > 1 \therefore f'(x) < 0, \quad \forall x > 0 \Rightarrow$

$\Rightarrow f(x)$ ***decreases on*** $(1,\infty)$***. Thus, if***

$$x < y, f(x) > f(y) \Rightarrow x^{\frac{1}{x-1}} > y^{\frac{1}{y-1}}\ \forall y > x > 1$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $1 < x < y$***, we get*** $0 < x - 1 < y - 1$ ***and*** $1 < x < y \rightarrow$

$\rightarrow 1^{(x-1)} < x^{(x-1)} < y^{(x-1)}$ ***and since*** $y^{(x-1)} < y^{(y-1)} \rightarrow$

$$\to 0 < 1^{(x-1)} < x^{(x-1)} < y^{(x-1)} \to \frac{1}{x^{(x-1)}} > \frac{1}{y^{(y-1)}}$$

Therefore, it is to be true.

161. ROMANIAN INEQUALITY – 1

In acute – angled $\Delta\ ABC, \omega -$ the Brocard angle:

$$\frac{R}{r} \geq \max\left\{\frac{1}{\sin\omega}, \frac{(a+b)(b+c)(c+a)}{16RS}, \frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\right\}$$

Proof by Soumava Chakraborty-Kolkata-India

Using Goldstone's inequality,

$$4R^2s^2 \geq \sum a^2b^2 \Rightarrow \frac{1}{2Rs} \leq \frac{1}{\sqrt{\sum a^2b^2}} \Rightarrow \frac{rs}{2Rs} \leq \frac{\Delta}{\sqrt{\sum a^2b^2}}$$

$$\Rightarrow \frac{2\Delta}{\sqrt{\sum a^2b^2}} \geq \frac{r}{R} \Rightarrow \sin\omega \geq \frac{r}{R} \Rightarrow \frac{R}{r} \geq \frac{1}{\sin\omega} \quad \textit{(1)}$$

Now, $\frac{R}{r} \geq \frac{(a+b)(b+c)(c+a)}{16RS}$ ***(2)***

$$\Leftrightarrow \frac{R}{r} \geq \frac{2abc + \sum ab\,(2s-c)}{16Rrs}$$

$$\Leftrightarrow 16R^2s \geq 8Rrs + 2s\left(\sum ab\right) - 12Rrs$$

$$\Leftrightarrow 8R^2 \geq s^2 + 4Rr + r^2 - 2Rr = s^2 + 2Rr + r^2$$

$$\Leftrightarrow s^2 \leq 8R^2 - 2Rr - r^2$$

Gerretsen $\Rightarrow s^2 \leq 4R^2 + 4Rr + 3r^2$

$\therefore$ ***to prove (2), it suffices to prove that:***

$$4R^2 + 4Rr + 3r^2 \leq 8R^2 - 2Rr - r^2$$

$$\Leftrightarrow 4R^2 - 6Rr - 4r^2 \geq 0 \Leftrightarrow 2R^2 - 3Rr - 2r^2 \geq 0$$

$\Leftrightarrow (R-2r)(2R+r) \geq 0 \to$ ***true,*** $\because R \geq 2r$ ***(Euler)***

$\Rightarrow$ ***(2) is true.***

Now, $\frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\frac{2\left(4ab^2+4bc^2+4ca^2\right)}{3\cdot 4abc}$

$$\leq \frac{b(a+b)^2+c(b+c)^2+a(c+a)^2}{3\cdot 2abc}\left(\because 4ab\underset{etc}{\leq}(a+b)^2\right)$$

$$=\frac{\left(\sum a^2b+\sum ab^2\right)+\sum ab^2+\sum a^3}{3\cdot 2abc}\leq\frac{\left(\sum a^3+3abc\right)+\sum ab^2+\sum a^3}{3\cdot 2abc}\quad \textit{(Schur)}$$

$$\leq\frac{\left(\sum a^3+3abc\right)+\sum a^3+\sum a^3}{3\cdot 2abc}$$

$$\left(\begin{array}{c} a^3+b^3+b^3\geq 3ab^2, b^3+c^3+c^3\overset{A-G}{\geq}3bc^2, c^3+a^3+a^3\overset{A-G}{\geq}3ca^2 \\ \Rightarrow 3\sum a^3\geq 3\sum ab^2\Rightarrow\sum ab^2\leq\sum a^3\end{array}\right)$$

$$=\frac{3\sum a^3+3abc}{3\cdot 2ab}=\frac{\sum a^3+abc}{2abc}$$

$$\therefore \frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\leq\frac{\sum a^3+abc}{2abc}\quad \textit{(3)}$$

Now $\frac{R}{r}\geq\frac{\sum a^3+abc}{2abc}$ ***(4)***

$$\Leftrightarrow\frac{R}{r}\geq\frac{\sum a^3-3abc+4abc}{2abc}\Leftrightarrow\frac{R}{r}\geq\frac{2s\left(\sum a^2-\sum ab\right)+16Rrs}{8Rrs}$$

$$\Leftrightarrow 4R^2\geq\sum a^2-\sum ab+8Rr\Leftrightarrow 4R^2\geq s^2-12Rr-3r^2+8Rr$$

$\Leftrightarrow s^2\leq 4R^2+4Rr+3r^2\rightarrow$ ***true by Gerretsen*** $\Rightarrow$ ***(4) is true***

(3) and (4) $\Rightarrow\frac{R}{r}\geq\frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ ***(5)***

(1), (2) and (5) $\Rightarrow$

$$\frac{R}{r}\geq\max\left\{\frac{1}{\sin\omega},\frac{(a+b)(b+c)(c+a)}{16RS},\frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\right\}$$

(Proved)

162. **Prove that, for positive a, b:**

$$\frac{a}{b\sqrt{2}}+\frac{b\sqrt{2}}{a}+2\left(\frac{\sqrt{a^2+b^2}}{b}+\frac{b}{a^2+b^2}\right)\ge\frac{9\sqrt{2}}{2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

In a right triangle with sides $a, b, \sqrt{a^2+b^2}$. Let θ be the acute angle opposite a: $a=\sqrt{a^2+b^2}\sin\theta$; $a: b=\sqrt{a^2+b^2}\cos\theta$. Then

$$LHS=\frac{1}{\sqrt{2}}\tan\theta+\sqrt{2}\cot\theta+2\cos\theta+2\sec\theta\overset{def}{=}f(\theta).$$

Solving $f'(\theta)=\frac{1}{\sqrt{2}\cos^2\theta}-\sqrt{2}\frac{1}{\sin^2\theta}-2\sin\theta+\frac{2\sin\theta}{\cos^2\theta}=0.$

we get successively $\frac{\sin^2\theta-2\cos^2\theta}{\sqrt{2}\cos^2\theta\sin^2\theta}-2\sin\theta\frac{\cos^2\theta-1}{\cos\theta}=0,$

$$\frac{3\sin^2\theta-2}{\sqrt{2}\cos^2\theta\sin^2\theta}+2\frac{\sin^3\theta}{\cos\theta}=0,\frac{2-3\sin^2\theta}{\sqrt{2}\cos^2\theta\sin^2\theta}=2\frac{\sin^3\theta}{\cos\theta},$$

$$2-3\sin^2\theta=2\sqrt{2}\sin^5\theta, 4+9\sin^4\theta-12\sin^2\theta=8\sin^{10}\theta.$$

With $t=\sin^2\theta>0, 8t^5-9t^4+12t-4=0,$

$(2t-1)(4t^4+2t^3+(t-2)^2)=0, t=\frac{1}{2}, \sin\theta=\frac{\sqrt{2}}{2}, \theta=\frac{\pi}{4}$. ***Now,***

$$f''(\theta)=\sqrt{2}\sec^2\theta\tan\theta+2\sqrt{2}\frac{\cot\theta}{\sin^2\theta}-2\cos\theta+2\sec^3\theta+2\tan^2\theta\sec\theta.$$

$f''\left(\frac{\pi}{4}\right)=11\sqrt{2}>0$, ***implying that $\theta=\frac{\pi}{4}$ is a minimum and that f never attends a maximum on $\left(0,\frac{\pi}{2}\right)$. Hence,***

$$f(\theta)\ge f\left(\frac{\pi}{2}\right)=4\sqrt{2}+\frac{1}{\sqrt{2}}=\frac{9\sqrt{2}}{2}.$$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

Note that $a\sin x+b\cos x\le\sqrt{a^2+b^2}$ ***so that***

$\frac{a\sin x}{\sqrt{a^2+b^2}}+\frac{b\cos x}{\sqrt{a^2+b^2}}\le 1$, ***with equality only when***

$a=\sqrt{a^2+b^2}\sin x$, $b=\sqrt{a^2+b^2}\cos x$, $x\in\left(0,\frac{\pi}{2}\right)$. ***The given inequality***

is equivalent to $f(x)=\frac{1}{\sqrt{2}}\tan x+\sqrt{2}\cot x+\frac{2}{\cos x}+2\cos x\ge\frac{9\sqrt{2}}{2}$.

where $f(x)=0$; $f'(x)=\frac{1}{\sqrt{2}\cos^2 x}+\frac{\sqrt{2}}{\sin^2 x}-\frac{2\sin x}{\cos^2 x}-2\sin x=0$,

or, $(2\sqrt{2}\sin^3 x-1)(\cos^2 x+1)$. $(2\sqrt{2}\sin^3 x-1)=0$ ***implies***

$$x=\frac{\pi}{4}\cdot f\left(\frac{\pi}{4}\right)=\frac{9\sqrt{2}}{2}.$$

Solution 3 by Su Tanaya-Kolkata-India

Using the AM-GM **inequality,**

$$3\cdot\frac{\frac{a}{b\sqrt{2}}+\frac{b}{\sqrt{a^2+b^2}}+\frac{b}{\sqrt{a^2+b^2}}}{3}+3\cdot\frac{\frac{b\sqrt{2}}{a}+\frac{\sqrt{a^2+b^2}}{b}+\frac{\sqrt{a^2+b^2}}{b}}{3}$$

$$=3\left(\frac{ab}{\sqrt{2}(a^2+b^2)}\right)^{\frac{1}{3}}+6\left(\frac{\sqrt{2}(a^2+b^2)}{8ab}\right)^{\frac{1}{3}}$$

$$=3\cdot 3\cdot\frac{\left(\frac{ab}{\sqrt{2}(a^2+b^2)}\right)^{\frac{1}{3}}+\left(\frac{\sqrt{2}(a^2+b^2)}{8ab}\right)^{\frac{1}{3}}+\left(\frac{\sqrt{2}(a^2+b^2)}{8ab}\right)^{\frac{1}{3}}}{3}$$

$$\ge 9\cdot\left[\left(\frac{ab}{\sqrt{2}(a^2+b^2)}\right)^{\frac{1}{3}}\left(\frac{\sqrt{2}(a^2+b^2)}{8ab}\right)^{\frac{2}{3}}\right]^{\frac{1}{3}}=9\cdot\left[\left(\frac{a^2+b^2}{ab}\right)^{\frac{1}{3}}\cdot(\sqrt{2})^{\frac{2}{3}-\frac{1}{3}-4}\right]^{\frac{1}{3}}\ge$$

$$\ge 9\cdot\left[2^{\frac{1}{3}}\cdot(\sqrt{2})^{\frac{1}{3}-4}\right]^{\frac{1}{3}}\quad \textbf{\textit{(because}}\ \frac{a^2+b^2}{2}\ge ab\textbf{\textit{)}}$$

$$= 9\cdot\left[\left(\sqrt{2}\right)^{\frac{2}{3}+\frac{1}{3}-4}\right]^{\frac{1}{3}} = 9\cdot\left(\sqrt{2}\right)^{-1} = \frac{9\sqrt{2}}{2}.$$

Solution 4 by Kunihiko Chikaya-Tokyo-Japan

$$LHS = \frac{1}{\sqrt{2}}\left(\frac{b}{a}+\frac{a}{b}\right) + 2\left(\frac{\sqrt{a^2+b^2}}{b}+\frac{b}{\sqrt{a^2+b^2}}\right) + \frac{1}{\sqrt{2}}\cdot\frac{b}{a} - \frac{2b}{\sqrt{a^2+b^2}}$$

$$\geq \frac{1}{\sqrt{2}}\cdot 2 + 2\cdot 2\sqrt{2} + \frac{1}{\sqrt{2}}\cdot\frac{b}{a} - \frac{2b}{\sqrt{2ab}} \text{ By The AM} - \text{GM inequality}$$

$$= 5\sqrt{2} + \frac{1}{\sqrt{2}}\left(\sqrt{\frac{b}{a}}-1\right)^2 - \frac{1}{\sqrt{2}} \geq \frac{9}{\sqrt{2}}$$

Equality: $\frac{b}{a} = \frac{a}{b} \wedge \frac{\sqrt{a^2+b^2}}{b} = \frac{2b}{\sqrt{a^2+b^2}} \wedge \sqrt{\frac{b}{a}} = 1 \ (a, b \in \mathbb{R}^+) \Leftrightarrow a = b > 0$

Solution 5 by Imad Zak-Saida-Lebanon

Homogeneous, so let $a^2 + b^2 = 4 \overset{AM-GM}{\geq} 2ab \Rightarrow ab \leq 2$ ***(1)***

The inequality becomes: $\frac{a}{b\sqrt{2}} + \frac{b\sqrt{2}}{a} + \frac{4}{b} + b \overset{??}{\geq} \frac{9\sqrt{2}}{4}$

$$LHS = \frac{a}{b\sqrt{2}} + \left(\frac{b\sqrt{2}}{2a}+\frac{b\sqrt{2}}{2a}\right) + \left(\frac{1}{b}+\frac{1}{b}+\frac{1}{b}+\frac{1}{b}\right) + \left(\frac{b}{2}+\frac{b}{2}\right)$$

$$\overset{AM-GM}{\geq} 9\cdot\sqrt[9]{\frac{ab^4\cdot 2}{b^5\cdot a^2 16\sqrt{2}}} = \frac{1}{8\sqrt{2}(ab)} \quad \text{use (1)}$$

$$\geq 9\sqrt[9]{\frac{1}{16\sqrt{2}}} = 2^{-\frac{9}{2}} = 9\cdot 2^{-\frac{1}{2}} = \frac{9\sqrt{2}}{2}. \text{ Q.E.D}$$

Solution 6 by Redwane El Mellas-Morocco

$$\text{Let } f(x>0) = \frac{x}{\sqrt{2}} + \frac{\sqrt{2}}{x} + 2\left(\sqrt{x^2+1}+\frac{1}{\sqrt{x^2+1}}\right)$$

$$\therefore f'(x) = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{x^2} + \frac{2x}{\sqrt{x^2+1}} - \frac{2x}{(x^2+1)\sqrt{x^2+1}} =$$

$$=\frac{x^2-2}{\sqrt{2x^2}}+\frac{2x^3}{(x^2+1)\sqrt{x^2+1}}=\frac{(x^2-2)(x^2+1)\sqrt{x^2+1}+2\sqrt{x^3}}{\sqrt{2x^2(x^2+1)\sqrt{x^3+1}}}=$$

$$=\frac{(x^2-2)^2(x^2+1)^3-8x^{10}}{\sqrt{2}\left((x^2-2)(x^2+1)\sqrt{x^2+1}-2\sqrt{2}x^3\right)x^2(x^2+1)\sqrt{x^2+1}}$$

$$=\frac{(x^4-4x^2+4)\left(x^5+3x^4+3x^2+1\right)-8x^{10}}{\sqrt{2}\left((x^2-2)(x^2+1)\sqrt{x^2+1}-2\sqrt{2}x^5\right)x^2(x^2+1)\sqrt{x^2+1}}$$

$$=\frac{-7x^{10}-x^5-5x^6+x^4+8x^2+4}{\sqrt{2}\left((x^2-2)(x^2+1)\sqrt{x^2+1}-2\sqrt{2}x^5\right)x^2(x^2+1)\sqrt{x^2+1}}$$

$$=\frac{(1-x)\left(7x\quad+7x+8x^7+8x^6+13x^5+13x^4+12x^3+12x^2+4x+4\right)}{\sqrt{2}\left((x^2-2)(x^2+1)\sqrt{x^2+1}-2\sqrt{2}x^5\right)x^2(x^2+1)\sqrt{x^2+1}}$$

Now let $g(x)=(x^2-2)(x^2+1)\sqrt{x^2+1}-2\sqrt{2}x^3$

If $0<x\leq 1$ ***clearly we have*** $g(x)<0$

If $x>1$: $\therefore\ x^2-2<2x^2\ \&\ \sqrt{x^2+1}<\sqrt{2x}\Rightarrow g(x)=2x^2(x^2+1)\sqrt{2}x-$

$-2\sqrt{2}x^5=2\sqrt{2}\left(x^6-x^5\right)<0$. ***So*** $g(x>0)<0$.

Then we get $f'(x>1)>0$, $f'(x<1)<0$ ***and*** $f'(1)=0$.

Finally $f(x>0)\geq f(1)=\frac{9\sqrt{2}}{2}$

As a special case $f\left(\frac{a}{b}\right)\geq\frac{9\sqrt{2}}{2}$ ***with equality if and only if*** $a=b$.

163. If $x\in\mathbb{R}$ then:

$$\left(\sqrt{x^2-x+1}-\sqrt{x^2+x+1}\right)^2+\left(\sqrt{x^2-x+1}-\sqrt{4x^2+3}\right)^2+\left(\sqrt{x^2+x+1}-\sqrt{4x^2+3}\right)^2<6x^2+2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\sqrt{x^2-x+1}-\sqrt{x^2+x+1}\right)^2+\left(\sqrt{x^2-x+1}-\sqrt{4x^2+3}\right)^2+\left(\sqrt{x^2+x+1}-\sqrt{4x^2+3}\right)^2\overset{(1)}{<}6x^2+2$$

$$\textbf{(1)}\Leftrightarrow 12x^2+10-(6x^2+2)$$

$$<2\sqrt{(x^2+x+1)(x^2+x+1)}+\sqrt{(x^2-x+1)(4x^2+3)}+$$

$$+\sqrt{(x^2+x+1)(4x^2+3)}\quad \textbf{(squaring (1))}$$

$$\Leftrightarrow (3x^2+4)^2<\{(x^2-x+1)(x^2+x+1)+(x^2-x+1)(4x^2+3)+(x^2+x+1)(4x^2+3)\}+$$

$$+2\left\{\begin{matrix}(x^2-x+1)\sqrt{(x^2+x+1)(4x^2+3)}+(4x^2+3)\sqrt{(x^2-x+1)(x^2+x+1)}+\\+(x^2+x+1)\sqrt{(4x^2+3)(x^2-x+1)}\end{matrix}\right\}\textbf{(squaring again)}$$

$$\Leftrightarrow (9x^4+24x^2+16)-(9x^4+15x^2+7)<$$

$$<2\left\{\begin{matrix}(x^2-x+1)\sqrt{(x^2+x+1)(4x^2+3)}+\\+(4x^2+3)\sqrt{(x^2-x+1)(x^2+x+1)}+\\+(x^2+x+1)\sqrt{(4x^2+3)(x^2-x+1)}\end{matrix}\right\}\quad \textbf{(2)}$$

$$\because x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\geq\frac{3}{4},4x^2+3\geq 3$$

$$\textbf{and } x^2-x+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\geq\frac{3}{4},$$

$$\therefore RHS\textbf{ of (2)}>2\left\{\begin{matrix}(x^2-x+1)\left(\frac{\sqrt{3}}{2}\right)\left(\sqrt{3}\right)+(4x^2+3)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\\+(x^2+x+1)\left(\sqrt{3}\right)\left(\frac{\sqrt{3}}{2}\right)\end{matrix}\right\}=$$

$$=3(2x^2+2)+\frac{3}{2}(4x^2+3)=12x^2+\frac{21}{2}>9x^2+9=LHS\textbf{ of (2)}$$

$$\Rightarrow \textbf{(2) is true (Proved)}$$

Solution 2 by Daniel Sitaru – Romania

$$a=\sqrt{4x^2+3};b=\sqrt{x^2-x+1};c=\sqrt{x^2+x+1}$$

$$a+b>c;a+c>b;b+c>a$$

In $\Delta\, ABC$ with sides a, b, c:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{-2x^2 - 1}{2\sqrt{x^4 + x^2 + 1}}$$

$$\sin A = \sqrt{\frac{3}{4(x^4 + x^2 + 1)}}$$

$$S = \frac{1}{2}bc\sin A = \frac{1}{2}\sqrt{x^4 + x^2 + 1}\cdot\frac{\sqrt{3}}{2\sqrt{x^4 + x^2 + 1}} = \frac{\sqrt{3}}{4}$$

By Hadwiger – Finsler's inequality:

$$\sum(a - b)^2 + 4S\sqrt{3} < a^2 + b^2 + c^2$$

By Hadwiger – Finsler's inequality:

$$\sum(a - b)^2 + 4S\sqrt{3} < a^2 + b^2 + c^2$$

$$\sum(a - b)^2 + 4\sqrt{3}\cdot\frac{\sqrt{3}}{4} < x^2 - x + 1 + x^2 + x + 1 + 4x^2 + 3$$

$$\sum(a - b)^2 < 6x^2 + 2$$

$$\left(\sqrt{x^2 - x + 1} - \sqrt{x^2 + x + 1}\right)^2 + \left(\sqrt{x^2 - x + 1} - \sqrt{4x^2 + 3}\right)^2 +$$

$$+\left(\sqrt{x^2 + x + 1} - \sqrt{4x^2 + 3}\right)^2 < 6x^2 + 2$$

164. If $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n > 0, n \in \mathbb{N}^*$ then:

$$\exp\left(\sum_{i=1}^{n}(x_i - y_i)\right) \ge \left(\frac{x_1}{y_1}\right)^{y_1}\cdot\left(\frac{x_2}{y_2}\right)^{y_2}\cdot\ldots\cdot\left(\frac{x_n}{y_n}\right)^{y_n}$$

Proposed by Abdallah El Farissi – Bechar – Algerie

Solution 1 by Ravi Prakash-New Delhi-India

Suppose $x > y$***, then*** $\frac{\ln x - \ln y}{x-y} = \frac{1}{t} < \frac{1}{y}$ ***for some*** t, $y < t < x$

[Lagsange's Mean value Th.] $\Rightarrow y\ln\left(\frac{x}{y}\right) < x - y \Rightarrow \left(\frac{x}{y}\right)^y < \exp(x-y)$

If $y > x$***, then*** $\frac{\ln x - \ln y}{x-y} = \frac{1}{t_1} < \frac{1}{x}$

$$[x < t_1 < y]$$

$$\Rightarrow x\ln\left(\frac{x}{y}\right) < x - y \Rightarrow \left(\frac{x}{y}\right)^x < \exp(x-y)$$

As $0 < \frac{x}{y} < 1$ ***and*** $y > x$

$$\left(\frac{x}{y}\right)^y < \left(\frac{x}{y}\right)^x < \exp(x-y)$$

For $x = y$, $\left(\frac{x}{y}\right)^y = \exp(x-y)$

Thus, $\exp(x-y) \ge \left(\frac{x}{y}\right)^y \quad \forall x, y > 0$

Take $x = x_i, y = y_i \quad (i = 1, 2, \dots, n)$

to obtain $\exp(x_i - y_i) \ge \left(\frac{x_i}{y_i}\right)^{y_i} \quad i = 1, 2, \dots, n$

$$\Rightarrow \prod_{i=1}^{n} \exp(x_i - y_i) \ge \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^{y_i} \Rightarrow \exp\left(\sum_{i=1}^{n}(x_i - y_i)\right) \ge \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^{y_i}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\exp\left(\sum_{i=1}^{n}(x_i - y_i)\right) \ge \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^{y_i}$$

We know $\frac{x}{1+x} \le \ln(1+x) \le x$ ***for all*** $x \in [0, \infty)$

Case 1: Let $x_i \ge y_i$ ***for all*** $i = 1, 2, 3, \dots, n$

$\therefore \frac{x_i}{y_i} \in [1, \infty)$ ***and replacing*** x ***by*** $x - 1$ ***in the statement***

$x \geq \ln(1 + x)$***. Then we have*** $x - 1 \geq \ln x$ ***where*** $x \in [1, \infty)$

putting $x = \frac{x_i}{y_i}$ ***where*** $i = 1, 2, 3, \dots, n$ ***in the statement we have***

$$\frac{x_i}{y_i} - 1 \geq \ln \frac{x_i}{y_i}$$

$$\Rightarrow x_i - y_i \geq \ln\left(\frac{x_i}{y_i}\right)^{y_i} \Rightarrow \sum_{i=1}^{n}(x_i - y_i) \geq \sum_{i=1}^{n} \ln\left(\frac{x_i}{y_i}\right)^{y_i}$$

$$\Rightarrow \sum_{i=1}^{n}(x_i - y_i) \geq \ln \prod_{i=1}^{n}\left(\frac{x_i}{y_i}\right)^{y_i} \Rightarrow \exp\left(\sum_{i=1}^{n}(x_i - y_i)\right) \geq \prod_{i=1}^{n}\left(\frac{x_i}{y_i}\right)^{y_i}$$

(Proved)

Case 2: Let $y_i \geq x_i$ ***for all*** $i = 1, 2, 3, \dots, n$

then we need to prove,

$$\prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{y_i} \geq \exp\left(\sum_{i=1}^{n}(y_i - x_i)\right)$$

we know that $\ln(1 + x) \geq \frac{x}{1+x}$ ***for all*** $x \in [0, \infty)$***. Now replacing***

x ***by*** $x - 1$ ***we have,*** $\ln x \geq 1 - \frac{1}{x}$ ***for all*** $x \in [1, \infty)$***. Now putting*** $x = \frac{y_i}{x_i}$

in the statement we have, $\ln \frac{y_i}{x_i} \geq 1 - \frac{x_i}{y_i} \Rightarrow \ln\left(\frac{y_i}{x_i}\right)^{y_i} \geq y_i - x_i$

$$\Rightarrow \sum_{i=1}^{n} \ln\left(\frac{y_i}{x_i}\right)^{y_i} \geq \sum_{i=1}^{n}(y_i - x_i) \Rightarrow \prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{y_i} \geq \exp\left(\sum_{i=1}^{n}(y_i - x_i)\right)$$

(proved)

Case 3: for $m < n$ ***let us assume*** $x_i \geq y_i$ ***where*** $i = 1, 2, 3, \dots, m$ ***and***

$y_j \geq x_j$

where $j = m+1, m+2, m+3, \dots, n$. Then applying the same process as done in Case 1 and Case 2.

Solution 3 by Marin Dincă – Romania

The inequality is equivalent to: $\ln\left(e^{\sum_{i=1}^{n}(x_i-y_i)}\right) \ge \ln \prod_{i=1}^{n}\left(\frac{x_i}{y_i}\right)^{y_i}$

or, $\sum_{i=1}^{n}(x_i-y_i) \ge \sum_{i=1}^{n} y_i(\ln x_i - \ln y_i)$

the function $f(x) = \ln x$, is concave, result:

$f(x) - f(y) \le (x-y)f'(y)$, ***or***

$\ln x - \ln y \le (x-y)\frac{1}{y}$, ***or:*** $y(\ln x - \ln y) \le x - y$ ***(1)***

for $x = x_i; y = y_i$ result:

$$y_i(\ln x_i - \ln y_i) \le x_i - y_i \Rightarrow \sum_{i=1}^{n} y_i(\ln x_i - \ln y_i) \le \sum_{i=1}^{n}(x_i - y_i)$$

165. **Let f, g, h be continuously differentiable functions on $(0,1)$ so that:**

i) $\forall x \in [0,1], 0 < g(x) \le f(x) \le h(x)$

ii) **g and h both have fixed points on $[0,1]$.**

iii) $f(0) = 0$

iv) $\forall x \in (0,1), x < f(x) < 1$

Prove that there are n distinct numbers $\alpha_i \in (0,1)$ with $i = 1,2,\dots,n$ such that

$$\sum_{i=1}^{n}\left(f'(\alpha_i) - \sqrt{f'(\alpha_i)}\right) > 0.$$

Proposed by by Anas Adlany-El Jadida-Morocco

Solution by proposer

First, we will prove three claims as a start.

Claim 0.1 For any $a_i > 0$ ($i = 1, \dots, n$) and $a_1 a_2 \dots a_n = 1$, we have

$$\sum_{i=1}^{n}\left(a_i-\sqrt{a_i}\right)>0$$

Proof. we have by AM-GM inequality:

$$\sum_{i=1}^{n}a_i\geq n \qquad (1)$$

hence, by Cauchy Schwartz's inequality and (1), we then have

$$\left(\sum_{i=1}^{n}a_i\right)^2\geq n\cdot\sum_{i=1}^{n}a_i=(1+1+\cdots+1+1)\cdot\sum_{i=1}^{n}a_i\geq\left(\sum_{i=1}^{n}a_i\right)^2$$

and this ends the proof.

Claim 0.2. f has a fixed point $(\neq 0)$.

Proof.

Let $\phi(x)=f(x)-x$ and define the fixed points of g and h as follows $h(a)=a$ and $g(b)=b$. Thus, since ϕ is continuous on $[0,1]$ and

$$\phi(a)\cdot\phi(b)=(f(a)-a)(f(b)-b)=\big(f(a)-h(a)\big)\big(f(b)-g(b)\big)<0$$

hence f also has a fixed point $(\neq 0)$.

Claim 0.3. There are n distinct numbers $a_i\in(0,1)$ with $i=1,2,\dots,n$ such that

$$\prod_{i=1}^{n}f'(\alpha_i)=1\ \left(\forall i\neq j;\ \alpha_i\neq\alpha_j\right).$$

Proof. Let $\psi(x)=f_n(x)-x$ with $f_n(x)=f\left(f\big(f(\dots f(x)..)\big)\right)$ [n - times]

Now since ψ is continuously defferentiable function and $\psi(0)=\psi(\gamma)$ (γ is the fixed point of f). Therefore, there is $\beta\in(0,1)$ such that

$$\psi'(\beta) = 0 \Leftrightarrow \prod_{i=1}^{n} f'(\alpha_i) = 1$$

with** $a_i = f_i(\beta)$ **and because of iv) the** $a_i's$ **must be distincts.

Now, let's go back to our problem and put** $a_i = f'(\alpha_i)$ **in Claim 0.1 we get the desired result immediately, and we are done.

166. In any scalene acute – angled $\Delta\ ABC$:

$$\sqrt{\sum (\sin A)^{2\cos A}} + \sqrt{\sum (\cos A)^{2\sin A}} > \frac{\sqrt{3}}{3}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India

***In any scalene acute – angled** $\Delta\ ABC$,*

$$\sqrt{\sum (\sin A)^{2\cos A}} + \sqrt{\sum (\cos A)^{2\sin A}} > \sqrt{3}$$

$$\{(\sin A)^{\cos A}\}^2 + \{(\sin B)^{\cos B}\}^2 + \{(\sin C)^{\cos C}\}^2$$

$$> \frac{1}{3}(\sin A^{\cos A} + \sin B^{\cos B} + \sin C^{\cos C})^2$$

$$\left(\because \sum x^2 > \frac{1}{3}\left(\sum x\right)^2, if\ x \neq y \neq z\right)$$

$$\therefore \sqrt{\sum (\sin A)^{2\cos A}} > \frac{1}{\sqrt{3}}\left(\sin A^{\cos A} + \sin B^{\cos B} + \sin C^{\cos C}\right) \quad \textbf{(1)}$$

***Again,** $\{(\cos A)^{\sin A}\}^2 + \{(\cos B)^{\sin B}\}^2 + \{(\cos C)^{\sin C}\}^2$*

$$> \frac{1}{3}\left(\cos A^{\sin A} + \cos B^{\sin B} + \cos C^{\sin C}\right)^2$$

$$\therefore \sqrt{\sum (\cos A)^{2\sin A}} > \frac{1}{\sqrt{3}}\left(\cos A^{\sin A} + \cos B^{\sin B} + \cos C^{\sin C}\right) \quad \textbf{(2)}$$

*($\because$ $\Delta\ ABC$ **is acute – angled,** $\therefore \cos A, \cos B, \cos C > 0$)*

$$\textbf{(1)} + \textbf{(2)} \Rightarrow LHS \overset{(i)}{>}$$

$$> \frac{1}{\sqrt{3}}\{(\cos A^{\sin A} + \sin A^{\cos A}) + (\cos B^{\sin B} + \sin B^{\cos B}) + (\cos C^{\sin C} + \sin C^{\cos C})\}$$

Now, $(\ln \sin A)(\cos A - 2) > 0$

($\because \ln \sin A < 0$ ***as*** $\sin A < 1$ ***and*** $\cos A - 2 < 0$ ***as*** $\cos A < 1 < 2$***)***

$$\Rightarrow \cos(\ln \sin A) > 2\ln \sin A$$

$$\Rightarrow \ln(\sin A^{\cos A}) > \ln(\sin^2 A) \Rightarrow \sin A^{\cos A} > \sin^2 A \quad \textbf{(3)}$$

Also, $(\ln \cos A)(\sin A - 2) > 0$

($\because \ln \cos A < 0$ ***as*** $\cos A < 1$ ***and*** $\sin A - 2 < 0$ ***as*** $\sin A < 1 < 2$***)***

$$\Rightarrow \sin A\,(\ln \cos A) > 2\ln \cos A$$

$$\Rightarrow \ln\left(\cos A^{\sin A}\right) > \ln(\cos^2 A) \Rightarrow \cos A^{\sin A} > \cos^2 A \quad \textbf{(4)}$$

$$\textbf{(3)} + \textbf{(4)} \Rightarrow \cos A^{\sin A} + \sin A^{\cos A} \overset{5}{>} \cos^2 A + \sin^2 A = 1$$

Similarly, $\cos B^{\sin B} + \sin B^{\cos B} > 1$ ***(6)***

and, $\cos C^{\sin C} + \sin C^{\cos C} > 1$ ***(7)***

(5) + ***(6)*** + ***(7)*** $\Rightarrow LHS > \frac{1}{\sqrt{3}}(1 + 1 + 1)$ ***(from (i))*** $= \frac{3}{\sqrt{3}} = \sqrt{3}$ ***(Proved)***

167. If $a, b, c > 0, m \geq 0$ **then:**

$$\frac{a}{(b+c)^{m+1}} + \frac{b}{(c+a)^{m+1}} + \frac{c}{(a+b)^{m+1}} \geq \frac{3^{m+1}}{2^{m+1}(a+b+c)^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Mihalcea Andrei Ștefan – Romania

$$\sum \frac{a}{(b+c)^{m+1}} = \sum \frac{a^{m+2}}{(ab+ac)^{m+1}} \overset{Radon}{\geq} \frac{(\sum a)^{m+2}}{2^{m+1}(\sum ab)^{m+1}} \geq \frac{3^{m+1}}{2^{m+1}(\sum a)^m}$$

$$\Leftrightarrow (\sum a)^{m+3} \geq 3^{m+1}(\sum ab)^{m+1} \Leftrightarrow (\sum a)^2 \geq 3\sum ab. \ \textit{true.}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$LHS=\frac{a}{b+c}\cdot\frac{1}{(b+c)^m}+\frac{b}{c+a}\cdot\frac{1}{(c+a)^m}+\frac{c}{a+b}\cdot\frac{1}{(a+b)^m}$$

WLOG, we may assume $a\ge b\ge c$. ***Then*** $\frac{a}{b+c}\ge\frac{b}{c+a}\ge\frac{c}{a+b}$ ***and***

$$\frac{1}{(b+c)^m}\ge\frac{1}{(c+a)^m}\ge\frac{1}{(a+b)^m}$$

$$\therefore LHS\ge\frac{1}{3}\left(\sum\frac{a}{b+c}\right)\sum\frac{1}{(b+c)^m}\ \textit{(Chebyshev)}$$

$$\overset{Nesbitt}{\ge}\frac{1}{3}\cdot\frac{3}{2}\left\{\frac{1^{m+1}}{(b+c)^m}+\frac{1^{m+1}}{(c+a)^m}+\frac{1^{m+1}}{(a+b)^m}\right\}\ge$$

$$\overset{Radon}{\ge}\frac{1}{2}\cdot\frac{3^{m+1}}{(2\sum a)^m}=\frac{3^{m+1}}{2^{m+1}(a+b+c)^m}\ \textit{(Proved)}$$

Solution 3 by Abdallah El Farissi – Bechar – Algerie

Let $f(x)=\frac{x}{(a+b+c-x)^{m+1}},\ x<a+b+c$. ***We have***

$f''(x)=\frac{(m+1)(2(a+b+c)+mx)}{(a+b+c-x)^{m+3}}\ge 0$ ***then f is convex function and***

$$\frac{a}{(b+c)^{m+1}}+\frac{b}{(a+c)^{m+1}}+\frac{c}{(a+b)^{m+1}}=f(a)+f(b)+f(c)\ge$$

$$\ge 3f\left(\frac{a+b+c}{3}\right)=\frac{3^{m+1}}{2^{m+1}(a+b+c)^m}$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

If $a,b,c>0;\ m\ge 0$ ***then:***

$$I=\frac{a}{(b+c)^{m+1}}+\frac{b}{(c+a)^{m+1}}+\frac{c}{(a+b)^{m+1}}\ge\frac{3^{m+1}}{2^{m+1}\cdot(a+b+c)^m}$$

$$\left.\begin{array}{c}a\ge b\ge c\\ \frac{1}{(b+c)^{m+1}}\ge\frac{1}{(c+a)^{m+1}}\ge\frac{1}{(a+b)^{m+1}}\end{array}\right\}\textit{Chebyshev}$$

$$I \geq \frac{1}{3} \cdot (a+b+c) \cdot \left(\sum \frac{1}{(a+b)^{m+1}}\right) \overset{Cauchy}{\geq}$$

$$\geq \frac{1}{3} \cdot (a+b+c) \cdot \frac{3}{\left(\sqrt[3]{(a+b)\cdot(b+c)\cdot(c+a)}\right)^{m+1}} \overset{Cauchy}{\geq}$$

$$\geq (a+b+c) \cdot \frac{1}{\left(\frac{a+b+b+c+c+a}{3}\right)^{m+1}} = \frac{3^{m+1}}{2^{m+1} \cdot (a+b+c)^m}$$

168. **Prove that if $a \in (0,\infty), n \in N, k = 1,2,\dots,n$, then the following inequalities hold:**

i) $2\prod_{i=1}^{n}(e^k - 1) \geq e^{\frac{n(n-1)}{2}}$ **and**

ii) $\prod_{i=1}^{n}(e^k + a^k - 2) \geq 2^{n-1} \cdot a^{\frac{11n^2-14n-1}{24}}$

Proposed by Anas Adlany - El Jadida – Morroco

Solution 1 by Said Ibnja – Marrakesh – Morocco

i. Define f on $(0,\infty)$ so that:

$$(\forall n \in N^*); f(x) = \frac{e^x}{\sqrt[n]{2}} - e^{x+1} + 1$$

Now, since f is derivable on $(0,\infty)$ we then find that

$$f'(x) = e^x\left(\frac{1}{\sqrt[n]{2}} - e\right) < 0$$

that is $\forall x > 0, f(x) < f(0) < 0 \Rightarrow \forall x > 0, \frac{e^x}{\sqrt[n]{2}} < e^{x+1} - 1.$

Now, let's go back to our problem and set $x = k-1$ for any $k = 2,3,\dots,n+1$ and then we get after multiplying these inequalities the desired result, and we are done.

Solution 2 by proposer

we will prove ii) and then we conclude the second.

We begin by recalling some well-known results.

1. $\forall x \geq 0, e^x \geq x + 1.$

2. $(\forall\, k \in N^*), a^k + k - 1 \geq ka^{k-1}$

3. $(\forall n \in N), 2(1 + 2 + 3 + \cdots + n) = n(n + 1)$

4. $\forall x > 0, \sin(x) < x$

5. $\sum_{k=1}^{n} \sin\left(\frac{k}{n}\right) > \frac{(11n-1)(n+1)}{24n}$***. Now, using 1. and 2. to get that***

$$e^k + a^k - 2 \geq a^k + k - 1 \geq ka^{k-1}$$

thus,

$$\prod_{k=1}^{n}(e^k + a^k - 2) \geq n!\, a^{\sum_{k=1}^{n} k-n} \geq 2^{n-1} \cdot a^{\frac{11n^2-14n-1}{24}}$$

where the last step follows from 3., 4., 5. and the fact that $n! \geq 2^{n-1}$***,***

hence proved

ii). It suffices to take $a = e$ ***in the previous inequality and the desired result follows immediately .***

169. If $a, b, c > 0$ then:

$$\frac{(\sqrt{a}+\sqrt{b})^2}{4} + \frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}{9} + \frac{(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d})^2}{16} < 4(a + b + c + d)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by SK Rejuan-West Bengal-India

$$a, b, c > 0$$

$$LHS=\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{4}+\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{9}+\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2}{10}=P$$

$$<\left(\frac{a+b}{2}\right)+\left(\frac{a+b+c}{3}\right)+\left(\frac{a+b+c+d}{4}\right)$$

By Cauchy inequality,

$$\left(\sqrt{a}+\sqrt{b}\right)^2<2(a+b)\Rightarrow\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{4}<\frac{1}{2}(a+b)$$

Similarly we get the others

$$\Rightarrow P<a\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+b\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+c\left(\frac{1}{3}+\frac{1}{4}\right)+d\frac{1}{4}=$$

$$=\frac{13}{12}a+\frac{13}{12}b+\frac{7}{12}c+\frac{1}{4}d$$

$$<4a+4b+4c+4d=4(a+b+c+d)\Rightarrow P<4\sum a$$

i.e. $\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{4}+\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{9}+\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2}{16}<4(a+b+c+d)$

(proved)

Solution 2 by Nirapada Pal-India

We know, $\frac{\sum_{i=1}^{n}a_i^m}{n}>\left(\frac{\sum_{i=1}^{n}a_i}{n}\right)^m$ ***for*** $m>1$

$$\therefore\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{4}+\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{9}+\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2}{16}=$$

$$=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2+\left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3}\right)^2+\left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}}{4}\right)^2<$$

$$<\frac{a+b}{2}+\frac{a+b+c}{3}+\frac{a+b+c+d}{4}$$

$$=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)a+\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)b+\left(\frac{1}{3}+\frac{1}{4}\right)c+\frac{1}{4}d<$$

$$< 4a + 4b + 4c + 4d = 4(a + b + c + d)$$

170. **If $x, y, z, t \in \mathbb{R}, x + y + z + t = 0$ then:**

$$2^x + 2^y + 2^z + 2^t + 8 \geq 3\left(\frac{1}{\sqrt[3]{2^x}} + \frac{1}{\sqrt[3]{2^y}} + \frac{1}{\sqrt[3]{2^z}} + \frac{1}{\sqrt[3]{2^t}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Shahlar Maharramov-Jebrail-Azerbaidian

Take $2^x = a, 2^y = b, 2^z = c, 2^x = d \Rightarrow$

$$x = \log_2 a\,, y = \log_2 b\,, z = \log_2 c\,, t = \log_2 d$$

$$\Rightarrow \log_2 a + \log_2 b + \log_2 c + \log_2 d = 0 \Rightarrow$$

$$\Rightarrow \log_2 abcd = 0 \Rightarrow abcd = 1$$

Then we have to prove if $abcd = 1$ ***then***

$$a + b + c + d + 8 \geq 3\left(\frac{1}{\sqrt[3]{a}} + \frac{1}{\sqrt[3]{b}} + \frac{1}{\sqrt[3]{c}} + \frac{1}{\sqrt[3]{d}}\right)$$

$$LHS\ \ a + b + c + d + 8 \geq 4\underbrace{\sqrt[4]{abcd}}_{1} + 8 = 12$$

$$RHS = 3\left(\frac{1}{\sqrt[3]{a}} + \frac{1}{\sqrt[3]{b}} + \frac{1}{\sqrt[3]{c}} + \frac{1}{\sqrt[3]{d}}\right) \overset{A-H}{\leq} 3\frac{16}{4\sqrt[4]{(abcd)^{\frac{1}{3}}}} = 12$$

$$\Rightarrow RHS \leq 12, \textbf{but } LHS \geq 12$$

Solution 2 by Marian Dincă – Romania

Apply Popoviciu inequality for $n = 4$ ***and the function convex*** $f(x) = 2^x$***,***

we obtain: $f(x) + f(y) + f(z) + f(t) + 4(4-2)f\left(\frac{x+y+z+t}{4}\right) \geq$

$$\sum_{cycl} 3f\left(\frac{x+y+z}{3}\right)$$

Solution 3 by Uche Eliezer Okeke-Nigeria

$$*\sum 2^x + 8 \geq 3\sum 2^{-\frac{x}{3}}$$

$$**\sum\left\{(3)2^{-\frac{x}{3}} - 2^x\right\} \leq 8$$

$$f(x) = (3)2^{-\frac{x}{3}} - 2^x;\ f''(x) \leq 0\ \forall x > 0$$

$$\Rightarrow \sum\left\{(3)2^{-\frac{x}{3}} - 2^x\right\} \overset{Jensen}{\leq} (4)(3)2^{-\frac{1}{3}\left(\frac{\sum x}{4}\right)} - 4\cdot 2^{\frac{\sum x}{4}} = 12 - 4 = 8$$

171. If $0 \leq x, y, z < \frac{\pi}{2}$ then:

$$2^{\sin x} + 2^{\sin y} + 2^{\sin z} + 2^{\tan x} + 2^{\tan y} + 2^{\tan z} \geq 6\sqrt[3]{2^{x+y+z}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaidian

It is known that $\sin x + \tan x \geq 2x,\ 0 \leq x < \frac{\pi}{2}$ ***(*)***

Then, we have $\left.\begin{array}{l} 2^{\sin x} + 2^{\tan x} \geq 2\cdot\sqrt{2^{\sin x+\tan x}} \\ 2^{\sin y} + 2^{\tan y} \geq 2\cdot\sqrt{2^{\sin y+\tan y}} \\ 2^{\sin z} + 2^{\tan z} \geq 2\cdot\sqrt{2^{\sin z+\tan z}} \end{array}\right\}\oplus$

$$LHS \geq 2\left(\sqrt{2^{\sin x+\tan x}} + \sqrt{2^{\sin y+\tan y}} + \sqrt{2^{\sin z+\tan z}}\right) \geq 2(2^x + 2^y + 2^z)$$

Hence $LHS \geq 2(2^x + 2^y + 2^z) \geq 2\cdot 3\sqrt[3]{2^{x+y+z}}$

Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece

$f(x) = \sin x + \tan x$***, convex for*** $x \in \left(0, \frac{\pi}{2}\right)$ ***tangen line*** $x_0 = 0$

$$y = f'(0)\cdot x + f(0);\ y = 2x;\ f'(x) = \cos x + \frac{1}{\cos^2 x}$$

$$f'(0) = 1 + 1 = 2$$

$f'(0) = 0 \Rightarrow f(x) \geq y\ \ \forall x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin x + \tan x \geq 2x$ ***(1)***

$$LHS = 2^{\sin x} + 2^{\sin y} + 2^{\sin z} + 2^{\tan x} + 2^{\tan y} + 2^{\tan z}$$

$$\overset{AM-GM}{\geq} 6 \cdot \sqrt[6]{2^{\sin x+\sin y+\sin z+\tan x+\tan y+\tan z}}$$

$$\overset{(1)}{\geq} 6 \cdot \sqrt[6]{2^{2x+2y+2z}} = 6 \cdot \sqrt[6]{2^{2(x+y+z)}} = 6 \cdot \sqrt[3]{2^{x+y+z}}$$

$$\text{“=”}\ x = y = z$$

Solution 3 by Dimitris Kastriotis-Athens-Greece

$$\sum\left(2^{\sin x} + 2^{\tan x}\right) \geq 6\sqrt[6]{2^{\sin x+\sin y+\sin y+\tan x+\tan y+\tan z}} \quad \textit{(1)}$$

$$\text{Let } f(t) = \cos t + \frac{1}{\cos^2 t}, t \in \left[0, \frac{\pi}{2}\right)$$

$$\min_{t\in\left[0,\frac{\pi}{2}\right)} f(t) = f(0) = 2 \Rightarrow \cos t + \frac{1}{\cos^2 t} \geq 2 \quad \forall t \in \left[0, \frac{\pi}{2}\right) \Rightarrow$$

$$\Rightarrow \int_0^x \left(\cos t + \frac{1}{\cos^2 t}\right) dx \geq 2\int_0^x dx \Rightarrow$$

$$\Rightarrow \sin x + \tan x \geq 2x \quad \forall x \in \left[0, \frac{\pi}{2}\right) \quad \textit{(2)}$$

$$\begin{Bmatrix}(1)\\(2)\end{Bmatrix} \Rightarrow \sum\left(2^{\sin x} + 2^{\tan x}\right) \geq 6\sqrt[6]{2^{2x+2y+2z}} = 6\sqrt[3]{2^{x+y+z}}$$

172. In $ABCD$ convexe quadrilateral: $AB = a, BC = b, CD = c, DA = d$.

Prove that:

$$\sum\sqrt{a^2 + b^2 + c^2} > 2\sqrt{3 \cdot AC \cdot BD}$$

Proposed by Daniel Sitaru – Romania

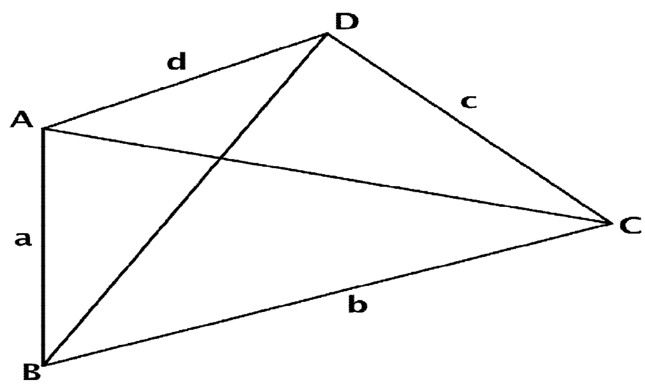
Solution 1 by SK Rejuan-West Bengal-India

$$(a^2 + b^2 + c^2) > \frac{1}{3}(a + b + c)^2 \quad \textit{[by mth power theorem]}$$

$$\Rightarrow \sqrt{a^2+b^2+c^2} > \frac{1}{\sqrt{3}}(a+b+c)$$

$$\Rightarrow \sum \sqrt{a^2+b^2+c^2} > \frac{1}{\sqrt{3}}\sum(a+b+c) = \frac{3}{\sqrt{3}}(a+b+c+d)$$

$$\Rightarrow \sum \sqrt{a^2+b^2+c^2} > \sqrt{3}(a+b+c) \quad \textbf{(1)}$$



For, $\Delta\, ABC, a+b > AC$; $\Delta BCD, b+c > BD$; $\Delta\, CDA, c+d > AC$

$\Delta DAB, d+a > BD$***. Adding the we get,*** $2(a+b+c+d) > 2(AC+BD)$

$$\Rightarrow \sqrt{3}(a+b+c+d) > \sqrt{3}(AC+BD) \quad \textbf{(2)}$$

Also by $AM > GM$ ***we get,*** $AC+BD > 2\sqrt{AC\cdot BD}$

$$\Rightarrow \sqrt{3}(AC+BD) > 2\sqrt{3\cdot AC\cdot BD} \quad \textbf{(3)}$$

From (1), (2) & (3) we get,

$$\sum \sqrt{a^2+b^2+c^2} > \sqrt{3}(a+b+c+d) > \sqrt{3}(AC+BD) > 2\sqrt{3\cdot AC\cdot BD}$$

$$\Rightarrow \sum \sqrt{a^2+b^2+c^2} > 2\sqrt{3\cdot AC\cdot BD}$$

Solution 2 by Myagmarsuren Yadamsuren – Mongolia

In $ABCD$ ***– CONVEXE quadrilater:***

$AB = a; BC = b; CD = c; DA = d$. ***Prove that***

$$\sum \sqrt{a^2 + b^2 + c^2} > 2 \cdot \sqrt{3 \cdot AC \cdot BD}$$

$$\sqrt{a^2 + b^2 + c^2} + \sqrt{b^2 + c^2 + d^2} + \sqrt{c^2 + d^2 + a^2} + \sqrt{d^2 + a^2 + b^2} \geq$$

$$\geq \sqrt{3 \cdot (a^2 + b^2 + c^2 + d^2)}$$

$$\Rightarrow \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{b^2 + c^2 + d^2} > 0$$ ***(apparently inequality)***

$$(a^2 + b^2 + c^2) + 2 \cdot \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{b^2 + c^2 + d^2} + (b^2 + c^2 + d^2) >$$

$$> (a^2 + 2b^2 + 2c^2 + d^2) \Rightarrow$$

$$\Rightarrow \left(\sqrt{a^2 + b^2 + c^2} + \sqrt{b^2 + c^2 + d^2}\right)^2 > (a^2 + 2b^2 + 2c^2 + d^2)$$

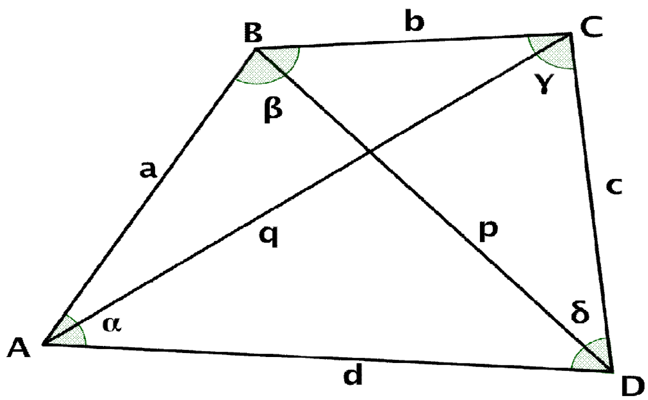
$$\begin{cases} \sqrt{a^2 + b^2 + c^2} + \sqrt{b^2 + c^2 + d^2} > \sqrt{a^2 + 2b^2 + 2c^2 + d^2} \\ \sqrt{c^2 + d^2 + a^2} + \sqrt{a^2 + b^2 + c^2} > \sqrt{2a^2 + b^2 + c^2 + 2d^2} \end{cases}$$

similarly

$$\sqrt{a^2 + 2b^2 + 2c^2 + d^2} + \sqrt{2a^2 + b^2 + c^2 + 2d^2} > \sqrt{3(a^2 + b^2 + c^2 + d^2)}$$

$$\sum \sqrt{a^2 + b^2 + c^2} > \sqrt{3(a^2 + b^2 + c^2 + d^2)}$$

To assure: $\sqrt{3 \cdot (a^2 + b^2 + c^2 + d^2)} > 2 \cdot \sqrt{3 \cdot AC \cdot BD}$ ***(*)***



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$$AB = a, BC = b, CD = c, DA = d, AC = q, BD = p$$

$$\cos(t_2):$$

$$\sum \begin{array}{l} p^2 = a^2 + d^2 - 2ad \cdot \cos\alpha \\ p^2 = b^2 + c^2 - 2bc \cdot \cos\gamma \\ q^2 = a^2 + b^2 - 2ab \cdot \cos\beta \\ q^2 = c^2 + d^2 - 2cd \cdot \cos\delta \end{array}$$

— — — — — — — — — — — — — — — — —

$$2 \cdot (p^2 + q^2) = 2 \cdot (a^2 + b^2 + c^2 + d^2) -$$

$$-2 \cdot (ad \cdot \cos\alpha + ab \cdot \cos\beta + bc \cdot \cos\gamma + cd \cdot \cos\delta)$$

$$p^2 + q^2 = a^2 + b^2 + c^2 + d^2 - (ad \cdot \cos\alpha + ab \cdot \cos\beta + bc \cdot \cos\gamma + ca \cdot \cos\delta)$$

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + (ad \cdot \cos\alpha + ab \cdot \cos\beta + bc \cdot \cos\gamma + cd \cdot \cos\delta) \overset{CBC}{\geq}$$

$$\geq p^2 + q^2 + \frac{1}{4}(ab + bc + cd + da) \cdot (\cos\alpha + \cos\beta + \cos\gamma + \cos\delta)$$

Let's: $\left.\begin{array}{r} \beta \geq \alpha \geq \gamma \geq \delta \\ \alpha + \gamma = 180° \\ \beta + \delta = 180° \end{array}\right\} \Rightarrow \cos\alpha + \cos\beta + \cos\gamma + \cos\delta \geq$

$$\geq 2 \cdot (\cos\alpha + \cos\gamma) = 4 \cdot \cos\frac{\alpha + \gamma}{2} \cdot \cos\frac{\alpha - \gamma}{2} =$$

$$= 4 \cdot \cos 90° \cdot \cos\frac{\alpha - \gamma}{2} = 0$$

There: $a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + x^2$

$$a^2 + b^2 + c^2 + d^2 > p^2 + q^2$$

(*) $\Rightarrow \sqrt{3 \cdot (a^2 + b^2 + c^2 + d^2)} > \sqrt{3 \cdot (p^2 + q^2)} \overset{Cauchy}{\geq} 2 \cdot \sqrt{3pq}$

173. In $ABCD$ cyclic quadrilater, $AB = a, BC = b, CD = c, DA = d$, s – semiperimeter:

$$\sin A \sin B \le \left(1-\frac{s}{a}\right)\left(1-\frac{s}{b}\right)\left(1-\frac{s}{c}\right)\left(1-\frac{s}{d}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 Adil Abdullayev – Baku – Azerbaidjian

Lemma 1. $S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$

Lemma 2.

$$\sin A = \frac{2S}{ad+bc}$$

$$\sin B = \frac{2S}{ab+cd}$$

$$LHS \overset{Lemma\ 2}{=} \frac{4S^2}{(ad+bc)(ab+cd)} \ \ldots (A)$$

$$RHS = \frac{(s-a)(s-b)(s-c)(s-d)}{abcd} \overset{Lemma\ 1}{=} \frac{S^2}{abcd} \ \ldots (B)$$

$$LHS \le RHS \overset{(A)-(B)}{\Leftrightarrow} \frac{4S^2}{(ad+bc)(ab+cd)} \le \frac{S^2}{abcd} \Leftrightarrow (ad+bc)(ad+cd) \ge 4abcd$$

... (C)

$$\begin{matrix} ad+bc \overset{AM-GM}{\ge} 2\sqrt{abcd} \\ ab+cd \overset{AM-GM}{\ge} 2\sqrt{abcd} \end{matrix} \Rightarrow (C)$$

Solution 2 by Ravi Prakash - New Delhi – India

$\sin A = \frac{2S}{ad+bc}$; $\sin B = \frac{2S}{ab+cd}$. Also $ad+bc \ge 2\sqrt{adbc}$

$$\sin A \sin B \le \frac{4S^2}{4abcd} = \frac{S^2}{abcd} = \frac{(s-a)(s-b)(s-c)(s-d)}{abcd} =$$

$$= \left(1-\frac{s}{a}\right)\left(1-\frac{s}{b}\right)\left(1-\frac{s}{c}\right)\left(1-\frac{s}{d}\right)$$

174. In $ABCD$ cyclic quadrilater, $AB = a, BC = b, CD = c, DA = d,$

$$S - area\,[ABCD]$$

$$\sin A + \sin B + \sin C + \sin D \le \frac{4S}{\sqrt{abcd}}$$

Proposed by Daniel Sitaru – Romania

Solution by Adil Abdullayev – Baku – Azerbaidjian

$$\begin{matrix} \sin A = \sin C \\ \sin B = \sin D \end{matrix} \Rightarrow \sin A + \sin B \le \frac{2S}{\sqrt{abcd}} \ldots \text{(A)}$$

$$\begin{matrix} \sin A = \frac{2S}{ad+bc} \\ \sin B = \frac{2S}{ab+cd} \end{matrix} \Rightarrow \text{(A)} \Leftrightarrow \frac{1}{ad+bc} + \frac{1}{ab+cd} \le \frac{1}{\sqrt{abcd}} \ldots \text{(B)}$$

$$ad + bc \overset{AM-GM}{\ge} 2\sqrt{abcd}$$

$$ab + cd \overset{AM-GM}{\ge} 2\sqrt{abcd}$$

$$\frac{1}{ad + bc} + \frac{1}{ab + cd} \le \frac{1}{2\sqrt{abcd}} + \frac{1}{2\sqrt{abcd}} = \frac{1}{\sqrt{abcd}}.$$

175. Let $A_1A_2A_3A_4$ be a tetrahedron and let M be its interior point. Denote respectively by S_i and d_i the are and distance from M to face opposite to vertex A_i. Prove that

$$\sum_{1\le i<j\le 4} S_iS_jd_id_j \le \frac{27}{8}V^2$$

where V is the volume of the tetrahedron.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Marian Dincă – Romania

$$\text{Let } V_p = volume\,[MA_iA_jA_k]$$

where: $p \neq i \neq j \neq k \neq p$ ***and*** $(i, j, k, p) = \{1, 2, 3, 4\}$

result:

$$\sum_{1 \le i < j \le 4} S_i S_j d_i d_j = \sum_{1 \le i < j \le 4} 9 V_i V_j \le 9 \binom{2}{4} \left(\frac{\sum_{k=1}^{4} V_k}{4} \right)^2 = \frac{27}{8} V^2$$

Mac – Laurin inequality

176. If $a, b \in (0, \infty)$ then:

$$\frac{2\sqrt{ab}}{a+b} + \frac{4ab}{(a+b)^2} + \frac{(a+b)^2}{4ab} + \frac{a+b}{2\sqrt{ab}} \le 2\left(\frac{a}{b} + \frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Peru

Si: $a, b \in < 0, \infty >$***. Probar:***

$$\frac{2\sqrt{ab}}{a+b} + \frac{4ab}{(a+b)^2} + \frac{(a+b)^2}{4ab} + \frac{a+b}{2\sqrt{ab}} \le 2\left(\frac{a}{b} + \frac{b}{a}\right)$$

Por: $MA \ge MG$

$$\frac{2\sqrt{ab}}{a+b} \le 1 \quad \textbf{(A)}$$

$$\frac{4ab}{(a+b)^2} \le 1 \quad \textbf{(B)}$$

Defínase: $x = \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}$ ***(C)*** $\rightarrow MA \ge MG: x \ge 2, x^2 - 2 = \frac{a}{b} + \frac{b}{a}$ ***(D)***

La desigualdad es equivalente:

$$1 + 1 + \frac{a}{4b} + \frac{b}{4a} + \frac{1}{2} + \frac{1}{2}\sqrt{\frac{a}{b}} + \frac{1}{2}\sqrt{\frac{b}{a}} \le 2\left(\frac{a}{b} + \frac{b}{a}\right)$$

$$2 + \frac{1}{4}(x^2 - 2) + \frac{1}{2}(x + 1) \le 2(x^2 - 2)$$

$$\Rightarrow 2\cdot 4+(x^2-2)+2(x+1)\le 8(x^2-2)\Rightarrow 7x^2-2x-24\ge 0\to$$

$$\to (x-2)(7x+12)\ge 0$$ ***(La desigualdad se mantiene)***

177. Prove that if: $z_1, z_2, \dots, z_n \in \mathbb{C}^*, n\in\mathbb{N}$ then:

$$\frac{\left|\sum_{i=1}^{n}(Re\, z_i + Im\, z_i)\right|}{\sum_{i=1}^{n}|z_i|}\le\sqrt{2}$$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren-Mongolia

$$|(a_1+\dots+a_n)+(b_1+\dots+b_n)| \overset{\text{Cauchy-Schwarz}}{\le}$$

$$\le\left|\sqrt{(1^2+1^2)\left((a_1+\dots+a_n)^2+(b_1+\dots+b_n)^2\right)}\right|=$$

$$=\left|\sqrt{2}\cdot\sqrt{(a_1+\dots+a_n)^2+(b_1+\dots+b_n)^2}\right|=\left|\sqrt{2}\left((a_1+\dots+a_n)+(b_1+\dots b_n)i\right)\right|=$$

$$=\sqrt{2}\cdot|(a_1+b_1i)+\dots+(a_n+b_ni)|\le$$

$$\le\sqrt{2}\cdot(|a_1+ib_1|+\dots+|a_n+ib_n|)=\sqrt{2}\cdot\left(\sum_{i=1}^{n}|z_i|\right)$$

178. If $F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n$ then:

$$2\left(F_n^4+F_{n+1}^4+F_{n+2}^4\right)>9(F_nF_{n+1}F_{n+2})^{\frac{4}{3}}$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

Solution by Marian Ursarescu-Romania

From Hölder's inequality $\Rightarrow F_n^4+F_{n+1}^4+F_{n+2}^4\ge\frac{(F_n+F_{n+1}+F_{n+2})^4}{27}\Rightarrow$

$$\Rightarrow 2\left(F_n^4+F_{n+1}^4+F_{n+2}^4\right)>\frac{2}{27}(F_n+F_{n+1}+F_{n+2})^4 \quad (1)$$

$$F_n+F_{n+1}+F_{n+2}>3\sqrt[3]{F_n\cdot F_{n+1}\cdot F_{n+2}} \quad (2)$$

From (1)+(2)$\Rightarrow 2\left(F_n^4+F_{n+1}^4+F_{n+2}^4\right)>\frac{2}{27}\left(3\sqrt[3]{F_n\cdot F_{n+1}\cdot F_{n+2}}\right)^4\Rightarrow$

$$\Rightarrow 2\left(F_n^4+F_{n+1}^4+F_{n+2}^4\right)>18\left(F_n\cdot F_{n+1}F_{n+2}\right)^{\frac{4}{3}}>9\left(F_nF_{n+1}F_{n+2}\right)^{\frac{4}{3}}$$

179. If $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n\in(0,\infty),n\in\mathbb{N}^*$ **then:**

$$\left(\sum_{i=1}^{n}\frac{x_i^2+y_i^2}{x_iy_i}\right)\left(\sum_{i=1}^{n}\frac{x_iy_i}{x_i^2+y_i^2}\right)\le\left(\sum_{i=1}^{n}\frac{x_i}{y_i}\right)\left(\sum_{i=1}^{n}\frac{y_i}{x_i}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash –New Delhi-India:

For $a,b,c,d>0$:

$$\frac{a}{b}\cdot\frac{d}{c}+\frac{bc}{ad}-\frac{a^2+b^2}{ab}\cdot\frac{cd}{c^2+d^2}-\frac{c^2+d^2}{cd}\cdot\frac{ab}{a^2+b^2}\ge 0$$

$$\Leftrightarrow\frac{a^2d^2+b^2c^2}{abcd}-\frac{(a^2+b^2)^2c^2d^2+(c^2+d^2)^2a^2b^2}{abcd(a^2+b^2)(c^2+d^2)}\ge 0$$

$$\Leftrightarrow(a^2d^2+b^2c^2)(a^2+b^2)(c^2+d^2)-[(a^2+b^2)^2c^2d^2+a^2b^2(c^2+d^2)^2]\ge 0$$

$$\Leftrightarrow(a^2d^2+b^2c^2)[a^2c^2+b^2c^2+a^2d^2+b^2d^2]$$

$$-(a^4+b^4+2a^2b^2)c^2d^2-(c^4+d^4+2c^2d^2)a^2b^2\ge 0$$

$$\Leftrightarrow a^4c^2d^2+a^2b^2c^4+a^2b^2c^2d^2+b^4c^4+$$

$$+a^4d^4+a^2b^2c^2d^2+a^2b^2d^4+b^4c^2d^2-$$

$$-[a^4c^2d^2+b^4c^2d^2+2a^2b^2c^2d^2+a^2b^2c^4+a^2b^2d^4+2a^2b^2c^2d^2]\ge 0$$

$$\Leftrightarrow(b^2c^2-a^2d^2)^2\ge 0$$

which is true. Consider

$$\left(\sum_{i=1}^{n}\frac{x_i}{y_i}\right)\left(\sum_{i=1}^{n}\frac{y_i}{x_i}\right)-\left(\sum_{i=1}^{n}\frac{x_i^2+y_i^2}{x_iy_i}\right)\left(\sum_{i=1}^{n}\frac{x_iy_i}{x_i^2+y_i^2}\right)=$$

$$= \sum_{i=1}^{n} \frac{x_i}{y_i} \cdot \frac{y_i}{x_i} + \sum_{i<j} \frac{x_i}{y_i} \cdot \frac{y_j}{x_j} + \sum_{i>j} \frac{x_i}{y_i} \cdot \frac{y_j}{x_j}$$

$$-\sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{x_i y_i} \cdot \frac{x_i y_i}{x_i^2 + y_i^2} - \sum_{i<j} \frac{x_i^2 + y_i^2}{x_i y_i} \cdot \frac{x_j y_j}{x_j^2 + y_j^2} - \sum_{i>j} \frac{x_i^2 + y_i^2}{x_i y_i} \cdot \frac{x_j y_j}{x_j^2 + y_j^2} =$$

$$= n - n + \sum_{i<j} \left(\frac{x_i y_j}{y_i x_j} + \frac{x_j}{y_j} \cdot \frac{y_i}{x_i} - \frac{x_1^2 + y_i^2}{x_i y_i} \cdot \frac{x_j y_j}{x_j^2 + y_j^2} - \frac{x_j^2 + y_j^2}{x_j y_j} \cdot \frac{x_i y_i}{x_i^2 + y_i^2} \right)$$

$$= \sum_{i<j} \left(x_j^2 y_i^2 - x_i^2 y_j^2 \right)^2 \frac{1}{x_i x_j y_i y_j \left(x_i^2 + y_i^2 \right)\left(x_j^2 + y_j^2 \right)} \geq 0$$

180. If $a_i > 0, i \in \overline{1,n}, a_1 + a_2 + \cdots + a_n = 1$ then:

$$\frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n}} \geq n + 1$$

Proposed by Regragui El Khammal-Morocco

Solution by Daniel Sitaru-Romania:

$$\frac{\sum_{i=1}^{n} a_i + a_k}{n+1} \overset{AM-HM}{\geq} \frac{n+1}{\sum_{i=1}^{n} \frac{1}{a_i} + \frac{1}{a_k}}, k \in \overline{1,n}$$

$$\frac{\sum_{i=1}^{n} \frac{1}{a_i} + \frac{1}{a_k}}{n+1} \geq \frac{n+1}{1 + a_k}$$

$$\frac{1}{n+1} \left(\sum_{k=1}^{n} \sum_{i=1}^{n} \frac{1}{a_i} + \sum_{k=1}^{n} \frac{1}{a_k} \right) \geq (n+1) \sum_{k=1}^{n} \frac{1}{1 + a_k}$$

$$\frac{(n+1)\sum_{k=1}^{n}\frac{1}{a_k}}{n+1} \geq (n+1)\sum_{k=1}^{n}\frac{1}{1+a_k}$$

$$\sum_{k=1}^{n}\frac{1}{a_k} \geq (n+1)\sum_{k=1}^{n}\frac{1}{1+a_k}$$

$$\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{\frac{1}{1+a_1}+\frac{1}{1+a_2}+\cdots+\frac{1}{1+a_n}} \geq n+1$$

181. If $a, r \in (0,\infty)$ then:

$$\sum_{k=1}^{n}\frac{k}{\left(\sum_{i=1}^{k}\left(\frac{1}{a+(i-1)r}\right)\right)} < (2a+(n-1)r)n, n \in \mathbb{N}^*$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Pal – Kolkata – India

$$P_k = \frac{k}{\sum_{i=1}^{k}\left(\frac{1}{a+(i-1)r}\right)} < \frac{\sum_{i=1}^{k}(a+(i-1)r)}{k} =$$

(by AM – HM)

$$= \frac{\frac{k}{2}(2a+(k-1)r)}{k} = a + \frac{(k-1)r}{2} = S_k$$

$$\sum_{k=1}^{n} P_k < \sum_{k=1}^{n} S_k =$$

$$= \sum_{k=1}^{n}\left(a+\frac{(k-1)r}{2}\right) = na + \frac{n(n-1)}{4}r < 2na + n(n-1)r =$$

$$= n(2a + (n-1)r)$$

$$\Rightarrow \sum_{k=1}^{n}\left(\frac{k}{\sum_{r=1}^{k}\left(\frac{1}{a+(i-1)r}\right)}\right) < (2a + (n-1)r)n$$

Proved

182. **If $a_i, b_i \in (0,\infty), i \in \overline{1,n}, n \in \mathbb{N}^*$ then:**

$$\frac{(2n)^n}{\prod_{i=1}^{n}(a_i+b_i)} \le \frac{1}{2}\left[\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)^n + \left(\sum_{i=1}^{n}\frac{1}{b_i}\right)^n\right]$$

Proposed by Daniel Sitaru – Romania

Solution by Saptak Bhattacharya-Kolkata-India

We have $\frac{\left(\sum\frac{1}{a_i}\right)^n+\left(\sum\frac{1}{b_i}\right)^n}{2} \ge \frac{\left\{\sum\left(\frac{1}{a_i}+\frac{1}{b_i}\right)\right\}^n}{2^n}$ *which is by AM* $\ge$ *HM*

$\ge 4^n\left(\sum\frac{1}{(a_i+b_i)}\right)^n$*. Thus, enough to show that*

$\left(\frac{\sum\frac{1}{(a_i+b_i)}}{n}\right)^n \ge \frac{1}{\prod(a_i+b_i)}$ *which clearly holds by AM* $\ge$ *GM*

183. **Let $n \in \mathbb{N}^*, a_1, \dots, a_n \in \left(0,\frac{\pi}{2}\right)$ such that**

$$\sum_{1\le k\le n} a_k \le n.$$

Prove that

$$\left(\sum_{1\le k\le n}\frac{1}{a_k}\right)\sin\left(\frac{n}{\sum_{1\le k\le n}\frac{1}{a_k}}\right) + \frac{n}{\pi} > n$$

Proposed by Mihalcea Andrei Stefan-Romania

Solution by Redwane El Mellass-Morroco

$$\text{Let } f\left(0 < x < \frac{\pi}{2}\right) = \frac{\sin(x)}{x} \text{ and } g(t \geq 0) = \sin(t) - t + \frac{t^3}{6}$$

$$\therefore f'(x) = \frac{\cos(x)\,(x - \tan(x))}{x^2} < 0(\tan(x) > x)$$

$$\therefore g'(t) = \cos(t) - 1 + \frac{t^2}{2} \text{ and } g''(t) = t - \sin(t) \geq 0(\sin(t) \leq t)$$

$$\text{So } g'(t) \geq g'(0) = 0 \Rightarrow g(t) \geq g(0) = 0.$$

$$\text{we get } f(0 < x \leq 1) \geq \sin(1) \text{ and } g(1) > 0 \Rightarrow \sin(1) > \frac{5}{6}$$

$$\text{So } f(0 < x \leq 1) > \frac{5}{6}.$$

$$\therefore \sum_{k=1}^{n} \frac{1}{a_k} \geq \frac{n^2}{\sum_{k=1}^{n} a_k} \geq n \Rightarrow 0 < \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}} \leq 1$$

$$\Rightarrow \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \sin\left(\frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}\right) + \frac{n}{\pi} = n\left(\frac{\sin\left(\frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}\right)}{\frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}} + \frac{1}{\pi}\right) >$$

$$> n\left(\frac{5}{6} + \frac{1}{\pi}\right) > n\left(\frac{5}{6} + \frac{1}{6}\right) = n.$$

184. If $a_k \in (0, \infty)$ where $k = 1, 2, 3, \dots, n$ and $a_1 + a_2 + \cdots + a_n = 1$ then:

$$\sum_{k=1}^{n} \frac{1}{a_k} \geq (n + 1)\left(\sum_{k=1}^{n} \frac{1}{a_k + 1}\right)$$

Proposed by Marin Chirciu – Romania

Solution by Soumitra Mandal - Chandar Nagore – India

$$(n+1)\left(\frac{1}{2a_1+a_2+\cdots+a_n}+\frac{1}{a_1+2a_2+\cdots+a_n}+\cdots+\frac{1}{a_1+a_2+\cdots+2a_n}\right)$$

$$\le\frac{1}{n+1}\left(\frac{2}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)+\frac{1}{n+1}\left(\frac{1}{a_1}+\frac{2}{a_2}+\cdots+\frac{1}{a_n}\right)+\cdots+\frac{1}{n+1}\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{2}{a_n}\right)$$

$$=\frac{1}{n+1}\left(\sum_{k=1}^{n}\frac{n+1}{a_k}\right)=\sum_{k=1}^{n}\frac{1}{a_k}$$

(Proved)

185. If $a,b,c\in(0,\infty)$ then:

$$(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)\ge 2\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash – New Delhi – India

$$(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)=3+\left(\frac{a^4}{b^4}+\frac{b^4}{a^4}\right)+\left(\frac{a^4}{c^4}+\frac{c^4}{a^4}\right)+\left(\frac{b^4}{c^4}+\frac{c^4}{b^4}\right)$$

If $\frac{a}{b}<1,\frac{a^4}{b^4}+\frac{b^4}{a^4}>2>2\frac{a}{b}$***. Similarly for any expression*** <1 ***on RHS***

If $\frac{a}{b}\ge 1,\frac{a^4}{b^4}+\frac{b^4}{a^4}+1>\frac{a^4}{b^4}+1$ ***and*** $\frac{a^4}{b^4}+1\ge 2\frac{a^2}{b^2}\ge 2\frac{a}{b}$.

Similarly for other expression ≥ 1 ***on RHS.***

In any case we get $(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)>2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$

Solution 2 by Rozeta Atanasova – Skopje

WLOG let $a\ge b\ge c\Rightarrow$

$(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{a^4}+\frac{1}{c^4}\right)\ge$ ***(Rearrangement inequality)***

$$\geq (a^2b^2+a^2c^2+b^2c^2)\left(\frac{1}{a^2b^2}+\frac{1}{a^2c^2}+\frac{1}{b^2c^2}\right)\geq \quad \textit{(CSB inequality)}$$

$$\geq \left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)^2=\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)\geq \textit{(Rearrangement inequality)}$$

$$\geq \left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)\left(\frac{a}{a}+\frac{c}{c}+\frac{b}{b}\right)=3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$$

186. **If $a,b,c\in(0,\infty)$ then:**

$$\sum\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}\geq\frac{48abc}{1+a+b+c}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios -Huarmey– Peru

Si: $a,b,c\in<0,\infty>$***. Probar que:***

$$\sum\begin{bmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{bmatrix}\geq\frac{48abc}{1+a+b+c}$$

$$\Rightarrow\sum(1+a)\begin{bmatrix}1+b & 1\\ 1 & 1+c\end{bmatrix}-\sum 1\begin{bmatrix}1 & 1\\ 1 & 1+c\end{bmatrix}+\sum 1\begin{bmatrix}1 & 1+b\\ 1 & 1\end{bmatrix}\geq\frac{48abc}{1+a+b+c}$$

$$\Rightarrow\sum(1+a)[(1+b)(1+c)-1]-\sum 1[1(1+c)-1]+\sum 1[1-(1+b)]\geq\frac{48abc}{1+a+b+c}$$

$$\Rightarrow\sum(1+a)(b+c+bc)-\sum c-\sum b=\sum bc+\sum ab+\sum ac+\sum abc=$$

$$=3ab+3ac+3bc+3abc$$

$$\Rightarrow 3(ab+ac+bc+abc)\geq\frac{48abc}{1+a+b+c}$$

Desde que: $a,b,c>0$***, dividimos la expersión*** $\div(abc)$***:***

$$\Rightarrow 3\left(\frac{1}{c}+\frac{1}{b}+\frac{1}{a}+1\right)(1+a+b+c)\geq 48$$

Por:

$$MA \geq MH \rightarrow \frac{1+a+b+c}{4} \geq \frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1} \rightarrow 3\left(\frac{1}{c}+\frac{1}{b}+\frac{1}{a}+1\right)(1+a+b+c) \geq 48$$

La igualdad se alcanza cuando: $a = b = c = 1$

187. If $a, b, c, d, e, f \in (0, \infty)$ then:

$$\begin{vmatrix} a & \sqrt{ad} & \sqrt{ae} \\ \sqrt{ad} & b+d & \sqrt{de}+\sqrt{bf} \\ \sqrt{ae} & \sqrt{de}+\sqrt{bf} & c+e+f \end{vmatrix} > 0$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios -Huarmey– Peru

Si: $a, b, c, d, e, f \in <0, \infty>$. ***Probar que:***

$$\begin{vmatrix} a & \sqrt{ad} & \sqrt{ae} \\ \sqrt{ad} & b+d & \sqrt{de}+\sqrt{bf} \\ \sqrt{ae} & \sqrt{de}+\sqrt{bf} & c+e+f \end{vmatrix} > 0$$

$\Rightarrow$ ***Desarrollando lo pedido se puedes expresar como:***

$$a\begin{bmatrix} b+d & \sqrt{de}+\sqrt{bf} \\ \sqrt{de}+\sqrt{bf} & c+e+f \end{bmatrix} - \sqrt{ad}\begin{bmatrix} \sqrt{ad} & \sqrt{de}+\sqrt{bf} \\ \sqrt{ae} & c+e+f \end{bmatrix} + \sqrt{ae}\begin{bmatrix} \sqrt{ad} & b+d \\ \sqrt{ae} & \sqrt{de}+\sqrt{bf} \end{bmatrix}$$

$$a\left((b+d)(c+e+f) - \left(\sqrt{de}+\sqrt{bf}\right)^2\right) - \sqrt{ad}\left(\sqrt{ad}(c+e+f) - \sqrt{ae}\left(\sqrt{de}+\sqrt{bf}\right)\right) +$$

$$+\sqrt{ae}\left(\left(\sqrt{ad}\right)\left(\sqrt{de}+\sqrt{bf}\right) - \sqrt{ae}(b+d)\right)$$

$$T_1 = a\left((b+d)(c+e+f) - \left(\sqrt{de}+\sqrt{bf}\right)^2\right)$$

$$T_1 = a\left(bc+be+bf+dc+de+df-de-bf-2\sqrt{bdfe}\right)$$

$$\Rightarrow T_1 = abc+abe+adc+adf-2a\sqrt{bdfe}$$

$$T_2 = -\sqrt{ad}\left(\sqrt{ad}(c+e+f) - \sqrt{ae}\left(\sqrt{de}+\sqrt{bf}\right)\right)$$

$$T_2=-\sqrt{ad}\left(\sqrt{adc}+\sqrt{ade}+\sqrt{adf}-e\sqrt{ad}-\sqrt{aebf}\right)$$

$$\Rightarrow T_2=-adc-adf+a\sqrt{bdfe}$$

$$T_3=\sqrt{ae}\left(\left(\sqrt{ad}\right)\left(\sqrt{de}+\sqrt{bf}\right)-\sqrt{ae}(b+d)\right)$$

$$T_3=\sqrt{ae}\left(d\sqrt{ae}+\sqrt{abdf}-b\sqrt{ae}-d\sqrt{ae}\right)$$

$$\Rightarrow T_3=a\sqrt{bdfe}-abe\rightarrow \textbf{Sumando:}$$

$$T_1+T_2+T_3=abc>0\Leftrightarrow a,b,c,d,e,f\in<0,\infty>.$$

188. If $\Delta(x)=\begin{vmatrix} x & a & b & c\\ a & x & b & c\\ a & b & x & c\\ a & b & c & x\end{vmatrix}, a,b,c\in(0,\infty)$ then:

$$\frac{\Delta'(a+b+c)}{\Delta(a+b+c)}\le\frac{1}{6\sqrt[3]{abc}}+\frac{1}{2}\sum\frac{1}{\sqrt{ab}}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

$$\Delta(x)=\begin{vmatrix} x & a & b & c\\ a & x & b & c\\ a & b & x & c\\ a & b & c & x\end{vmatrix}=(x+a+b+c)\begin{vmatrix} 1 & a & b & c\\ 1 & x & b & c\\ 1 & b & x & c\\ 1 & b & c & x\end{vmatrix}=$$

$$=(x+a+b+c)\begin{vmatrix} 1 & a & b & c\\ 0 & x-a & 0 & 0\\ 0 & b-a & x-b & 0\\ 0 & b-a & c-b & x-c\end{vmatrix}$$

$$\Delta(x)=(x+a+b+c)(x-a)(x-b)(x-c)$$

$$\ln\Delta(x)=\ln(x+a+b+c)+\ln(x-a)+\ln(x-b)+\ln(x-c)$$

$$\frac{\Delta'(x)}{\Delta(x)}=\frac{1}{x+a+b+c}+\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}$$

$$\frac{\Delta'(a+b+c)}{\Delta(a+b+c)}=\frac{1}{2(a+b+c)}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\le$$
$$\le\frac{1}{6(abc)^{\frac{1}{3}}}+\frac{1}{2}\sum\frac{1}{\sqrt{ab}}$$

189. $A_{2n+1}=\begin{pmatrix} a & b & 0 & \cdots & 0 & 0\\ 0 & a & b & \cdots & 0 & 0\\ 0 & 0 & a & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & a & b\\ b & 0 & 0 & \cdots & 0 & a\end{pmatrix}\in M_{2n+1}(\mathbb{R}_+^*), n\in\mathbb{N}^*,\ \Omega_{2n+1}=\det(A_{2n+1})$

Prove that: $\frac{\Omega_{2n+7}}{\Omega_{2n+5}}\ge\frac{\Omega_{2n+3}}{\Omega_{2n+1}}$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren – Mongolia

$$\det(A_{2n+1})=a^{2n+1}+b^{2n+1}=\Omega_{2n+1}$$

$$\left(\frac{a}{b}\right)^2+\left(\frac{b}{a}\right)^2\ge 2\ \textit{(True)}\Rightarrow\left(\frac{a}{b}\right)^2+\left(\frac{b}{a}\right)^2-1\ge 1\ \Big|\cdot\left(\frac{a}{b}+\frac{b}{a}>0\right)$$

$$\left(\frac{a}{b}\right)^3+\left(\frac{b}{a}\right)^3\ge\frac{a}{b}+\frac{b}{a}\Big|\cdot(x\cdot y>0)\quad\left(\frac{a}{b}\right)^3\cdot xy+\left(\frac{b}{a}\right)^3\cdot xy\ge\left(\frac{a}{b}+\frac{b}{a}\right)xy$$

$$(x^2+y^2)+\left(\frac{a}{b}\right)^3\cdot xy+\left(\frac{b}{a}\right)^3\cdot xy\ge(x^2+y^2)+\frac{a}{b}\cdot xy+\left(\frac{b}{a}\right)\cdot xy$$

$$(a^3\cdot x+b^3\cdot y)\left(\frac{1}{a^3}\cdot x+\frac{1}{b^3}\cdot y\right)\ge(ax+by)\cdot\left(\frac{1}{a}\cdot x+\frac{1}{b}\cdot y\right)$$

$$\frac{a^3\cdot x+b^3\cdot y}{a\cdot x+b\cdot y}\ge\frac{\frac{1}{a}\cdot x+\frac{1}{b}\cdot y}{\frac{1}{a^3}\cdot x+\frac{1}{b^3}\cdot y}\quad\begin{bmatrix}x=a^{2n+4}\\ y=b^{2n+4}\end{bmatrix}$$

$$\frac{a^{2n+7}+b^{2n+7}}{a^{2n+5}+b^{2n+5}}\ge\frac{a^{2n+3}+b^{2n+3}}{a^{2n+1}+b^{2n+1}}\Rightarrow\frac{\Omega_{2n+7}}{\Omega_{2n+5}}\ge\frac{\Omega_{2n+3}}{\Omega_{2n+1}}$$

190. If $ABEC$ cyclic quadrilater, H_1 – orthocenter of ΔBEC, H_2 – orthocenter of $\Delta\ ABC$ then: $H_1H_2 = BE$.

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Daniel Sitaru-Romania

Let O – be the circumcentre:

$$\overrightarrow{OH_1} \overset{SYLVESTER}{=} \overrightarrow{OA} + \overrightarrow{OE} + \overrightarrow{OC}, \overrightarrow{OH_2} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

$$H_1H_2 = \left|\overrightarrow{H_1H_2}\right| = \left|\overrightarrow{OH_2} - \overrightarrow{OH_1}\right| = \left|\overrightarrow{OA} + \overrightarrow{OE} + \overrightarrow{OC} - \overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OC}\right| =$$

$$= \left|\overrightarrow{OE} - \overrightarrow{OB}\right| = \left|\overrightarrow{BE}\right| = BE$$

191. If $a, b, c \in (0, \infty)$ then:

$$\sum \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} \geq \frac{48abc}{1+a+b+c}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Si: $a, b, c \in < 0, \infty >$. Probar que:

$$\sum \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} \geq \frac{48abc}{(1+a+b+c)}$$

$$\Rightarrow \sum (1+a)\begin{bmatrix} 1+b & 1 \\ 1 & 1+c \end{bmatrix} - \sum 1 \begin{bmatrix} 1 & 1 \\ 1 & 1+c \end{bmatrix} + \sum 1 \begin{bmatrix} 1 & 1+b \\ 1 & 1 \end{bmatrix} \geq \frac{48}{1+a+b+c}$$

$$\Rightarrow \sum (1+a)\left[(1+b)(1+c) - 1\right] - \sum 1\left[1(1+c) - 1\right] + \sum 1\left[1 - (1+b)\right] \geq$$

$$\geq \frac{48abc}{1+a+b+c}$$

$$\Rightarrow \sum (1+a)(b+c+bc) - \sum c - \sum b = \sum bc + \sum ab + \sum ac + \sum abc =$$

$$= 3ab + 3ac + 3bc + 3abc$$

$$\Rightarrow 3(ab+ac+bc+abc) \geq \frac{48}{1+a+b+c}$$

Desde que: $a, b, c > 0$***, dividimos la expresión*** $\div (abc)$***:***

$$\Rightarrow 3\left(\frac{1}{c}+\frac{1}{b}+\frac{1}{a}+1\right)(1+a+b+c) \geq 48$$

Por: MA $\geq$ ***MH*** $\rightarrow \frac{1+a+b+c}{4} \geq \frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1} \rightarrow 3\left(\frac{1}{c}+\frac{1}{b}+\frac{1}{a}+1\right)(1+a+b+c) \geq 48$

(LQQD). La igualdad se alcanza cuando: $a=b=c=1$***.***

Solution 2 by Saptak Bhattacharya-Kolkata-India

$$\sum \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} \geq \frac{48abc}{1+a+b+c}$$

$$\Leftrightarrow ab+bc+ca+abc \geq \frac{16abc}{1+a+b+c}$$

$\Leftrightarrow (1+a+b+c)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 16$; ***which is true by AM*** $\geq$ ***HM***

192. If $a, b, c \in (0, \infty), a \neq b \neq c \neq a$

$$\Delta_1 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a^2 & b^2 & c^2 \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \\ bc & ca & ab \end{vmatrix}$$

then:

$$\frac{\Delta_1 - \Delta_2}{(b-a)(a-c)(b-c)} \geq 12\sqrt[6]{(abc)^5}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

$$\Delta_2 = \begin{vmatrix} a^2 & b^2 & c^2 \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \\ bc & ca & ab \end{vmatrix}$$

$$R_2 \to R_2 + R_1$$

$$= (a^2+b^2+c^2)\begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ bc & ca & ab \end{vmatrix} = \frac{a^2+b^2+c^2}{abc}\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ abc & abc & abc \end{vmatrix} =$$

$$= (a^2+b^2+c^2)\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = -(a^2+b^2+c^2)\Delta_1$$

$$\Delta_1 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^3-b^3 \\ 0 & b-c & b^3-c^3 \\ 1 & c & c^3 \end{vmatrix} =$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a^2+b^2+ab \\ 1 & b^2+c^2+cb \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 1 & a^2+b^2+ab \\ 0 & c^2-a^2+b(c-a) \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(c+a+b)$$

$$\therefore \frac{\Delta_1 - \Delta_2}{(a-b)(b-c)(c-a)}$$

$$= (a+b+c) + (a+b+c)(a^2+b^2+c^2) \geq$$

$$\geq 3(abc)^{\frac{1}{3}} + 3(abc)^{\frac{1}{3}}(a^2+b^2+c^2) = 3(abc)^{\frac{1}{3}}(4)(a^2b^2c^2)^{\frac{1}{4}} = 12(abc)^{\frac{5}{6}}$$

193. In $\Delta\, ABC$ the following relationship holds:

$$\frac{2\begin{vmatrix} s+a^2 & ab & ac \\ ab & s+b^2 & bc \\ ac & bc & s+c^2 \end{vmatrix}}{s^2} \geq 8\sqrt{3}S + 3\sqrt[3]{4RS}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

In triángulo ABC. Probar que:

$$2\begin{bmatrix} s+a^2 & ab & ac \\ ab & s+b^2 & bc \\ ac & bc & s+c^2 \end{bmatrix} \geq 8\sqrt{3}Ss^2 + 3\sqrt[3]{4RS}s^2$$

$2s$ = perimetro, S = Área de región triángular

$$2(s+a^2)\begin{bmatrix} s+b^2 & bc \\ bc & s+c^2 \end{bmatrix} - 2ab\begin{bmatrix} ab & bc \\ ac & s+c^2 \end{bmatrix} + 2ac\begin{bmatrix} ab & s+b^2 \\ ac & bc \end{bmatrix} \geq$$

$$\geq 8\sqrt{3}Ss^2 + 3\sqrt[3]{abc}s^2$$

$$2(s+a^2)[(s+b^2)(s+c^2) - b^2c^2] - 2ab[(s+c^2)(ab) - c^2ab] +$$

$$+2ac[b^2ac - (s+b^2)(ac)] \geq 8\sqrt{3}Ss^2 + 3\sqrt[3]{abc}s^2$$

$$\Rightarrow 2(s+a^2)\left[s\left(s+(b^2+c^2)\right)\right] - 2ab[sab] + 2ac[-sac] \geq$$

$$\geq 8\sqrt{3}Ss^2 + 3\sqrt[3]{abc}s^2$$

Divididiendo $(\div s^2)$ a la desigualdad:

$$\Rightarrow \frac{2s^2 + 2sa^2 + 2s(b^2+c^2) + 2a^2b^2 + 2a^2c^2 - 2a^2b^2 - 2a^2c^2}{s} \geq$$

$$\geq 8\sqrt{3}S + 3\sqrt[3]{abc}$$

$$\Rightarrow 2s + 2(a^2+b^2+c^2) \geq 3\sqrt[3]{abc} + 8\sqrt{3}S$$

$2s = a+b+c \geq 3\sqrt[3]{abc} \Leftrightarrow$ ***Válido por:*** $(MA \geq MG)$

Por lo cual falta demostrar que: $a^2+b^2+c^2 \geq 4\sqrt{3}S$

$$\Rightarrow 4R^2(\operatorname{sen}^2 A + \operatorname{sen}^2 B + \operatorname{sen}^2 C) \geq 4\sqrt{3}(2R^2 \operatorname{sen} A \operatorname{sen} B \operatorname{sen} C) \rightarrow$$

$$\rightarrow \frac{\operatorname{sen}(B+C)}{\operatorname{sen} B \operatorname{sen} C} + \frac{\operatorname{sen}(A+C)}{\operatorname{sen} A \operatorname{sen} C} + \frac{\operatorname{sen}(A+B)}{\operatorname{sen} A \operatorname{sen} B} \geq 2\sqrt{3}$$

$$\Rightarrow 2(\cot A + \cot B + \cot C) \geq 2\sqrt{3} \rightarrow \cot A + \cot B + \cot C \geq \sqrt{3} \rightarrow$$

→(Válido en un triángulo ABC)

Solution 2 by Soumava Chakraborty- Kolkata – India

To show:

$$\frac{2}{s^2}\Big((s+a^2)\{(s+b^2)(s+c^2)-b^2c^2\}+ab\{abc^2-ab(s+c^2)\}$$

$$+\{acb^2-ac(s+b^2)\}\Big)\ge 8\sqrt{3}S+3\sqrt[3]{4RS}$$

$$LHS=\frac{2}{s^2}\Big((s+a^2)(s^2+sb^2+sc^2)+a^2b^2(-s)+a^2c^2(-s)\Big)$$

$$=\frac{2}{s^2}\Big(s^3+s^2(a^2+b^2+c^2)\Big)=2s+2(a^2+b^2+c^2)$$

$$=(a+b+c)+2(a^2+b^2+c^2)$$

Now, $a+b+c\ge 3\sqrt[3]{abc}\ (AM\ge GM)=3\sqrt[3]{4RS}$ ***(1)***

Now, $2(a^2+b^2+c^2)\ge 8\sqrt{3}S$

$$\Leftrightarrow (a^2+b^2+c^2)^2\ge 48S^2=48s(s-a)(s-b)(s-c)$$

$$\Leftrightarrow (a^2+b^2+c^2)^2\ge 48\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{a+b-c}{2}\right)$$

$$\Leftrightarrow (a^2+b^2+c^2)^2\ge 3((a+b)^2-c^2)(c^2-(a-b)^2)$$

$$\Leftrightarrow (a^2+b^2+c^2)^2\ge 3(c^2(a+b)^2-(a^2-b^2)^2-c^4+c^2(a-b)^2)$$

$$\Leftrightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2\ge 3(2c^2(a^2+b^2)+2a^2b^2-a^4-b^4-c^4)$$

$$\Leftrightarrow 4a^4+4b^4+4c^4-4a^2b^2-4b^2c^2-4c^2a^2\ge 0$$

$$\Leftrightarrow 2a^4+2b^4+2c^4-2a^2b^2-2b^2c^2-2c^2a^2\ge 0$$

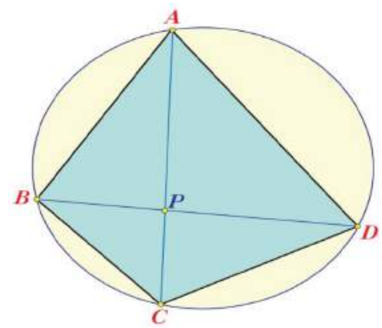
$$\Leftrightarrow (a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)\ge 0$$

which is true

$2(a^2+b^2+c^2)\ge 8\sqrt{3}S$ ***(2)***

$$(1)+(2)\Rightarrow a+b+c+2(a^2+b^2+c^2)\ge 8\sqrt{3}S+3\sqrt[3]{4RS}$$

194.

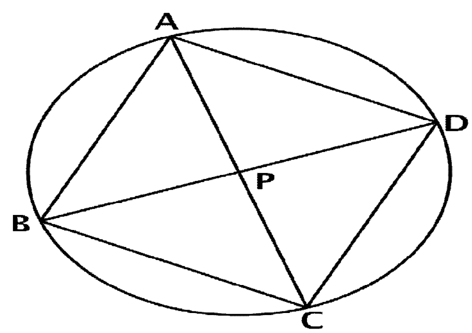


$ABCD$ harmonic quadrangle, $(AB \cdot CD = AD \cdot BC)$, P – intersection of diagonals. Prove that: AP is symmedian line of ABD

Reference: "Some Properties of the Harmonic Quadrilateral",Ion Patrascu, Florentin Smarandache

Design by Abdilkadir Altintas-Afyonkarashisar-Turkey

Solution by Marian Ursarescu-Romania



$$\Delta PAB \sim \Delta PDC \Rightarrow \frac{PB}{PC} = \frac{PA}{PD} = \frac{AB}{CD} \quad (1)$$

$$\Delta PAD \sim \Delta PBC \Rightarrow \frac{PD}{PC} = \frac{PA}{PB} = \frac{BC}{AD} \quad (2)$$

But $AB \cdot CD = AD \cdot BC \Rightarrow \frac{AB}{AD} = \frac{BC}{CD}$ **(3)**

From (1)+(2)+(3)$\Rightarrow \frac{PB}{PD} = \frac{AB^2}{AD^2} \Rightarrow P$ is symmedian line of ΔABD.

195. If $a_1, a_2, \dots, a_n \in [0,1), n \in \mathbb{N}^*$ then:

$$\prod_{k=1}^{n} \frac{1+a_k}{1-a_k} \geq \left(\frac{1+\sqrt[n]{a_1 a_2 \dots a_n}}{1-\sqrt[n]{a_1 a_2 \dots a_n}}\right)^n$$

Proposed by Regragui El Khammal-Morocco

Solution by Daniel Sitaru -Romania:

$$1-a_1 \geq 1-a_2 \geq \dots \geq 1-a_n; \frac{1}{1-a_1} \leq \frac{1}{1-a_2} \leq \dots \leq \frac{1}{1-a_n}$$

$$\frac{1+a_1}{1-a_1} \leq \frac{1+a_2}{1-a_2} \leq \dots \leq \frac{1+a_n}{1-a_n}$$

$$\prod_{k=1}^{n} \frac{1+a_k}{1-a_k} \geq \left(\frac{1+a_1}{1-a_1}\right)^n = f(a_1), (1)$$

$$f:[0,1) \to \mathbb{R}, f(x) = \frac{1+x}{1-x}, f'(x) = \frac{-2x}{(1-x)^2} < 0, f \textit{ decreasing}$$

$$a_1 \leq \sqrt[n]{a_1 a_2 \dots a_n}; f(a_1) \geq f\left(\sqrt[n]{a_1 a_2 \dots a_n}\right)$$

By (1):

$$\prod_{k=1}^{n} \frac{1+a_k}{1-a_k} \geq \left(\frac{1+a_1}{1-a_1}\right)^n = \left(f(a_1)\right)^n \geq \left(f\left(\sqrt[n]{a_1 a_2 \dots a_n}\right)\right)^n$$

$$\prod_{k=1}^{n} \frac{1+a_k}{1-a_k} \geq \left(\frac{1+\sqrt[n]{a_1 a_2 \dots a_n}}{1-\sqrt[n]{a_1 a_2 \dots a_n}}\right)^n$$

196. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\sum_{cyc} (\tan x + 2\sin x) > 3(x+y+z)$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

Si: $x, y, z \in< 0, \frac{\pi}{2} >$. ***Probar que:*** $\sum(\tan x + 2\operatorname{sen} x) > 3(x + y + z)$

$$(\tan x + 2\operatorname{sen} x - 3x) + (\tan y + 2\operatorname{sen} y - 3y) + (\tan z + 2\operatorname{sen} z - 3z) > 0$$

Consideremos:

$f(x) = \tan x + 2\operatorname{sen} x - 3x$. ***Realizamos la primera derivada:***

$f'(x) = \sec^2 x + 2\cos x - 3$. ***Realizamos la segunda derivada.***

$$f''(x) = -2\operatorname{sen} x + \frac{2\operatorname{sen} x}{\cos^3 x} = 2\operatorname{sen} x\frac{(1-\cos^3 x)}{\cos^3 x} > 0\ \forall x \in< 0, \frac{\pi}{2} >$$

Desde que: $f(0) = f'(0) = 0$ y $f''(x) > 0$, ***se concluye que:*** $f(x) > 0$

$$\tan x + 2\operatorname{sen} x - 3x > 0 \quad (A)$$

$$\tan y + 2\operatorname{sen} y - 3y > 0 \quad (B)$$

$$\tan z + 2\operatorname{sen} z - 3z > 0 \quad (C)$$

Sumando: $(A) + (B) + (C)$

$$(\tan x + 2\operatorname{sen} x - 3x) + (\tan y + 2\operatorname{sen} y - 3y) + (\tan z + 2\operatorname{sen} z - 3z) > 0$$

(LQQD)

197. Prove that:

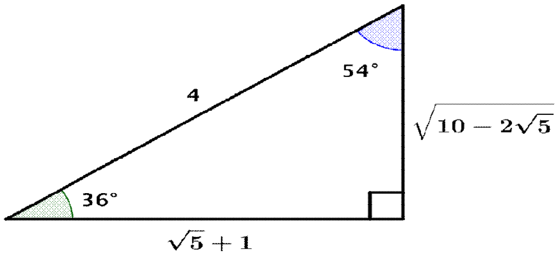
$$\left(\frac{1}{3} + \frac{1}{6\sin\frac{\pi}{5}}\right)\left(\frac{1}{6} + \frac{1}{3\cos\frac{\pi}{5}}\right) > \frac{5}{18}$$

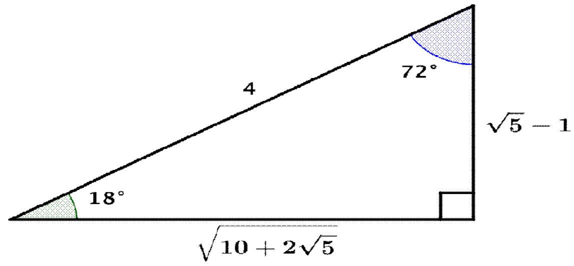
Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

Probar que si es (V) ó (F), lo siguiente:

$\left(\frac{1}{3} + \frac{1}{6\operatorname{sen} 36}\right)\left(\frac{1}{6} + \frac{1}{3\cos 36}\right) > \frac{5}{18} \rightarrow$ ***Triángulos aproximados:***





Desarrollando lo pedido:

$$\frac{1}{18}+\frac{1}{36\,\text{sen}\,36}+\frac{1}{9\cos 36}+\frac{1}{18\,\text{sen}\,36\cos 36}>\frac{5}{18}\rightarrow$$

$$\rightarrow\frac{1}{36\,\text{sen}\,36}+\frac{1}{9\cos 36}+\frac{1}{18\,\text{sen}\,36\cos 36}>\frac{4}{18}$$

Multiplicando a la epresión $36\,\text{sen}\,36\cos 36$:

$$\cos 36+4\,\text{sen}\,36+2>8\,\text{sen}\,36\cos 36\rightarrow\cos 36+4\,\text{sen}\,36+2>4\,\text{sen}\,72$$

$$2+\cos 36>4(\text{sen}\,72-\text{sen}\,36)\rightarrow 2+\cos 36>4(2\cos 54\,\text{sen}\,18)$$

$$2+\frac{\sqrt{5}+1}{4}>4\left(\frac{\sqrt{5}-1}{4}\right)2\cos 54\rightarrow\frac{\sqrt{5}+9}{4}>\left(\sqrt{5}-1\right)\left(2\frac{\sqrt{10-2\sqrt{5}}}{4}\right)$$

Multiplicando a la expresión: $\left(\sqrt{5}+1\right)$:

$$\frac{\sqrt{5}+9}{4}\left(\sqrt{5}+1\right)>\left(\sqrt{5}+1\right)\left(\sqrt{5}-1\right)\frac{\sqrt{10-2\sqrt{5}}}{2}\rightarrow$$

$$\rightarrow 14+10\sqrt{5}>8\sqrt{10-2\sqrt{5}}\rightarrow 7+5\sqrt{5}>4\sqrt{10-2\sqrt{5}}\Leftrightarrow 10-2\sqrt{5}>0$$

Elevando al cuadrado:

$$\left(7+5\sqrt{5}\right)^2>16\left(10-2\sqrt{5}\right)\rightarrow 49+70\sqrt{5}+125>160-32\sqrt{5}$$

$$\rightarrow 174+70\sqrt{5}>160-32\sqrt{5}\rightarrow 14+102\sqrt{5}>0$$

(Dicha expresión es positiva) $\rightarrow$ ***(V)***

198. Prove that:

$$\frac{1}{\sin x}+\frac{2\sqrt{2}}{\cos x}\geq 3\sqrt{3}\quad\left(0<x<\frac{\pi}{2}\right)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Demostrar que para x en el intervalo $<0,\frac{\pi}{2}>$

$$\csc x+2\sqrt{2}\sec x\geq 3\sqrt{3}\quad(1)$$

Sea: $\sin x=a\ \wedge\cos x=b\rightarrow a^2+b^2=1$

Por: $MA\geq MG$

$$a^2+\frac{b^2}{2}+\frac{b^2}{2}\geq 3\sqrt[3]{a^2\frac{b^4}{4}}\rightarrow\frac{1}{3}\geq\sqrt[3]{\frac{b^4}{4}a^2}\rightarrow$$

→ Elevando al cubo e invirtiendo tenemos:

$\frac{3^3}{4}\leq\frac{1}{a^2}\frac{1}{b^4}\rightarrow$ ***Extraemos raíz cuadrada:*** $\frac{3\sqrt{3}}{2}\leq\frac{1}{a}\frac{1}{b^2}$ ***(2)***

El equivalente de (1) es: $\frac{1}{a}+\frac{\sqrt{2}}{b}+\frac{\sqrt{2}}{b}\geq 3\sqrt{3}$***. Por*** $MA\geq MG$***:***

$\frac{1}{a}+\frac{\sqrt{2}}{b}+\frac{\sqrt{b}}{2}\geq 3\sqrt[3]{\frac{1}{a}\frac{2}{b^2}}$ ***(3) → De (2) ∧ (3) por transitividad tenemos:***

$$\frac{1}{a}+\frac{\sqrt{2}}{b}+\frac{\sqrt{2}}{b}\geq 3\sqrt[3]{\frac{1}{a}\frac{2}{b^2}}\geq 3\sqrt{3}$$

Solution 2 by Kunihiko Chikaya-Tokyo-Japan

$$\frac{1}{\sin x}+\frac{2\sqrt{2}}{\cos x}\geq 3\sqrt{3}\quad\left(0<x<\frac{\pi}{2}\right)$$

By AM-GM $\sin x,\cos x\in(0,1)$

$$\frac{1}{\sin x}+2\sqrt{2}\frac{1}{\cos x}\geq -3\sin x+2\sqrt{3}+2\sqrt{2}\left(-\frac{3}{2}\cos x+\sqrt{6}\right)$$

$$= 6\sqrt{3} - 3\left(\sqrt{2}\cos x + 1\cdot\sin x\right) \ge 6\sqrt{3} - 3\sqrt{\left(\sqrt{2}\right)^2 + 1^2} = 3\sqrt{3}$$

By Cauchy – Schwarz. Equality: $\cos x = \sqrt{\frac{2}{3}}, \sin x = \frac{1}{\sqrt{3}}$

Solution 3 by Soumava Chakraborty-Kolkata-India

Prove that $\frac{1}{\sin x} + \frac{2\sqrt{2}}{\cos x} \ge 3\sqrt{3}\ \forall x \in \left(0, \frac{\pi}{2}\right)$

Let $f(x) = \operatorname{cosec} x + 2\sqrt{2}\operatorname{cosec} x - 3\sqrt{3}$

$$f'(x) = -\operatorname{cosec} x \cot x + 2\sqrt{2}\sec x \tan x$$

$f''(x) = \operatorname{cosec}^3 x + \cot^2 x \operatorname{cosec} x + 2\sqrt{2}(\sec^3 x + \sec x \tan^2 x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$

when $f'(x) = 0, f(x)$ ***attains a minima and*** $f(x)$ ***never attains a maxima in*** $\left(0, \frac{\pi}{2}\right)$***, point at which*** $f(x)$ ***attains a minima is the point at which*** $f(x)$ ***attains its minimum value***

$$f'(x) = 0 \Rightarrow 2\sqrt{2}\cdot\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}$$

$$\Rightarrow 2\sqrt{2}\sin^3 x = \cos^3 x \Rightarrow \sqrt{2}\sin x = \cos x$$

$\Rightarrow \tan x = \frac{1}{\sqrt{2}} \Rightarrow \cos^2 x = \frac{2}{3}$ ***and*** $\sin^2 x = \frac{1}{3}$

$f_{\min} = \frac{1}{1\sqrt{3}} + \frac{2\sqrt{2}}{\frac{\sqrt{2}}{\sqrt{3}}} = 3\sqrt{3}$ ***at*** $x = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \frac{1}{\sin x} + \frac{2\sqrt{2}}{\cos x} \ge 3\sqrt{3}$***, equality at*** $x = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

199. $\frac{p^3}{\cos\theta} + \frac{q^3}{\sin\theta} \ge (p^2 + q^2)^{\frac{3}{2}} \quad \left(0 < \theta < \frac{\pi}{2}\right)$

p, q **are positive constants**

Proposed by Kunihiko Chikaya-Tokyo-Japan

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Solution by Kunihiko Chikaya-Tokyo-Japan

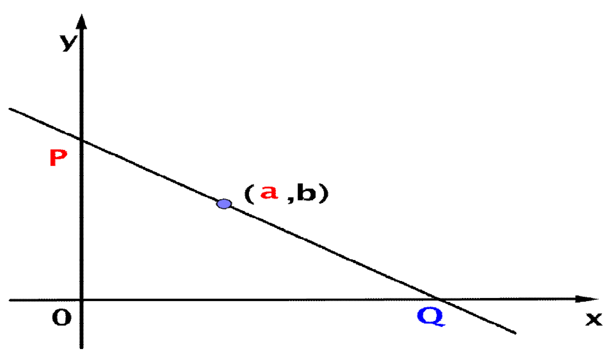
$$\frac{p^3}{\cos\theta} + \frac{q^3}{\sin\theta} \geq p^3\left(\frac{2\sqrt{p^2+q^2}}{p} - \frac{p^2+q^2}{p^2}\cos\theta\right)$$

$$\textit{Equality}\ \binom{p}{q} = \binom{\cos\theta}{\sin\theta} + q^3\left(\frac{2\sqrt{p^2+q^2}}{q} - \frac{p^2+q^2}{q^2}\sin\theta\right)$$

$$\Leftrightarrow \tan\theta = \frac{q}{p} = \sqrt[3]{\frac{b}{a}} = 2(p^2+q^2)\sqrt{p^2+q^2} - (p^2+q^2)(p\cos\theta + q\sin\theta)$$

$$\geq (p^2+q^2)^{\frac{3}{2}}$$

$$p = a^{\frac{1}{3}}, q = b^{\frac{1}{3}}\ (a, b > 0) \Rightarrow \frac{a}{\cos\theta} + \frac{b}{\sin\theta} \geq \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} = \left|\overrightarrow{PQ}\right|_{min}$$



200. Prove that if $A, B, C, D > 0, A + B + C + D = \frac{\pi}{4}$

$$\Omega_1 = \sum \tan A + \sum \tan A \tan B - \sum \tan A \tan B \tan C$$

$$\Omega_2 = \frac{\sin^2(A+B)\sin^2(C+D)}{\cos^2 A \cos^2 B \cos^2 C \cos^2 D}$$

then:

$$16(\Omega_1 - 1) \leq \Omega_2$$

Proposed by Daniel Sitaru-Romania

Solution by Kevin Soto Palacios –Huarmey-Peru:

Si: $A, B, C, D > 0, A + B + C + D = \frac{\pi}{4}$

$$A_1 = \sum \tan A + \sum \tan A \tan B - \sum \tan A \tan B \tan C$$

$$A_2 = \frac{\text{sen}^2(A+B)\,\text{sen}^2(C+D)}{(\cos A \cos B)^2(\cos C \cos D)^2} = (\tan A + \tan B)^2(\tan C + \tan D)^2$$

Probar que: $16(A_1 - 1) \le A_2$

$$\tan\big((A+B) + (C+D)\big) = \tan\frac{\pi}{4}$$

$$\frac{\tan(A+B) + \tan(C+D)}{1 - \tan(A+B)\tan(C+D)} = 1$$

$$\Rightarrow \tan(A+B) + \tan(C+D) = 1 - \tan(A+B)\tan(C+D)$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} + \frac{\tan C + \tan D}{1 - \tan C \tan D} = 1 - \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right)\left(\frac{\tan C + \tan D}{1 - \tan C \tan D}\right)$$

Multiplicamos: $(1 - \tan A \tan B)(1 - \tan C \tan D) \ne 0$

$$(\tan A + \tan B)(1 - \tan C \tan D) + (\tan C + \tan D)(1 - \tan A \tan B) =$$

$$= (1 - \tan A \tan B)(1 - \tan C \tan D) - (\tan A + \tan B)(\tan C + \tan D)$$

$$\to A_1 = \sum \tan A + \sum \tan A \tan B - \sum \tan A \tan B \tan C$$

$$= \tan A \tan B \tan C \tan D + 1$$

$$\Rightarrow 16(A_1 - 1) \le A_2 \to 16 \tan A \tan B \tan C \tan D \le (\tan A + \tan B)^2(\tan C + \tan D)^2 \to$$

$$\to (\textit{Válido por}: MA \ge MG)$$

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Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru