Some Relations On Triangles Whose Centers Are On Its Incircle

Abdilkadir Altintaş
Afyonkarahisar, Emirdağ, Turkey
e-mail: kadiraltintas1977@gmail.com

Abstract. In this work we present a collection of some relations on triangles whose centers are on its incircle .

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1. Problems

First problem is an exercise problem in reference [1] on page 76.

Problem 1.1. *Let ABC be a triangle with side lengths are a, b, c and centroid G. Show that the centroid of triangle ABC lies on the incircle if and only if:*

$$5(a^2 + b^2 + c^2) = 6(ab + bc + ca)$$

Proof. In homogenous barycentric coordinates respect to triangle ABC, incircle has equation;

$$a^2yz + b^2zx + c^2xy - (x + y + z)\left((s - a)^2x + (s - b)^2y + (s - c)^2z\right) = 0$$

Putting centroid $G = (1 : 1 : 1)$ in this equation we get;

$$\frac{1}{4}\left(-5a^2 - 5b^2 + 6bc - 5c^2 + 6a(b + c)\right) = 0$$

Hence desired results follows. □

Second problem is an exercise problem in reference [2].

Problem 1.2. *Let ABC be a triangle with side lengths are a, b, c and Nagel point Na. Show that the nagel point of triangle ABC lies on the incircle if one its sides length is $1/4$ of its perimeter.*

Proof. Nagel point has coordinates $Na = (s - a : s - b : s - c)$. Putting this point in the equation of incircle we get;

$$(s-a)^4 + (s-b)^4 + (s-c)^4 - 2(s-b)^2(s-c)^2 - 2(s-c)^2(s-a)^2 - 2(s-a)^2(s-b)^2 = 0$$

$$s(s + a - b - c)(s - a + b - c)(s - a - b + c) = 0$$

$$(3a - b - c)(3b - c - a)(3c - a - b) = 0$$

The desired triangles are therefore those where the sum of two of the sides is three times the third side. In other words one side length is $1/4$ of its perimeter. □

Third problem is in reference [3].

Problem 1.3. *Let ABC be a triangle with side lengths are a, b, c and Spieker point Sp. Show that the Spieker point of triangle ABC lies on the incircle if:*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{10}{a+b+c}$$

Proof. Spieker point has coordinates $Sp=(b+c:c+a:a+b)$. Putting this point in the equation of incircle we get;

$$2(b^2c^2+c^2a^2+a^2b^2)+a^3(b+c)+b^3(c+a)+c^3(a+b)=5abc(a+b+c)$$

leads to the well-earned final result

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{10}{a+b+c}$$

Its simple example is triangle $a=1, b=c=2$. □

Problem 1.4. *Let ABC be a triangle with side lengths are a, b, c and circumcenter O. Show that O lies on the incircle if:*

$$\cos A+\cos B+\cos C=\sqrt{2}$$

Proof.

$$OI^2=R(R-2r)$$
$$R^2-2Rr=r^2$$
$$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

So $R^2-2Rr=r^2$ is equivalent to

$$1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}-16\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}=0$$

which becomes

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{4}\left(\sqrt{2}-1\right)$$

We discard the solution $-(\sqrt{2}+1)/4$, since all the sines are necessarily positive.) We can also write last equation as

$$\cos A+\cos B+\cos C=\sqrt{2}$$

the Law of Cosines, and a bit of clean-up, then provides this form of the relation:

$$(-a+b+c)(a-b+c)(a+b-c)=2abc\left(\sqrt{2}-1\right)$$

□

Problem 1.5. ***[31 st Indian N.M.O, Jan. 2016]*** *Let ABC be an isosceles triangle with $AB=AC$ and orthocenter H. If orthocenter of triangle ABC lies on the incircle, compute $\frac{AB}{BC}$.*

REFERENCES

[1] P.Yiu, *Introduction to the Geometry of the Triangle*, Department of Mathematics, Florida Atlantic University, 2013.
[2] W.Gallatly, *Modern Geometry of Triangle*, 1910.
[3] O.Bottema, *Topics in Elementary Geometry*, Springer, 2007.
[4] *https://math.stackexchange.com*