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A MASTER THEOREM OF SERIES AND AN EVALUATION OF A CUBIC HARMONIC SERIES

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Abstract. In the actual paper we present and prove a new powerful theorem, a *Master Theorem of Series*, with important implications and numerous applications in the area of calculation of series with the generalized harmonic numbers. Using the mentioned theorem, we calculate one of the classical advanced cubic harmonic series by elementary series manipulations.

1. Introduction

The central theorem of the paper, a *Master Theorem of Series*, says that if k is a positive integer with $\mathscr{M}(k)=m(1)+m(2)+\cdots+m(k)$, and $m(k)$ are real numbers, where $\lim_{k\to\infty} m(k)=0$, then the following double equality holds

$$\sum_{k=1}^{\infty}\frac{\mathscr{M}(k)}{(k+1)(k+n+1)}=m(1)\left(\frac{H_n}{n}-\frac{1}{n+1}\right)+\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k+1)}{j+k+1}$$
$$=\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k)}{j+k},$$

where $H_n=1+\frac{1}{2}+\ldots+\frac{1}{n}$ denotes the nth harmonic number.

Using the stated *Master Theorem of Series* we will prove elementarily that

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3=\frac{1}{2}\left(\frac{93}{8}\zeta(6)-5\zeta^2(3)\right),$$

a classical result in the area of calculation of Euler sums which in [5] can be found evaluated by means of complex analysis.

Also similar series were previously calculated by means of real methods (see [1] and [2]), using a couple of special logarithmic integrals, elementary manipulations of series and the well-known Euler's identity in (6) (see [6]). With the new approach, *The Master Theorem of Series* will allow us to get the desired results without using integrals, but only by using elementary manipulations of series and the well-known Euler's identity in (6). We mention that in the evaluation of the cubic Euler sum we also employ auxiliary results which are not new and they exist in the mathematical literature. The series present in Lemma 5 and Lemma 6 are known and evaluated in [5], and the Corollary 1 can be found evaluated in [3].

We state below the first theorem we are going to prove.

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THEOREM 1. (A Master Theorem of Series) *If k is a positive integer with $\mathscr{M}(k) = m(1)+m(2)+\cdots+m(k)$, and $m(k)$ are real numbers, where $\lim_{k\to\infty} m(k) = 0$, then the following double equality holds*

$$\sum_{k=1}^{\infty} \frac{\mathscr{M}(k)}{(k+1)(k+n+1)} = m(1)\left(\frac{H_n}{n} - \frac{1}{n+1}\right) + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k+1)}{j+k+1}$$
$$= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k)}{j+k}.$$

Proof. Considering the partial sum of the series and using that

$$\frac{1}{n}\sum_{j=1}^{n}\frac{1}{(j+k)(j+k+1)} = \frac{1}{(k+1)(k+n+1)},$$

we get

$$\begin{aligned}
\sum_{k=1}^{N} \frac{\mathscr{M}(k)}{(k+1)(k+n+1)} &= \frac{1}{n}\sum_{k=1}^{N}\sum_{j=1}^{n}\frac{\mathscr{M}(k)}{(j+k)(j+k+1)} \\
&= \frac{1}{n}\sum_{k=1}^{N}\sum_{j=1}^{n}\left(\frac{\mathscr{M}(k)}{j+k} - \frac{\mathscr{M}(k)}{j+k+1}\right) \\
&= \frac{1}{n}\sum_{k=1}^{N}\sum_{j=1}^{n}\left(\frac{\mathscr{M}(k)}{j+k} - \frac{\mathscr{M}(k+1)-m(k+1)}{j+k+1}\right) \\
&= \frac{1}{n}\sum_{k=1}^{N}\sum_{j=1}^{n}\left(\frac{\mathscr{M}(k)}{j+k} - \frac{\mathscr{M}(k+1)}{j+k+1} + \frac{m(k+1)}{j+k+1}\right) \\
&= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{N}\left(\frac{\mathscr{M}(k)}{j+k} - \frac{\mathscr{M}(k+1)}{j+k+1} + \frac{m(k+1)}{j+k+1}\right) \\
&= \frac{1}{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{N}\left(\frac{\mathscr{M}(k)}{j+k} - \frac{\mathscr{M}(k+1)}{j+k+1}\right) + \sum_{k=1}^{N}\frac{m(k+1)}{j+k+1}\right).
\end{aligned} \tag{1}$$

For the first inner sum in (1), we have

$$\begin{aligned}
\sum_{k=1}^{N}\left(\frac{\mathscr{M}(k)}{j+k} - \frac{\mathscr{M}(k+1)}{j+k+1}\right) &= \frac{\mathscr{M}(1)}{j+1} - \frac{\mathscr{M}(N+1)}{j+N+1} \\
&= \frac{m(1)}{j+1} - \frac{\mathscr{M}(N+1)}{j+N+1}.
\end{aligned} \tag{2}$$

Then, by plugging (2) in (1), we obtain that

$$\begin{aligned}
\sum_{k=1}^{N} \frac{\mathscr{M}(k)}{(k+1)(k+n+1)} &= \frac{1}{n}\sum_{j=1}^{n}\left(\frac{m(1)}{j+1} - \frac{\mathscr{M}(N+1)}{j+N+1} + \sum_{k=1}^{N}\frac{m(k+1)}{j+k+1}\right) \\
&= m(1)\left(\frac{H_n}{n} - \frac{1}{n+1}\right) - \frac{1}{n}\sum_{j=1}^{n}\frac{\mathscr{M}(N+1)}{j+N+1} + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{N}\frac{m(k+1)}{j+k+1}.
\end{aligned} \tag{3}$$

Letting now $N \to \infty$ in (3), and using the Stolz–Cesàro theorem to show that $\lim_{N\to\infty} \frac{\mathscr{M}(N+1)}{j+N+1} = 0$, we obtain that

$$\sum_{k=1}^{\infty} \frac{\mathscr{M}(k)}{(k+1)(k+n+1)} = m(1)\left(\frac{H_n}{n} - \frac{1}{n+1}\right) + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{m(k+1)}{j+k+1}$$
$$= m(1)\left(\frac{H_n}{n} - \frac{1}{n+1}\right) + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=2}^{\infty} \frac{m(k)}{j+k}$$
$$= m(1)\left(\frac{H_n}{n} - \frac{1}{n+1}\right) + \frac{1}{n}\sum_{j=1}^{n}\left(-\frac{m(1)}{j+1} + \sum_{k=1}^{\infty} \frac{m(k)}{j+k}\right)$$
$$= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{m(k)}{j+k}. \quad \square$$

COROLLARY 1. *Let $n \geqslant 1$ be a positive integer. The following equality holds*

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{(k+1)(k+n+1)} = \begin{cases} \dfrac{H_n^2 + H_n^{(2)}}{2n}, & p = 1 \\ \dfrac{(-1)^{p-1}}{n}\left(\sum_{i=1}^{n} \dfrac{H_i}{i^p} + \sum_{i=2}^{p} (-1)^{i-1}\zeta(i) H_n^{(p-i+1)}\right), & p \geqslant 2, \end{cases}$$

where $H_k^{(p)} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$ is the kth harmonic number of order p.

Proof. The result is obtained immediately if using our *Master Theorem of Series*, the second equality, where we set $\mathscr{M}(k) = H_k^{(p)}$, $m(k) = \frac{1}{k^p}$. $\square$

COROLLARY 2. *Let $n \geqslant 1$ be a positive integer. The following equality holds*

$$\sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(k+n+1)} = \frac{H_n^3 + 3\zeta(2)H_n + 3H_nH_n^{(2)} + 2H_n^{(3)}}{3n} - \frac{1}{n}\sum_{i=1}^{n} \frac{H_i}{i^2},$$

where $H_k^{(p)} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$ is the kth harmonic number of order p.

Proof. By employing *The Master Theorem of Series*, the first equality, where we set $\mathscr{M}(k) = H_k^2$ and $m(k) = H_k^2 - H_{k-1}^2$, we get

$$\sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(k+n+1)} = (H_1^2 - H_0^2)\left(\frac{H_n}{n} - \frac{1}{n+1}\right) + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{H_{k+1}^2 - H_k^2}{j+k+1}$$
$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{2H_k + 1/(k+1)}{(k+1)(j+k+1)}$$
$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{2}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{H_k}{(k+1)(j+k+1)}$$
$$+ \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{1}{(k+1)^2(j+k+1)},$$

and making use of Corollary 1, the case $p = 1$, we get

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(k+n+1)} &= \frac{H_n}{n} - \frac{1}{n+1} + \frac{1}{n}\sum_{j=1}^{n} \frac{H_j^2 + H_j^{(2)}}{j} \\
&\quad + \frac{1}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{\infty}\left(\frac{1}{(k+1)^2} - \frac{1}{(k+1)(j+k+1)}\right) \\
&= \frac{H_n}{n} - \frac{1}{n+1} + \frac{H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}}{3n} \\
&\quad + \frac{1}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{\infty}\left(\frac{1}{(k+1)^2} - \frac{1}{(k+1)(j+k+1)}\right) \\
&= \frac{H_n}{n} - \frac{1}{n+1} + \frac{H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}}{3n} \\
&\quad + \frac{1}{n}\sum_{j=1}^{n}\frac{1}{j}\left(\zeta(2) - 1 + \frac{1}{j+1} - \frac{H_j}{j}\right) \\
&= \frac{H_n}{n} - \frac{1}{n+1} + \frac{H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}}{3n} \\
&\quad + (\zeta(2)-1)\frac{H_n}{n} + \frac{1}{n+1} - \frac{1}{n}\sum_{j=1}^{n}\frac{H_j}{j^2} \\
&= \frac{H_n^3 + 3\zeta(2)H_n + 3H_nH_n^{(2)} + 2H_n^{(3)}}{3n} - \frac{1}{n}\sum_{j=1}^{n}\frac{H_j}{j^2},
\end{aligned}$$

where in the calculations we made use of Lemma 2, and our proof is finalized. □

LEMMA 1. *Let $n, p \geqslant 1$ be a positive integer. The following equality holds*

$$\sum_{k=1}^{n} \frac{H_k^{(p)}}{k^p} = \frac{1}{2}((H_n^{(p)})^2 + H_n^{(2p)}),$$

where $H_k^{(p)} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$ is the kth harmonic number of order p.

Proof. The result is straightforward by Abel's summation formula (see [4, p. 55]) where we set $a_k = 1/k^p$ and $b_k = H_k^{(p)}$. □

LEMMA 2. *Let $n \geqslant 1$ be a positive integer. The following equality holds*

$$\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} = \frac{1}{3}(H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}),$$

where $H_k^{(p)} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$ is the kth harmonic number of order p.

Proof. Using Abel's summation formula (see [4, p. 55]) with $a_k = \frac{1}{k}$ and $b_k = H_k^2 + H_k^{(2)}$, we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} &= H_n(H_{n+1}^2 + H_{n+1}^{(2)}) - 2\sum_{k=1}^{n} \left(\frac{H_k^2}{k+1} + \frac{H_k}{(k+1)^2} \right) \\ &= H_n(H_{n+1}^2 + H_{n+1}^{(2)}) - 2\sum_{k=1}^{n} \left(\frac{(H_{k+1} - 1/(k+1))^2}{k+1} + \frac{H_{k+1} - 1/(k+1)}{(k+1)^2} \right) \\ &= H_n(H_{n+1}^2 + H_{n+1}^{(2)}) - 2\sum_{k=1}^{n} \frac{H_{k+1}^2}{k+1} + 2\sum_{k=1}^{n} \frac{H_{k+1}}{(k+1)^2} \\ &= H_n(H_{n+1}^2 + H_{n+1}^{(2)}) - 2\sum_{k=1}^{n+1} \frac{H_k^2}{k} + 2\sum_{k=1}^{n+1} \frac{H_k}{k^2} \\ &= H_n(H_{n+1}^2 + H_{n+1}^{(2)}) - 2\sum_{k=1}^{n} \frac{H_k^2}{k} + 2\sum_{k=1}^{n} \frac{H_k}{k^2} - 2\frac{H_{n+1}^2}{n+1} + 2\frac{H_{n+1}}{(n+1)^2}. \end{aligned} \tag{4}$$

Then, we apply Abel's summation formula for $\sum_{k=1}^{n} \frac{H_k}{k^2}$, where we set $a_k = 1/k^2$ and $b_k = H_k$, and then we get

$$\begin{aligned}\sum_{k=1}^{n} \frac{H_k}{k^2} &= H_{n+1}H_n^{(2)} - \sum_{k=1}^{n} \frac{H_k^{(2)}}{k+1} \\ &= H_{n+1}H_n^{(2)} - \sum_{k=1}^{n} \frac{H_{k+1}^{(2)} - \frac{1}{(k+1)^2}}{k+1} \\ &= H_{n+1}H_n^{(2)} - \sum_{k=1}^{n} \frac{H_{k+1}^{(2)}}{k+1} + \sum_{k=1}^{n} \frac{1}{(k+1)^3} \\ &= H_{n+1}H_n^{(2)} - \sum_{k=1}^{n+1} \frac{H_k^{(2)}}{k} + \sum_{k=1}^{n+1} \frac{1}{k^3} \\ &= H_{n+1}H_n^{(2)} - \sum_{k=1}^{n} \frac{H_k^{(2)}}{k} - \frac{H_{n+1}^{(2)}}{n+1} + H_n^{(3)} + \frac{1}{(n+1)^3}. \end{aligned} \tag{5}$$

Then, by combining the results from (4) and (5), we obtain

$$\begin{aligned}\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} &= -2\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} + H_n(H_{n+1}^2 + H_{n+1}^{(2)}) + 2H_{n+1}H_n^{(2)} \\ &\quad - 2\frac{H_{n+1}^{(2)}}{n+1} + 2H_n^{(3)} + \frac{2}{(n+1)^3} - 2\frac{H_{n+1}^2}{n+1} + 2\frac{H_{n+1}}{(n+1)^2} \\ &= -2\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} + H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}, \end{aligned}$$

whence we get

$$\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} = \frac{1}{3}(H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}),$$

and our lemma is proved. □

LEMMA 3. *Let $n \geqslant 1$ be a positive integer. The following equality holds*

$$\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(k+1)(k+n+1)} = \frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{3n},$$

where $H_k^{(p)} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$ is the kth harmonic number of order p.

Proof. The result is obtained immediately by combining Corollary 1 and Corollary 2. □

LEMMA 4. *The following equality holds*

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17}{4}\zeta(4).$$

Proof. Using the case $p = 1$ of Corollary 1 and multiplying both sides of the equality by $1/n$, we get

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)(k+n+1)n} = \frac{H_n^2 + H_n^{(2)}}{2n^2}.$$

Summing both sides of the equality above from $n = 1$ to ∞, we obtain that

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{H_k}{(k+1)(k+n+1)n} \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{H_k}{(k+1)(k+n+1)n} \\
&= \sum_{k=1}^{\infty} \frac{H_k H_{k+1}}{(k+1)^2} = \sum_{k=1}^{\infty} \frac{(H_{k+1} - 1/(k+1))H_{k+1}}{(k+1)^2} \\
&= \sum_{k=1}^{\infty} \frac{(H_k - 1/k)H_k}{k^2} = \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} - \sum_{k=1}^{\infty} \frac{H_k}{k^3} \\
&= \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} - \frac{5}{4}\zeta(4) = \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} - \frac{5}{4}\zeta(4),
\end{aligned}$$

whence we get that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{5}{2}\zeta(4) + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \frac{5}{2}\zeta(4) + \frac{1}{2}\zeta^2(2) + \frac{1}{2}\zeta(4) = \frac{17}{4}\zeta(4),$$

where above we made use of the well-known linear Euler sum identity

$$2\sum_{k=1}^{\infty}\frac{H_k}{k^n}=(n+2)\zeta(n+1)-\sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1),\ \ n\in\mathbb{N},\ \ n\geqslant 2, \tag{6}$$

and Lemma 1, the case $p=2$, and the proof of the lemma is complete. □

LEMMA 5. *The following equality holds*

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4}=\zeta^2(3)-\frac{\zeta(6)}{3}.$$

Proof. To calculate our series, we first start with a slightly different series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\zeta(4)-1-\frac{1}{2^4}-\cdots-\frac{1}{n^4}\right)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^2(n+k)^4}. \tag{7}$$

Considering the partial fraction decomposition in the right-hand side of (7)

$$\frac{1}{n^2(n+k)^4}=\frac{1}{k^4n^2}-\frac{4}{k^5}\left(\frac{1}{n}-\frac{1}{n+k}\right)+\frac{3}{k^4(n+k)^2}+\frac{2}{k^3(k+n)^3}+\frac{1}{k^2(n+k)^4},$$

we get

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^2(n+k)^4}=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^4n^2}-4\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^5}\left(\frac{1}{n}-\frac{1}{n+k}\right)+3\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^2}$$
$$+2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^3(k+n)^3}+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^2(n+k)^4}$$

that leads immediately to

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^2}=\frac{25}{12}\zeta(6)-\zeta^2(3), \tag{8}$$

where we used that due to symmetry $\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^2(n+k)^4}=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^2(n+k)^4}$, then the fact that $2\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\frac{1}{k^3n^3}=\zeta^2(3)-\zeta(6)$ which is straightforward if noting that $\sum_{k=1}^{\infty}\sum_{n=1}^{k-1}\frac{1}{k^3n^3}+\sum_{k=1}^{\infty}\frac{1}{k^6}+\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\frac{1}{k^3n^3}=\zeta^2(3)$, and finally the identity in (6). Therefore, based upon the result in (8), we have

$$\frac{25}{12}\zeta(6)-\zeta^2(3)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^2}=\sum_{k=1}^{\infty}\frac{1}{k^4}\left(\zeta(2)-H_k^{(2)}\right)$$
$$=\zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^4}-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4}=\frac{7}{4}\zeta(6)-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4},$$

whence we obtain that

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4}=\zeta^2(3)-\frac{\zeta(6)}{3},$$

and the proof of the lemma is complete. □

LEMMA 6. *The following equality holds*

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}=\frac{97}{24}\zeta(6)-2\zeta^2(3),$$

where $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ *denotes the* n*th harmonic number.*

Proof. Using the case $p=1$ of Corollary 1 and then multiplying both sides of the equality by $1/n^3$, we have

$$\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+n+1)n^3}=\frac{H_n^2+H_n^{(2)}}{2n^4}.$$

Then, taking the sum over both sides of the relation above from $n=1$ to ∞, we get that

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}&=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+n+1)n^3}\\
&=\sum_{k=1}^{\infty}\frac{H_k}{k+1}\sum_{n=1}^{\infty}\left(\frac{1}{n^3(k+1)}-\frac{1}{n^2(k+1)^2}+\frac{1}{(k+1)^2n(n+k+1)}\right)\\
&=\zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2}-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3}+\sum_{k=1}^{\infty}\frac{H_kH_{k+1}}{(k+1)^4}\\
&=\zeta(3)\sum_{k=1}^{\infty}\frac{H_{k+1}-1/(k+1)}{(k+1)^2}-\zeta(2)\sum_{k=1}^{\infty}\frac{H_{k+1}-1/(k+1)}{(k+1)^3}\\
&\quad+\sum_{k=1}^{\infty}\frac{(H_{k+1}-1/(k+1))H_{k+1}}{(k+1)^4}\\
&=\zeta(3)\sum_{k=1}^{\infty}\frac{H_k-1/k}{k^2}-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k-1/k}{k^3}+\sum_{k=1}^{\infty}\frac{(H_k-1/k)H_k}{k^4}\\
&=\zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{k^2}-\zeta(3)\sum_{k=1}^{\infty}\frac{1}{k^3}-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^3}+\zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^4}\\
&\quad-\sum_{k=1}^{\infty}\frac{H_k}{k^5}+\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}\\
&=\frac{3}{2}\zeta^2(3)-\frac{35}{16}\zeta(6)+\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}=\frac{3}{2}\zeta^2(3)-\frac{35}{16}\zeta(6)+\sum_{n=1}^{\infty}\frac{H_n^2}{n^4},
\end{aligned}$$

whence we obtain that

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}=\frac{35}{8}\zeta(6)-3\zeta^2(3)+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=\frac{97}{24}\zeta(6)-2\zeta^2(3),$$

where in the calculations we made use of Euler's identity in (6) and Lemma 5. □

Now we state and prove the second theorem.

THEOREM 2. (An advanced cubic harmonic series) *The following equality holds*

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3=\frac{1}{2}\left(\frac{93}{8}\zeta(6)-5\zeta^2(3)\right),$$

where $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ *denotes the* n*th harmonic number.*

Proof. We make use of Lemma 3

$$\sum_{k=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{(k+1)(k+n+1)}=\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{3n}.$$

which we multiply by $1/n^2$, and summing both sides from $n=1$ to ∞, we get that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{(k+1)(k+n+1)n^2}=\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}+\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3},$$

or if changing the summations order, we obtain

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{(k+1)(k+n+1)n^2}=\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}+\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}.$$

Since we have that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2(k+n+1)}&=\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}-\frac{1}{k+n+1}\right)\\&=\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{1}{n(k+n+1)}\\&=\frac{\zeta(2)}{k+1}-\frac{H_{k+1}}{(k+1)^2},\end{aligned}$$

then we get

$$\begin{aligned}&\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}+\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}\\&=\sum_{k=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{k+1}\left(\frac{\zeta(2)}{k+1}-\frac{H_{k+1}}{(k+1)^2}\right)\\&=\zeta(2)\sum_{k=1}^{\infty}\left(\frac{H_k}{k+1}\right)^2-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(k+1)^2}-\sum_{k=1}^{\infty}\frac{H_k^2H_{k+1}}{(k+1)^3}+\sum_{k=1}^{\infty}\frac{H_k^{(2)}H_{k+1}}{(k+1)^3}\\&=\zeta(2)\sum_{k=1}^{\infty}\frac{(H_{k+1}-1/(k+1))^2}{(k+1)^2}-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(k+1)^2}-\sum_{k=1}^{\infty}\frac{(H_{k+1}-1/(k+1))^2H_{k+1}}{(k+1)^3}\end{aligned}$$

$$+\sum_{k=1}^{\infty}\frac{(H_{k+1}^{(2)}-1/(k+1)^2)H_{k+1}}{(k+1)^3}$$
$$=\zeta(2)\sum_{k=1}^{\infty}\left(\frac{H_{k+1}}{k+1}\right)^2-2\zeta(2)\sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^3}+\zeta(2)\sum_{k=1}^{\infty}\frac{1}{(k+1)^4}$$
$$-\zeta(2)\sum_{k=1}^{\infty}\frac{H_{k+1}^{(2)}}{(k+1)^2}+\zeta(2)\sum_{k=1}^{\infty}\frac{1}{(k+1)^4}-\sum_{k=1}^{\infty}\left(\frac{H_{k+1}}{k+1}\right)^3+2\sum_{k=1}^{\infty}\frac{H_{k+1}^2}{(k+1)^4}$$
$$-\sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^5}+\sum_{k=1}^{\infty}\frac{H_{k+1}H_{k+1}^{(2)}}{(k+1)^3}-\sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^5}$$
$$=\zeta(2)\sum_{k=1}^{\infty}\left(\frac{H_k}{k}\right)^2-2\zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^3}+\zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^4}-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2}+\zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^4}$$
$$-\sum_{k=1}^{\infty}\left(\frac{H_k}{k}\right)^3+2\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}-\sum_{k=1}^{\infty}\frac{H_k}{k^5}+\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^3}-\sum_{k=1}^{\infty}\frac{H_k}{k^5}$$
$$=\zeta^2(3)-\frac{35}{8}\zeta(6)+\zeta(2)\sum_{k=1}^{\infty}\left(\frac{H_k}{k}\right)^2-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2}-\sum_{k=1}^{\infty}\left(\frac{H_k}{k}\right)^3$$
$$+2\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}+\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^3}$$
$$=\frac{97}{12}\zeta(6)-3\zeta^2(3)-\sum_{k=1}^{\infty}\left(\frac{H_k}{k}\right)^3+\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^3}$$
$$=\frac{97}{12}\zeta(6)-3\zeta^2(3)-\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3},$$

whence we obtain that

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3=\frac{3}{4}\left(\frac{97}{12}\zeta(6)-3\zeta^2(3)-\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}\right)=\frac{1}{2}\left(\frac{93}{8}\zeta(6)-5\zeta^2(3)\right),$$

where in the calculations we made use of Euler's identity in (6), the cases $p=2,3$ of Lemma 1, Lemma 4 and Lemma 6. □

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