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INEQUALITY IN TRIANGLE 271
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1. In acute-angled ΔABC

$$\mathbf{2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\tan B\tan C\geq 9\sqrt{3}.}$$

Proposed by Daniel Sitaru - Romania

Proof.

Using the known identities in triangle $\sum\cot\frac{A}{2}=\frac{p}{r}$ *and* $\prod\tan A=\frac{2pr}{p^2-(2R+r)^2}$,

the inequality we have to prove can be written: $2\cdot\frac{p}{r}+\frac{2pr}{p^2-(2R+r)^2}\geq 9\sqrt{3}$.

Using Mitrinović's inequality $p\geq 3\sqrt{3}\cdot r$ *we have* $\frac{p}{r}\geq 3\sqrt{3}$ *and* $pr\geq 3\sqrt{3}\cdot r^2$

it's enough to prove that

$$\frac{2r^2}{p^2-(2R+r)^2}\geq 1\Leftrightarrow p^2\leq 4R^2+4Rr+3r^2 \text{ (Gerretsen's inequality).}$$

The inequality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be developed:

2. In acute-angled ΔABC

$$\mathbf{n\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\tan B\tan C\geq (n+1)\cdot 3\sqrt{3},\ \text{where } n\geq 0.}$$

Proof.

We use the known inequalities in triangle $\sum\cot\frac{A}{2}=\frac{p}{r}$ *and* $\prod\tan A=\frac{2pr}{p^2-(2R+r)^2}$.

We have

(i) $\sum\cot\frac{A}{2}\geq 3\sqrt{3}\Leftrightarrow\frac{p}{r}\geq 3\sqrt{3}\Leftrightarrow p\geq 3\sqrt{3}\cdot r$ *(Mitrinović's inequality);*

(ii)

$\prod\tan A\geq 3\sqrt{3}\Leftrightarrow\frac{2pr}{p^2-(2R+r)^2}\geq 3\sqrt{3}$, *which follows from* $pr\geq 3\sqrt{3}\cdot r^2$ *and*

$$\frac{2r^2}{p^2-(2R+r)^2} \geq 1 \Leftrightarrow p^2 \leq 4R^2+4Rr+3r^2 \text{ (Gerretsen's inequality).}$$

From (i), (ii) *and* $n \geq 0$ *the conclusion is obtained.*

The inequality holds if and only if the triangle is equilateral.

□

3. In acute-angled ΔABC

$$\boldsymbol{n\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+k\tan A\tan B\tan C \geq (n+k)\cdot 3\sqrt{3}}, \textbf{ where } \boldsymbol{n \geq 0, k \geq 0.}$$

Proof.

We use the known identities in triangle $\sum \cot\frac{A}{2} = \frac{p}{r}$ *and* $\prod \tan A = \frac{2pr}{p^2-(2R+r)^2}$.

We have

(i) $\sum \cot\frac{A}{2} \geq 3\sqrt{3} \Leftrightarrow \frac{p}{r} \geq 3\sqrt{3} \Leftrightarrow p \geq 3\sqrt{3}\cdot r$ *(Mitrinović's inequality);*

(ii)

$\prod \tan A \geq 3\sqrt{3} \Leftrightarrow \frac{2pr}{p^2-(2R+r)^2} \geq 3\sqrt{3}$, *which follows from* $pr \geq 3\sqrt{3}\cdot r^2$ *and*

$$\frac{2r^2}{p^2-(2R+r)^2} \geq 1 \Leftrightarrow p^2 \leq 4R^2+4Rr+3r^2 \text{ (Gerretsen's inequality)}$$

From (i), (ii) *and* $n \geq 0, k \geq 0$ *the conclusion is obtained.*

The inequality holds if and only if the triangle is equilateral or $n = k = 0$.

□

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