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PROBLEM 120
RMM TRIANGLE MARATHON
101-200

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1. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{7}{R} - \frac{2}{r}$$

Proposed by Mehmet Şahin - Ankara - Turkey

Remark.

*Inequality **1** can be developed:*

2. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{\alpha}{R} - \frac{\beta}{r}, \text{ where } \alpha - 2\beta = 3 \text{ and } \beta \geq -2.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the forumulas $r_a = \frac{S}{p-a}$ *and* $r = \frac{S}{p}$ *we obtain*

$$\sum \frac{1}{r_a - r} = \frac{1}{r} \sum \frac{p-a}{a} = \frac{1}{r} \cdot \frac{p^2 + r^2 - 8Rr}{4Rr} = \frac{p^2 + r^2 - 8Rr}{4Rr^2}.$$

The inequality can be written

$\frac{p^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{\alpha}{R} - \frac{\beta}{r} \Leftrightarrow p^2 + r^2 - 8Rr \geq 4r(\alpha r - \beta R)$, *which follows from Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$$16Rr - 5r^2 + r^2 - 8Rr \geq 4r(\alpha r - \beta R) \Leftrightarrow (\beta + 2)R \geq (\alpha + 1)r \Leftrightarrow R \geq 2r,$$

because $\alpha - 2\beta = 3$ *and* $\beta \geq -2$.

The equality holds if and only if the triangle is equilateral.

For $\alpha = 7$ *and* $\beta = 2$ *we obtain inequality **1**, namely **Problem 120** from **RMM Triangle Marathon 101-200**, proposed by Mehmet Şahin - Ankara - Turkey.*

□

Remark.

*Inequality **1** can be strengthened:*

3. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{9}{4R - 2r}.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

Using the proven inequality at ***2****:* $\sum \frac{1}{r_a - r} = \frac{p^2 + r^2 - 8Rr}{4Rr^2}$, *inequality can be written:*

$\frac{p^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{9}{4R - 2r}$, *which follows from Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$.

It remains to prove that: $\frac{16Rr - 5r^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{9}{4R - 2r} \Leftrightarrow \frac{2R - r}{Rr} \geq \frac{9}{4R - 2r} \Leftrightarrow$

$\Leftrightarrow 8r^2 - 17Rr + 2r^2 \geq 0 \Leftrightarrow (R - 2r)(8R - r) \geq 0$, *obviously from Euler's inequality:* $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality ***3.*** *is stronger then inequality* ***1.****:*

4. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{9}{4R - 2r} \geq \frac{7}{R} - \frac{2}{r}.$$

Proof.

The first inequality is ***3.****, and the second inequality is equivalent with:*

$$\frac{9}{4R - 2r} \geq \frac{7r - 2R}{Rr} \Leftrightarrow 8R^2 - 23Rr + 14r^2 \geq 0 \Leftrightarrow (R - 2r)(8R - 7r) \geq 0,$$

obviously from Euler's inequality: $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality ***3.*** *can be developed:*

5. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{1}{xR - yr}, \text{ where } 2x - y = \frac{2}{3} \text{ and } x \geq 0.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the proven identity at ***2.****:* $\sum \frac{1}{r_a - r} = \frac{p^2 + r^2 - 8Rr}{4Rr^2}$, *the inequality can be written:*

$\frac{p^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{1}{xR - yr}$, *which follows from Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$ *and*

the observation that $xR - yr > 0$, *for* $2x - y = \frac{2}{3}$ *and* $x \geq 0$.

It remains to prove that: $\frac{16Rr - 5r^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{1}{xR - yr} \Leftrightarrow \frac{2R - r}{Rr} \geq \frac{1}{xR - yr} \Leftrightarrow$

$\Leftrightarrow (2R - r)(xR - yr) \geq Rr \Leftrightarrow 2xR^2 - (x + 2y + 1)Rr + yr^2 \geq 0 \Leftrightarrow (R - 2r)(4xR - yr) \geq 0$,

obviously from Euler's inequality: $R \geq 2r$ *and* $2x - y = \frac{2}{3}, x \geq 0$.

The equality holds if and only if the triangle is equilateral.

For $x=\frac{4}{9}$ *and* $y=\frac{2}{3}$ *we obtain inequality **3.** proposed by*
George Apostolopoulos - Messolonghi - Greece

□

Remark.

*Inequality **5.** is stronger than inequality **2.**:*

6. In ΔABC

$$\frac{1}{r_a-r}+\frac{1}{r_b-r}+\frac{1}{r_c-r}\geq\frac{1}{xR-yr}\geq\frac{\alpha}{R}-\frac{\beta}{r},$$

where $2x-y=\frac{2}{3}, x\geq 0$ **and** $\alpha-2\beta=3, \beta\geq 0.$

Proof.

*First inequality is **5.**, and the second inequality is equivalent with:*

$\frac{1}{xR-yr}\geq\frac{\alpha}{R}-\frac{\beta}{r}\Leftrightarrow Rr\geq(xR-yr)(\alpha r-\beta R)\Leftrightarrow\beta xR^2+(1-\alpha x-\beta y)Rr+\alpha yr^2\geq 0\Leftrightarrow$

$\Leftrightarrow(R-2r)(2\beta xR-\alpha yr)\geq 0$, *obviously from Euler's inequality:* $R\geq 2r$ *and* $2x-y=\frac{2}{3}$, $x\geq 0$, *and* $\alpha-2\beta=3, \beta\geq 0$, *which lead to* $(2x-y)(\alpha-2\beta)=2$, *wherefrom* $-\alpha y-4\beta x=2(1-\alpha x-\beta y)$, *thus motivating the last inequality.*

The equality holds if and only if the triangle is equilateral.

For $x=\frac{4}{9}, y=\frac{2}{3}, \alpha=7$ *and* $\beta=2$ *its obtained the double inequality **4.***

□

Remark.

We can propose inequalities with sums having the form $\sum\frac{a^n}{r_a-r}$, *where* $n=1,2,3,4,5.$

7. In ΔABC

$$3\sqrt{3}\leq\sum\frac{a}{r_a-r}\leq 3\sqrt{3}\cdot\frac{R}{2r}$$

Proof.

Using the formulas $r_a=\frac{S}{p-a}$ *and* $r=\frac{S}{p}$ *we obtain* $\sum\frac{a}{r_a-r}=\frac{1}{r}\sum(p-a)=\frac{p}{r}.$

The double inequality follows from Mitrinović's inequalities: $3\sqrt{3}\cdot r\leq p\leq\frac{3\sqrt{3}}{2}\cdot R.$

The equality holds if and only if the triangle is equilateral.

□

8. In ΔABC

$$18r\leq\sum\frac{a^2}{r_a-r}\leq 9R.$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^2}{r_a - r} = \frac{1}{r} \sum a(p-a) = \frac{1}{r} \cdot 2r(4R+r) = 2(4R+r).$$

The double inequality follows from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

9. In ΔABC

$$\mathbf{12pr \leq \sum \frac{a^3}{r_a - r} \leq 6pR.}$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^3}{r_a - r} = \frac{1}{r} \sum a^2(p-a) = \frac{1}{r} \cdot 4pr(R+r) = 4p(R+r).$$

The double inequality follows from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

10. In ΔABC

$$\mathbf{(6r)^3 \leq \sum \frac{a^4}{r_a - r} \leq (3R)^3.}$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^4}{r_a - r} = \frac{1}{r} \cdot \sum a^3(p-a) = \frac{1}{r} \cdot 2r\left[p^2(2R+3r) - r(4R+r)^2\right] = 2p^2(2R+3r) - 2r(4R+r)^2.$$

The first inequality follows from the above identity, Gerretsen's inequality:

$p^2 \geq 16Rr - 5r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain $2p^2(2R+3r) - 2r(4R+r)^2 \geq 2(16Rr - 5r^2)(2R+3r) - 2r(4R+r)^2 =$

$= 4r(8R^2 + 15Rr - 8r^2) \geq 4r \cdot 54r^2 = 216r^3 = (6r)^3$.

For the second inequality we use the above identity, Gerretsen's inequality:

$p^2 \leq 4R^2 + 4Rr + 3r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain $2p^2(2R+3r) - 2r(4R+r)^2 \leq 2(4R^2+4Rr+3r^2)(2R+3r) - 2r(4R+r)^2 =$

$= 16R^3 + 8R^2r + 20Rr^2 + 16r^3 \leq 16R^3 + 4R^3 + 5R^3 + 2R^3 = 27R^3 = (3R)^3$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

11. In ΔABC

$$\mathbf{18p\cdot(2r)^3\le\sum\frac{a^5}{r_a-r}\le 18p\cdot R^3.}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the formulas $r_a=\dfrac{S}{p-a}$ *and* $r=\dfrac{S}{r}$ *we obtain*

$$\sum\frac{a^5}{r_a-r}=\frac{1}{r}\cdot\sum a^4(p-a)=\frac{1}{r}\cdot 4pr\Big[p^2(R+2r)-r(12R^2+11Rr+2r^2)\Big]=$$
$$=4p\Big[p^2(R+2r)-r(12R^2+11Rr+2r^2)\Big].$$

The first inequality follows from the above identity, Gerretsen's inequality: $p^2\ge 16Rr-5r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$4p\Big[p^2(R+2r)-r(12R^2+11Rr+2r^2)\Big]\ge 4pr\Big[(16Rr-5r^2)(R+2r)-r(12R^2+11Rr+2r^2)\Big]$$
$$=16pr(R^2+4Rr-3r^2)\ge 16pr\cdot(4r^2+8r^2-3r^2)=16pr\cdot 9r^2=144pr^3=18p\cdot(2r)^3.$$
$$=4r(8R^2+15Rr-8r^2)\ge 4r\cdot 54r^2=216r^3=(6r)^3.$$

For the second inequality we use the above identity, Gerretsen's inequality: $p^2\le 4R^2+4Rr+3r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$4p\Big[p^2(R+2r)-r(12R^2+11Rr+2r^2)\Big]\le 4p\Big[(4R^2+4Rr+3r^2)(R+2r)-r(12R^2+11Rr+2r^2)\Big]=$$
$$=16p(R^3+r^3)\le 16p\cdot\left(R^3+\frac{R^3}{8}\right)=18p\cdot R^3$$

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

Remark.

We can propose inequalities with sums having the form $\sum\dfrac{a^n(b+c)}{r_a-r}$, *where* $n=1,2,3,4$.

12. In ΔABC

$$\mathbf{(6r)^2\le r\sum\frac{a(b+c)}{r_a-r}\le(3R)^2.}$$

Proof.

Using the formulas $r_a=\dfrac{S}{p-a}$ *and* $r=\dfrac{S}{p}$ *we obtain*

$$\sum\frac{a(b+c)}{r_a-r}=\frac{1}{r}\sum(b+c)(p-a)=\frac{1}{r}\cdot 2(p^2-r^2-4Rr)=\frac{2(p^2-r^2-4Rr)}{r}.$$

The first inequality follows from the above identity, Gerretsen's inequality: $p^2\ge 16Rr-5r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$r\sum\frac{a(b+c)}{r_a-r}=r\cdot\frac{2(p^2-r^2-4Rr)}{r}=2(p^2-r^2-4Rr)\ge 2(16Rr-5r^2-r^2-4Rr)=$$

$$= 12r(2R - r) \geq 12R \cdot 3r = (6r)^2.$$

For the second inequality we use the above identity, Gerretsen's inequality: $p^2 \leq 4R^2 + 4Rr + 3r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain

$$r \sum \frac{a(b+c)}{r_a - r} = r \cdot \frac{2(p^2 - r^2 - 4Rr)}{r} = 2(p^2 - r^2 - 4Rr) \geq 2(4R^2 + 4Rr + 3r^2 - r^2 - 4Rr) =$$
$$= 8R^2 + 4r^2 \leq 9R^2 = (3R)^2.$$

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

13. In ΔABC

$$\mathbf{36\sqrt{3} \cdot Rr \leq \sum \frac{a^2(b+c)}{r_a - r} \leq 18\sqrt{3} \cdot R^2.}$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^2(b+c)}{r_a - r} = \frac{1}{r} \sum a(b+c)(p-a) = \frac{1}{r} \cdot 12pRr = 12pR.$$

The double inequality follows from Mitrinović's inequalities: $3\sqrt{3} \cdot r \leq p \leq \dfrac{3\sqrt{3}}{2} \cdot R$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

14. In ΔABC

$$\mathbf{(6r)^4 \leq 3 \sum \frac{a^3(b+c)}{r_a - r} \leq (3R)^4.}$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^3(b+c)}{r_a - r} = \frac{1}{r} \sum a^2(b+c)(p-a) = \frac{1}{r} \cdot \left[2p^2(2Rr + r^2) + 2r^2(4R+r)^2\right] =$$
$$= \frac{2p^2(2Rr + r^2) + 2r^2(4R+r)^2}{r}.$$

The first inequality follows from the above identity, Gerretsen's inequality: $p^2 \geq 16Rr - 5r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain

$$3 \sum \frac{a^3(b+c)}{r_a - r} = 3 \cdot \frac{2p^2(2Rr + r^2) + 2r^2(4R+r)^2}{r} \geq 3 \cdot \frac{2(16Rr - 5r^2)(2Rr + r^2) + 2r^2(4R+r)^2}{r}$$
$$= 3 \cdot 4r^2(24R^2 + 7Rr - 2r^2) \geq 12r^2(24 \cdot 4r^2 + 7r \cdot 2r - 2r^2) = 12r^2 \cdot 108r^2 = 1296r^4 = (6r)^4.$$

For the second inequality we use the above identity, Gerretsen's inequality: $p^2 \leq 4R^2 + 4Rr + 3r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain

$$3\sum\frac{a^3(b+c)}{r_a-r}=3\cdot\frac{2p^2(2Rr+r^2)+2r^2(4R+r)^2}{r}\le 3\cdot\frac{2(4R^2+4Rr+3r^2)(2Rr+r^2)+2r^2(4R+r)^2}{r}$$

$$=3\cdot r(16R^3+56R^2r+36Rr^2+8r^3)\le 3r(16R^3+28R^3+9R^3+R^3)\le\frac{3R}{2}\cdot 54R^3=81R^4=(3R)^4$$

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

15. In ΔABC

$$p(12r)^3\le 6\sum\frac{a^4(b+c)}{r_a-r}\le p(6R)^3.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the formulas $r_a=\frac{S}{p-a}$ *and* $r=\frac{S}{p}$ *we obtain*

$$\sum\frac{a^4(b+c)}{r_a-r}=\frac{1}{r}\sum a^3(b+c)(p-a)=\frac{1}{r}\cdot\left[4p^3(Rr+r^2)+4p(-4R^2r^2+3Rr^3+r^4)\right]=$$

$$=4p^3(R+r)+4p(-4R^2r+3Rr^2+r^3).$$

The first inequality follows from the above identity, Gerretsen's inequality: $p^2\ge 16Rr-5r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$6\sum\frac{a^4(b+c)}{r_a-r}=6\left[4p^3(R+r)+4p(-4R^2r+3Rr^2+r^3)\right]=24p\left[p^2(R+r)-4R^2r+3Rr^2+r^3\right]$$

$$\ge 24p\left[(16Rr-5r^2)(R+r)-4R^2r+3Rr^2+r^3\right]=48pr(12R^2+14Rr-4r^2)\ge$$

$$\ge 48pr(24r^2+14r^2-2r^2)=48pr\cdot 36r^2=1728pr^3=p(12r)^3.$$

For the second inequality we use the above identity, Gerretsen's inequality: $p^2\le 4R^2+4Rr+3r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$6\sum\frac{a^4(b+c)}{r_a-r}=6\left[4p^3(R+r)+4p(-4R^2r+3Rr^2+r^3)\right]=24p\left[p^2(R+r)-4R^2r+3Rr^2+r^3\right]\le$$

$$\le 24p\left[(4R^2+4Rr+3r^2)(R+r)-4R^2r+3Rr^2+r^3\right]=12p\cdot(8R^3+8R^2r+20Rr^2+8r^3)\le$$

$$\le 12p(8R^3+4R^3+5R^3+R^3)=12p\cdot 18R^3=216pR^3=p(6R)^3.$$

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

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