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PROBLEM 125 - TRIANGLE MARATHON 101 - 200

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1. In ΔABC

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2}\ge\frac{8r}{3R}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Proof.

Using the known identity in triangle $\sum a^2=2(p^2-r^2-4Rr)$ *and the remarkable inequality* $\sum l_a^2\le p^2$, *which follows from* $l_a\le\sqrt{p(p-a)}$, *we obtain*

$\dfrac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2}\ge\dfrac{2(p^2-r^2-4Rr)}{p^2}\ge\dfrac{8r}{3R}$, *where the last inequality is equivalent with:*
$3R(p^2-r^2-4Rr)\ge p^2r\Leftrightarrow p^2(3R-4r)\ge 3R(r^2+4Rr)$, *true from Gerretsen's inequality* $p^2\ge 16Rr-5r^2$. *It remains to prove that:*
$(16Rr-5r^2)(3R-4r)\ge 3R(r^2+4Rr)\Leftrightarrow 18R^2-41Rr+10r^2\ge 0\Leftrightarrow (R-2r)(18R-5r)\ge 0$,
obviously from Euler's inequality $R\ge 2r$.
The equality holds if and only if the triangle is equilateral.

□

Remark

The inequality can be strengthened:

2. In ΔABC

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2}\ge\frac{18Rr}{p^2}.$$

Proof.

Using the known identity in triangle $\sum a^2=2(p^2-r^2-4Rr)$, *and the remarkable inequality* $\sum l_a^2\le p^2$, *which follows from* $l_a\le\sqrt{p(p-a)}$, *we obtain*

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2}\ge\frac{2(p^2-r^2-4Rr)}{p^2}\ge\frac{18Rr}{p^2},$$

where the last inequality is equivalent with: $p^2\ge r^2+13Rr$, *true from Gerretsen's inequality* $p^2\ge 16Rr-5r^2$.
It remains to prove that:

$$16Rr-5r^2\ge r^2+13Rr\Leftrightarrow 3Rr\ge 6r^2\Leftrightarrow R\ge 2r,\ (\textit{Euler's inequality}).$$

The equality holds if and only if the triangle is equilateral.

□

Remark

Inequality 2. is stronger then inequality 1.:

3. In ΔABC

$$\boldsymbol{\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{18Rr}{p^2} \geq \frac{8r}{3R}}$$

Proof.

See inequality 2. and Mitrinović's inequality: $p^2 \leq \frac{27R^2}{4}$.

The equality holds if and only if the triangle is equilateral.

□

Remark

Also, inequality 2. can be strengthened:

4. In ΔABC

$$\boldsymbol{\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{4}{3}.}$$

Proof.

Using the known identity known in triangle $\sum a^2 = 2(p^2 - r^2 - 4Rr)$ *and the remarkable inequality* $\sum l_a^2 \leq p^2$, *which follows from* $l_a \leq \sqrt{p(p-a)}$, *we obtain* $\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{2(p^2-r^2-4Rr)}{p^2} \geq \frac{4}{3}$, *where the last inequality is equivalent with:*

$$3(p^2 - r^2 - 4Rr) \geq 2p^2 \Leftrightarrow p^2 \geq 3r^2 + 12Rr,$$

true from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$.

It remains to prove that:

$$16Rr - 5r^2 \geq 3r^2 + 12Rr \Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r, \text{ (Euler's inequality).}$$

The equality holds if and only if the triangle is equilateral.

□

Remark

Inequality 4. is stronger than inequality 2.:

5. In ΔABC

$$\boldsymbol{\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{4}{3} \geq \frac{18Rr}{p^2}.}$$

Proposed by Marin Chirciu - Romania

Proof.

See inequality 4. and inequality: $2p^2 \geq 27Rr$, *true from Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$. *It remains to prove that:* $2(16Rr - 5r^2) \geq 27Rr \Leftrightarrow 5Rr \geq 10r^2 \Leftrightarrow R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

We can write the triple inequality:

6. In ΔABC

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{4}{3} \geq \frac{18Rr}{p^2} \geq \frac{8r}{3R}$$

Proof.

*See inequality **5.** and inequality **3.***

Equality holds if and only if the triangle is equilateral.

□

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