SQUARED DISTANCE SUMS

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Abstract. In this work we present the squared distance sums of a point X to the vertices of ABC in terms of s, R and r.

Keywords. Squared Distance, Centers

1. Introduction

In this work a, b, c is usual notations for side lengths of a triangle.s, r, R are semi perimeter, inradius and circumradius respectively. I, H, N, Sp, Na are notations for incenter, orthocenter, nine-point center, Spieker point and Nagel point of ABC respectively. Squared distance sums of these points evaluated in homogenous barycentric coordinates.

The coordinates $(u : v : w)$ are often normalized such that the values sum to 1, that is $u + v + w = 1$ in order to provide easier calculations. To normalize coordinates $(u : v : w)$, simply multiply each value by $\frac{1}{u+v+w}$. Therefore, the normalized coordinates of $P(u : v : w)$ are [3]:

$$\left\{\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w}\right\}$$

Let displacement vector of $PQ = (x, y, z)$ in normalized barycentric coordinates than,

$$PQ^2 = -a^2yz - b^2zx - c^2xy$$

The barycentric coordinates of some centers are as follows;

$$I = \{a : b : c\}$$

$$N = \left\{\left(b^2-c^2\right)^2 - a^2\left(b^2+c^2\right) : a^4 - b^2c^2 + c^4 - a^2\left(b^2+2c^2\right) : a^4 + b^4 - b^2c^2 - a^2\left(2b^2+c^2\right)\right\}$$

$$H = \left\{a^4 - \left(b^2-c^2\right)^2 : -a^4 + b^4 + 2a^2c^2 - c^4 : -a^4 + 2a^2b^2 - b^4 + c^4\right\}$$

$$S_p = \{b + c : c + a : a + b\}$$

$$N_a = \{s - a : s - b : s - c\}$$

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2. Squared Distance Sums

Problem 2.1. *Let I be incenter of ABC. Prove;*

$$AI^2+BI^2+CI^2=s^2+r^2-8rR$$

Proof.

$$AI^2+BI^2+CI^2=\frac{a^2(b+c)+bc(b+c)+a\left(b^2-3bc+c^2\right)}{a+b+c}$$

which can be written as,

$$\frac{2s\left(s^2+r^2-8rR\right)}{2s}=s^2+r^2-8rR \tag{1}$$

□

Problem 2.2. *Let H be orthocenter of ABC. Prove;*

$$AH^2+BH^2+CH^2=2\left(6R^2+4rR+r^2-s^2\right)$$

Proof.

$$AH^2+BH^2+CH^2=\frac{-a^6+a^4\left(b^2+c^2\right)-\left(b^2-c^2\right)^2\left(b^2+c^2\right)+a^2\left(b^4-6b^2c^2+c^4\right)}{a^4+\left(b^2-c^2\right)^2-2a^2\left(b^2+c^2\right)}$$

which can be written as,

$$\frac{32s^2r^2\left(s^2-r^2-4rR-6R^2\right)}{-16s^2r^2}=2\left(6R^2+4rR+r^2-s^2\right) \tag{2}$$

□

Problem 2.3. *Let N be nine-point center of ABC. Prove;*

$$AN^2+BN^2+CN^2=\frac{2s^2-2r^2-8rR+3R^2}{4}$$

Proof.

$$AN^2+BN^2+CN^2=\frac{a^6-a^4\left(b^2+c^2\right)+\left(b^2-c^2\right)^2\left(b^2+c^2\right)-a^2\left(b^4+9b^2c^2+c^4\right)}{4\left(a^4+\left(b^2-c^2\right)^2-2a^2\left(b^2+c^2\right)\right)}$$

which can be written as,

$$\frac{-16s^2r^2\left(2s^2-2r^2-8rR+3R^2\right)}{-64s^2r^2}=\frac{2s^2-2r^2-8rR+3R^2}{4} \tag{3}$$

□

Problem 2.4. *Let S_p be Spieker point of ABC. Prove;*

$$AS_p^2+BS_p^2+CS_p^2=\frac{3s^2-r^2-16rR}{4}$$

Proof.

$$AS_p^2 + BS_p^2 + CS_p^2 = \frac{a^3 + b^3 + 2b^2c + 2bc^2 + c^3 + 2a^2(b+c) + a\left(2b^2 - 3bc + 2c^2\right)}{4(a+b+c)}$$

which can be written as,

$$\frac{2s\left(3s^2 - r^2 - 16rR\right)}{8s} = \frac{3s^2 - r^2 - 16rR}{4} \tag{4}$$

□

Problem 2.5. *Let N_a be Nagel point of ABC. Prove;*

$$AN_a^2 + BN_a^2 + CN_a^2 = 2\left(s^2 + 3r^2 - 12rR\right)$$

Proof.

$$AN_a^2 + BN_a^2 + CN_a^2 = -\frac{a^3 + b^3 - 3b^2c - 3bc^2 + c^3 - 3a^2(b+c) - 3a\left(b^2 - 4bc + c^2\right)}{a+b+c}$$

which can be written as,

$$\frac{4s\left(s^2 + 3r^2 - 12rR\right)}{2s} = 2\left(s^2 + 3r^2 - 12rR\right) \tag{5}$$

□

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