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INEQUALITY IN TRIANGLE 295

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1. Prove that in any triangle ABC

$$\sum \frac{1}{(II_a)^2} + \sum \frac{1}{(I_bI_c)^2} \leq \frac{1}{4r^2}$$

Proposed by Daniel Sitaru - Romania

Proof.

Using the formulas $II_a = 4R\sin\frac{A}{2}, I_bI_c = 4R\cos\frac{A}{2}$ *and the known identities in triangle:*

$$\sum \frac{1}{\sin^2\frac{A}{2}} = \frac{p^2+r^2-8Rr}{r^2}, \sum \frac{1}{\cos^2\frac{A}{2}} = \frac{p^2+(4R+r)^2}{p^2}, \text{ we obtain:}$$

$$\sum \frac{1}{(II_a)^2} = \frac{p^2+r^2-8Rr}{16R^2r^2} \text{ and } \sum \frac{1}{(I_bI_c)^2} = \frac{p^2+(4R+r)^2}{16R^2p^2}.$$

We write the inequality:

$$\frac{p^2+r^2-8Rr}{16R^2r^2} + \frac{p^2+(4R+r)^2}{16R^2p^2} \leq \frac{1}{4r^2} \Leftrightarrow p^2(4R^2+8Rr-2r^2-p^2) \geq r^2(4R+r)^2,$$

which follows from Gerretsen's inequality $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$(16Rr-5r^2)(4R^2+8Rr-2r^2-4R^2-4Rr-3r^2) \geq r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow (16R-5r)(4R-5r) \geq (4R+r)^2 \Leftrightarrow 4R^2-9Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(4R-r) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

It can also be shown an inequality having an opposite sense for the above sum:

2. Prove that in any triangle ABC

$$\sum \frac{1}{(II_a)^2} + \sum \frac{1}{(I_bI_c)^2} \geq \frac{1}{R^2}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the above identities $\sum \frac{1}{(II_a)^2} = \frac{p^2+r^2-8Rr}{16R^2r^2}$ *and* $\sum \frac{1}{(I_bI_c)^2} = \frac{p^2+(4R+r)^2}{16R^2p^2}$,

We obtain:

$$\sum \frac{1}{(II_a)^2} = \frac{p^2+r^2-8Rr}{16R^2r^2} \geq \frac{16Rr-5r^2+r^2-8Rr}{16R^2r^2} = \frac{8Rr-4r^2}{16R^2r^2} =$$

$$= \frac{2R-r}{4R^2r} \geq \frac{3r}{4R^2r} = \frac{3}{4R^2},$$

where the first inequality follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$,

and the second from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

We've obtained the helpful result:

Lemma 1.

Prove that in any triangle *ABC*

$$\sum \frac{1}{(II_a)^2} \geq \frac{3}{4R^2}.$$

Then $\sum \frac{1}{(I_bI_c)^2} = \frac{p^2+(4R+r)^2}{16R^2p^2} \geq \frac{p^2+3p^2}{16R^2p^2} = \frac{1}{4R^2}$,

which follows from Doucet's inequality: $(4R+r)^2 \geq 3p^2$.

The equality holds if and only if the triangle is equilateral.

We've obtained the following helpful result:

Lemma 2.

Prove that in any triangle *ABC*

$$\sum \frac{1}{(I_bI_c)^2} \geq \frac{1}{4R^2}.$$

*Adding the inequality obtained from **Lemma 1** and **Lemma 2** we obtain conclusion **2**.*

□

Remark.

Finally it can be written the double inequality:

Prove that in any triangle *ABC*

$$\frac{1}{R^2} \leq \sum \frac{1}{(II_a)^2} + \sum \frac{1}{(I_bI_c)^2} \leq \frac{1}{4r^2}.$$

Proof.

*See **1** and **2**.*

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

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