Math Adventures on CutTheKnot Math 101-150

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DANIEL SITARU

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MATH ADVENTURES
ON
CutTheKnotMath

101 - 150

By Alexander Bogomolny and Daniel Sitaru

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Proposed by

Daniel Sitaru - Romania
Hung Nguyen Viet - Hanoi - Vietnam
Henry Ricardo - USA
Leonard Giugiuc - Romania
Abdilkadir Altintas - Afyonkarashisar - Turkey
Francisco Javier Garcia Capitan - Spain
Anish Ray - Santaragachi - India
Mehmet Şahin - Ankara - Turkey

Solution by

Daniel Sitaru - Romania, Alexander Bogomolny – USA
Nassim Nicholas Taleb – USA
Henry Ricardo - USA
Soumitra Mandal - India , Amit Itagi
Leonard Giugiuc – Romania, Ravi Prakash - India
Kevin Soto Palacios - Peru, Soumava Chakraborty - India
Abdul Aziz - Indonesia, Marian Dincă - Romania
Miquel Ochoa Sanchez - Peru , Andreea Aquaviva
Seyran Ibrahimov - Azerbaidian, Chris Kyriazis - Greece
Myagmarsuren Yadamsuren - Mongolia
Mehmet Şahin - Turkey , George Apostolopoulos - Greece
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Su Tanaya - India, Marian Daniel Vasile - Romania
Vasile Cîrtoaje - Romania, Srinivas Vemuri
Marjan Milanovic - Serbia , Shivam Sharma - India
Nirapada Pal - India, Nikolaos Skoutaris - Greece
Nguyen Thanh Nho - Vietnam, Uche Eliezer Okeke - Nigeria
Geanina Tudose - Romania

101. Dan Sitaru's Inequality with Tangents II

Prove that in any acute ΔABC,

$$\sum_{cycl} \tan A \tan B + 45 \le 2\tan^2 A \tan^2 B \tan^2 C$$

Proposed by Daniel Sitaru

Solution 1(by proposer).

Let be $x = \cot A \cot B; y = \cot B \cot C; z = \cot C \cot A$. ***It follows*** that $x + y + z = 1$. Further,

$$45 + \sum_{cycl} \tan \tan B = 45 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{xy + yz + zx + 45xyz}{xyz}$$

$$= \frac{(1 - (x + y + z) + xy + yz + zx - xyz) + 46xyz}{xyz}$$

$$= \frac{(1 - x)(1 - y)(1 - z) + 46xyz}{xyz}$$

$$\overbrace{\le}^{AM-GM} \frac{\frac{1}{27}(1 - x + 1 - y + 1 - z)^3 + \frac{46}{27}(x + y + z)}{xyz}$$

$$= \frac{\frac{1}{27} \cdot 8 + \frac{46}{27}}{xyz} = \frac{2}{xyz} = \frac{2}{\prod_{cycl} \cot^2 A} = 2\prod_{cycl} \tan^2 A$$

□

Solution 2 (by Kevin Soto Palacios).

Set $x = \tan A > 0, y = \tan B > 0, z = \tan C > 0$. ***Recollect that*** $x + y + z = xyz$. The required inequality is equivalent to

$$xy + yz + zx + 45 \le 2(x + y + z)^2,$$

which, in turn, is equivalent to

$$2(x^2 + y^2 + z^2) + 3(xy + yz + zx) \ge 45.$$

Since, by the ***rearrangement inequality***, $x^2 + y^2 + z^2 \ge xy + yz + zx$, suffice it to prove that $5(xy + yz + zx) \ge 45$, i.e., that $xy + yz + zx \ge 9$, which follows from $\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = 1$, by the ***Cauchy - Schwarz inequality***. □

Solution 3 (by Soumava Chakraborty).

$$2\left(\prod_{cycl} \tan A\right)^2 = 2\left(\sum_{cycl} \tan A\right)^2 \ge 6\sum_{cycl} \tan A \tan B$$

because $(x + y + z)^2 \ge 3(xy + yz + zx)$. To continue, by the AM-GM inequality,

$$\sum_{cycl} \tan A \tan B \ge 3\sqrt[3]{\tan^2 A \tan^2 B \tan^2 C}$$

$$= 3\sqrt[3]{\left(\sum_{cycl} \tan A\right)^2} \ge 3\sqrt[3]{(3\sqrt{3})^2} = 9,$$

because $\tan(x)$, being a convex function on

$$\left(0, \frac{\pi}{2}\right), \sum_{cycl} \tan A \geq 3 \tan\left(\frac{A+B+C}{3}\right)$$

Finally.

$$2\left(\prod_{cycl} \tan A\right)^2 \geq 6 \sum_{cycl} \tan A \tan B$$

$$= \sum_{cycl} \tan A \tan B + 5 \sum_{cycl} \tan A \tan B \geq \sum_{cycl} \tan A \tan B + 5 \cdot 9.$$

□

Acknowledgment (by Alexander Bogomolny)
The problem from the ***Romanian Mathematical Magazine*** has been kindly posted by Dan Sitaru at the ***CutTheKnotMath*** page. Dan also has communicated his solution (Solution 1) in a latex file. Solution 2 is by Kevin Soto Palacios; Solution 3 is by Soumava Chakraborty.

102. A Cyclic Inequality in Three Variables XVII

Prove that, for $x, y, z > 0$,

$$\left(\sum_{cycl} \frac{x^2}{y^2}\right)^5 \geq 9\left(\sum_{cycl} \frac{x^3}{y^2 z}\right)\left(\sum_{cycl} \frac{x}{\sqrt{yz}}\right)\left(\sum_{cycl} \frac{y}{z}\right)$$

Proposed by Daniel Sitaru

Solution 1 (by Leonard Giugiuc).

Let $\frac{x}{y} = a^2$, $\frac{y}{z} = b^2$, $\frac{z}{x} = c^2$. Then $abc = 1$ and the required inequality becomes

$$\left(\sum_{cycl} a^4\right)^5 \geq 9\left(\sum_{cycl} \frac{a^4}{c^2}\right)\left(\sum_{cycl} \frac{a}{c}\right)\left(\sum_{cycl} a^2 b^2\right)$$

But, since $a + b + c \geq 3$, *Hölder's inequality gives* $\sum_{cycl} a^4 \geq \sum_{cycl} a^3$. *Also,* $\sum_{cycl} \frac{a}{c} = \sum_{cycl} a^2 b$ *and* $\sum_{cycl} a^3 \geq \sum_{cycl} a^2 b$ *, so that* $\sum_{cycl} a^4 \geq \sum_{cycl} \frac{a}{c}$.
By the ***Rearrangement inequality***, $\sum_{cycl} a^4 \geq \sum_{cycl} a^2 b^2$.

On the other hand, $\sum_{cycl} \frac{a^4}{c^2} = \sum_{cycl} a^6 b^2$ and, according to an inequality by ***Vasile Cîrtoaje***, $(\sum_{cycl} a^4)^2 \geq 3 \sum_{cycl} a^6 b^2$. Finally, $\sum_{cycl} a^4 \geq 3$, by the ***AM-GM inequality.*** The required inequality results as the product of

$$\sum_{cycl} a^4 \geq \sum_{cycl} \frac{a}{c},$$

$$\sum_{cycl} a^4 \geq \sum_{cycl} a^2 b^2,$$

$$\left(\sum_{cycl} a^4\right)^2 \geq \left[3 \sum_{cycl} a^6 b^2\right] = 3 \sum_{cycl} \frac{a^4}{c^2},$$

$$\sum_{cycl} a^4 \geq 3.$$

□

Solution 2 (by Nassim Nicholas Taleb).

Prove that, for $x, y, z > 0$,

$$\left(\sum_{cycl} \frac{x^2}{y^2}\right)^5 \geq 9\left(\sum_{cycl} \frac{x^3}{y^2 z}\right)\left(\sum_{cycl} \frac{x}{\sqrt{yz}}\right)\left(\sum_{cycl} \frac{y}{z}\right)$$

1) $\left(\sum_{cycl} \frac{x^2}{y^2}\right) \geq \left(\sum_{cycl} \frac{x}{y}\right)$ *since* $\sum_{cycl} x^2 y^4 \geq \sum_{cycl} x y^2 z^3$ *(rearrangement inequality)*

2) $\left(\sum_{cycl} \frac{x^2}{y^2}\right) \geq \left(\sum_{cycl} \frac{x}{\sqrt{yz}}\right)$ *(variant of the rearrangement)*

3) $\left(\sum_{cycl} \frac{x^2}{y^2}\right) \geq 3$ *(AM-GM) since*

$$\frac{1}{3}\left(\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2}\right) \geq \left(\frac{x^2}{y^2} \cdot \frac{y^2}{z^2} \cdot \frac{z^2}{x^2}\right)^{\frac{1}{3}}, \frac{x^2}{y^2} \cdot \frac{y^2}{z^2} \cdot \frac{z^2}{x^2} = 1$$

4) Finally we need to prove that $\left(\sum_{cycl} \frac{x^2}{y^2}\right)^2 \geq 3 \sum_{cycl} \frac{x^3}{y^2 z}$

$$\sum_{cycl} x^4 y^8 + 2\sum_{cycl} x^2 y^6 z^4 \geq 3\sum_{cycl} x^3 y^7 z^2$$

by some mysterious inequality theorem I failed miseraaaably to find but that we can show via calculus or some expansion of the rearrangement inequality.

Postscript. Finnally found the mysterious inequality with help from Maestro Alexander Bogomolny (it necessitates a minor change of variable)

1. If x, y, z are real numbers, then

$$(x^2 + y^2 + z^2)^2 \geq 3(x^3 y + y^3 z + z^3 x)$$

(Vasile Cîrtoaje, GM-B, 7-8, 1992)

□

Acknowledgment (by Alexander Bogomolny)
This problem has been kindly posted at the ***CutTheKnotMath page*** by Leo Giugiuc, along with his solution. The problem by Daniel Sitaru has been previously posted at the ***Romanian Mathematical Magazine.*** Solution 2 is by N. N. Taleb.

103. A Cyclic Inequality in Three Variables XVIII

Prove that, for $a, b, c \geq 0$,

$$\left(\sum_{cycl} \sqrt{ab}\right)^6 \leq 27 \prod_{cycl} (a^2 + ab + b^2).$$

Proposed by Daniel Sitaru

Solution 1 (by Kevin Soto Palacios).

First note that
1. $a^2+ab+b^2 \geq \frac{3}{4}(a+b)^2 \geq 0 \Leftrightarrow (a-b)^2 \geq 0$,
2. $a,b,c>0 \Rightarrow \frac{9}{8}\prod_{cycl}(a+b) \geq (\sum_{cycl} a)(\sum_{cycl} ab)$,
3. $a,b,c \in \mathbb{R} \Rightarrow (a+b+c)^2 \geq 3(ab+bc+ca)$.

It follows that

$$\prod_{cycl}(a^2+ab+b^2) \geq 27\Big(\frac{3}{4}\Big)^2 \prod_{cycl}(a+b)^2$$
$$\geq 9(a+b+c)^2(ab+bc+ca)^2.$$

However,

$$9(a+b+c)^2(ab+bc+ca)^2 \geq 27(ab+bc+ca)^3.$$

Thus, suffice it to prove that

$$27(ab+bc+ca)^3 \geq (\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^6.$$

But, by the ***Cauchy - Schwarz inequality***,

$$27(ab+bc+ca)^3 \geq 28\left(\frac{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2}{3}\right)^3 \geq (\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^6.$$

□

Solution 2 (by Soumitra Mandal).

We know that $a^2+ab+b^2 \geq \frac{3}{4}(a+b)^2+\frac{(a-b)^2}{4} \geq \frac{3}{4}(a+b)^2$. Similarly, $b^2+bc+c^2 \geq \frac{3}{4}(b+c)^2$ and $c^2+ca+a^2 \geq \frac{3}{4}(c+a)^2$, implying

$$27\prod_{cycl}(a^2+ab+b^2) \geq 27\cdot\Big(\frac{3}{4}\Big)^3\prod_{cycl}(a+b)^2$$

$$\geq 27\cdot\Big(\frac{3}{4}\Big)^3\frac{64}{81}\left(\sum_{cycl} a\right)^2\left(\sum_{cycl} ab\right)^2 \geq 27\left(\prod_{cycl} ab\right)^3 \geq \left(\sum_{cycl}\sqrt{ab}\right)^6,$$

where we used the following inequalities:
1. $9\sum_{cycl}(a+b) \geq 8(\sum_{cycl} a)(\sum_{cycl} ab)$,
2. $(a+b+c)^2 \geq 3(ab+bc+ca)$,
3. $\frac{ab+bc+ca}{3} \geq \Big(\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{3}\Big)^2$.

□

Solution 3 (by Daniel Sitaru).

Lemma 1

For $x,y,z>0$,

$$\left(\sum_{cycl} xy\right)^3 \leq \prod(x^2+xy+y^2) \tag{1}$$

Indeed, from Hölder's inequality:

$$\prod_{cycl}(x^2+xy+y^2) = (xy+y^2+x^2)(y^2+yz+z^2)(x^2+z^2+xz)$$
$$\geq (\sqrt[3]{xy\cdot y^2\cdot x^2}+\sqrt[3]{y^2\cdot yz\cdot z^2}+\sqrt[3]{x^2\cdot z^2\cdot xz})^3 = (xy+yz+zx)^3.$$

Lemma 2
For $x, y \geq 0$,

$$\left(\sum_{cycl} xy\right)^3 \leq 27 \prod_{cycl}(x^2 - xy + y^2). \tag{2}$$

Indeed,

$$(x-y)^2 \geq 0 \Rightarrow 2(x-y)^2 \geq 0 \Rightarrow 2x^2 - 4xy + 2y^2 \geq 0$$

and, finally,

$$3x^2 - 3xy + 3y^2 \geq x^2 + xy + y^2.$$

or,

$$x^2 - xy + y^2 \geq \frac{1}{3}(x^2 + xy + y^2)$$

Similarly,

$$y^2 - yz + z^2 \geq \frac{1}{3}(y^2 + yz + z^2) \text{ and}$$

$$z^2 - zx + x^2 \geq \frac{1}{3}(z^2 + zx + x^2)$$

By multiplying the last three relationships,

$$\prod_{cycl}(x^2 + xy + y^2) \leq 27 \prod_{cycl}(x^2 - xy + y^2). \tag{3}$$

Multiplying by (1) and (2),

$$\left(\sum_{cycl} xy\right)^6 \leq 27 \prod_{cycl}(x^2 + xy + y^2) \cdot \prod_{cycl}(x^2 - xy + y^2)$$

$$= 27 \prod_{cycl}\left((x^2 + y^2)^2 - x^2y^2\right) = 27 \prod_{cycl}(x^4 + y^4 + x^2y^2).$$

Thus,

$$\left(\sum_{cycl} xy\right)^6 = 27 \prod_{cycl}(x^4 + y^4 + x^2y^2) \tag{4}$$

Setting now in (4) $x = \sqrt{a}; y = \sqrt{b}; z = \sqrt{c}$, we get

$$\left(\sum_{cycl} \sqrt{ab}\right)^6 \leq 27 \prod_{cycl}(a^2 + ab + b^2)$$

The equality holds if $a = b = c$. □

Solution 4 (by Soumava Chakraborty).

Case 1: Exactly one of $a, b, c = 0$
With $a = 0$, the given inequality reduces to $27b^2c^2(b^2 + bc + c^2) \geq b^3c^3$, which is obvious. Cases $b = 0$ and $c = 0$ are handled similarly.
Case 2: At least two of $0 \geq 0$.
Case 3: $a, b, c > 0$

Using ***Wu's inequality***, $RHS \geq 27(ab+bc+ca)^3$. Thus, suffice it to prove that

$$27(ab+bc+ca)^3 \geq (\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^6 \Leftrightarrow$$

$$3(ab+bc+ca) \geq (\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2 \Leftrightarrow 3\sum_{cycl} x^2 \geq \left(\sum_{cycl} x\right)^2,$$

where $x=\sqrt{ab}, y=\sqrt{bc}, z=\sqrt{ca}$. The latter is true by *Hölder's inequality.* □

Solution 5 (by Nassim Nicholas Taleb).

We have, by the Power − Mean Inequality, $\left(\frac{1}{3}(\sum_{cycl}(ab))^{\frac{1}{2}}\right)^2 \leq \frac{1}{3}\sum_{cycl} ab$. Hence

$$LHS = \left(\sum_{cycl}(ab)^{\frac{1}{2}}\right)^6 \leq 3^3\left(\sum_{cycl} ab\right)^3$$

and we need to prove that $3^3(\sum_{cycl} ab)^3 \leq RHS$, or

$$\left(\sum_{cycl} a^4b^2+\sum_{cycl} a^2b^4+\sum_{cycl} a^4bc\right)-\left(\sum_{cycl} ab^2c^3+\sum_{cycl} a^2b^2c+\sum_{cycl} a^2b^2c^2\right)\geq 0,$$

which is true by ***rearrangement***. □

Acknowledgment (by Alexander Bogomolny)

This problem has been kindly posted at the ***CutTheKnotMath page*** by Daniel Sitaru and then again by Kevin Soto Palacios, along with his solution (Solution 1). Solution 2 is by Soumitra Mandal; Solution 3 is by Daniel Sitaru; Solution 4 is by Soumava Chakraborty; Solution 5 is by Nassim Nicholas Taleb. The problem is by Dan Sitaru and has been previously published at the ***Romanian Mathematical Magazine.***

104. A Cyclic Inequality in Three Variables XXI

Prove that, for $a,b,c>0$,

$$\frac{abc}{7\sqrt{7}} \leq \prod_{cycl}\frac{a^2-ab+b^2}{\sqrt{a^2+5ab+b^2}}.$$

Proposed by Daniel Sitaru

Solution 1 (by Leonard Giugiuc).

Note that $\sqrt{\frac{ab}{7}} \leq \frac{a^2-ab+b^2}{\sqrt{a^2+5ab+b^2}}$. Indeed, that is equivalent to $(a-b)^2(7a^2-ab+7b^2)\geq 0$, which is true, with equality for $a=b$.
Taking the product of three such inequalities we obtain the required one.
Equality when $a=b=c$. □

Solution 2 (Kevin Soto Palacios).

Observe that

$$a^2-ab+b^2=\frac{3}{4}(a-b)^2+\frac{1}{4}(a+b)^2\geq\frac{1}{4}(a+b)^2,\text{ and}$$

$$a^2+5ab+b^2=-\frac{3}{4}(a-b)^2+\frac{7}{4}(a+b)^2\leq\frac{7}{4}(a+b)^2.$$

Also, for $a,b,c>0, (a+b)(b+c)(c+a)\geq 8abc$. Combining that

$$\prod_{cycl}\frac{a^2-ab+b^2}{\sqrt{a^2+5ab+b^2}}\geq\frac{\left(\frac{1}{4}\right)^3\prod_{cycl}(a+b)^2}{\left(\frac{\sqrt{7}}{2}\right)^3\prod_{cycl}(a+b)}$$

$$=\frac{1}{56\sqrt{7}}(a+b)(b+c)(c+a)\geq\frac{abc}{7\sqrt{7}}$$

□

Solution 3 (by Soumava Chakraborty).

$$a^2-ab+b^2=\frac{3}{4}(a-b)^2+\frac{1}{4}(a+b)^2\geq\frac{1}{4}(a+b)^2.\text{ Similarly,}$$

$$b^2-bc+c^2\geq\frac{(b+c)^2}{4}\text{ and }c^2-ca+a^2\geq\frac{(c+a)^2}{4}.\text{ Thus,}$$

$$\prod_{cycl}(a^2-ab+b^2)\geq\frac{(a+b)^2(b+c)^2(c+a)^2}{64}$$

$$=\frac{\prod_{cycl}(a+b)\cdot\prod_{cycl}(a+b)}{64}\geq\frac{8abc}{64}(a+b)(b+c)(c+a)$$

$$=\frac{(a+b)(b+c)(c+a)}{8}$$

Suffice it to prove that

$$\frac{abc\prod_{cycl}(a+b)}{8}\geq\frac{abc}{7\sqrt{7}}\prod_{cycl}\sqrt{a^2+5ab+b^2}$$

which is the same as

$$\text{(a)}\qquad\prod_{cycl}\frac{\sqrt{7}(a+b)}{2}\geq\prod_{cycl}\sqrt{a^2+5ab+b^2}$$

Suffice it to prove that $\frac{\sqrt{7}(a+b)}{2}\geq\sqrt{a^2+5ab+b^2}$, which is equivalent to

$$7(a^2+b^2+2ab)\geq 4(a^2+5ab+b^2)\Leftrightarrow 3(a-b)^2\geq 0,$$

which is true. □

Solution 4 (by Soumitra Mandal).

The required inequality is equivalent to

$$7^3\prod_{cycl}(a^2-ab+b^2)^2\geq(abc)^2\prod_{cycl}(a^2+5ab+b^2).$$

Now,

$$7(a^2-ab+b^2)^2-ab(a^2+5ab+b^2)$$

$$= 7(a^2+b^2)^2 - 15ab(a^2+b^2) + 2(ab)^2 = (a-b)^2(7a^2+7b^2-ab) \geq 0,$$

so $7(a^2-ab+b^2)^2 \geq ab(a^2+5ab+b^2)$. Rotating a, b, c and taking the product yields the required inequality. □

Solution 5 (by Ravi Prakash).

Consider

$$7(a^2-ab+b^2)^2 - ab(a^2+5ab+b^2)$$
$$= 7(a^2-ab+b^2)^2 - ab(a^2-ab+b^2) - 6a^2b^2$$
$$= 7(a^2-ab+b^2)^2 - 7ab(a^2-ab+b^2) + 6ab(a^2-ab+b^2) - 6a^2b^2$$
$$= 7(a^2-ab+b^2)(a^2-ab+b^2-ab) + 6ab(a^2-ab+b^2-ab)$$
$$= (a-b)^2[(a^2-ab+b^2) + 6(a^2+b^2)] \geq 0$$

Hence, $\frac{a^2-ab+b^2}{\sqrt{a^2+5ab+b^2}} \geq \frac{\sqrt{ab}}{\sqrt{7}}$. Rotating a, b, c and taking the product yields the required inequality. □

Acknowledgment (by Alexander Bogomolny)

This problem form the **Romanian Mathematical Magazine**, has been kindly posted at the ***CutTheKnotMath page*** by Daniel Sitaru. Solution 1 is by Leo Giugiuc; Solution 2 is by Kevin Soto Palacios; Solution 3 by Soumava Chakraborty; Solution 4 by Soumitra Mandal; Solution 5 by Ravi Prakash.

105. A Cyclic Inequality in Three Variables XXIII

Prove that, for $a, b, c > 0$,

$$3(a^2+b^2+c^2)^2 \geq 8abc(a+b+c) + \sum_{cycl}(a^2+b^2-c^2)^2$$

Proposed by Daniel Sitaru

Solution 1 (by Abdul Aziz).

$$(ab-bc) \geq 0 \Leftrightarrow$$
$$a^2b^2 + b^2c^2 \geq 2ab^2c. \text{ Similarly,}$$
$$b^2c^2 + c^2a^2 \geq 2abc^2 \text{ and } c^2a^2 + a^2b^2 \geq 2a^2bc.$$

Adding up, $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$. It's not hard to see that this inequality is equivalent to the original one (just carry out the implied arithmetic and cancel similar terms.) □

Solution 2 (by Alexander Bogomolny).

We'll start with $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$. This is equivalent to

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq a+b+c.$$

Assume, ***WLOG,*** that $a \geq b \geq c$. Then $ab \geq ac \geq bc$ and $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}$. By the ***Rearrangement inequality***, it then follows that

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = (ab)\cdot\frac{1}{c} + (bc)\cdot\frac{1}{a} + (ac)\cdot\frac{1}{b}$$

$$= (ab)\cdot\frac{1}{c} + (ac)\cdot\frac{1}{b} + (bc)\cdot\frac{1}{a} \geq (ab)\cdot\frac{1}{b} + (ac)\cdot\frac{1}{a} + (bc)\cdot\frac{1}{c}$$
$$= a + b + c.$$

□

Solution 3 (by Kevin Soto Palacios , Soumava Chakraborty).

In simplification of the above argument, observe that
$x^2 + y^2 + z^2 \geq xy + yz + zx$ and let $x = bc, y = ca, z = ab$ to obtain
$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$. □

Solution 4 (by Amit Itagi).

Another way of looking at $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$: This is true due to ***Muirhead's theorem*** because the triple $(2, 2, 0)$ majorizes $(2, 1, 1)$. □

Acknowledgment (by Alexander Bogomolny)

Dan Sitaru has kindly posted the problem from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath*** page and later added the solution (Solution 1) by Abdul Aziz. Solution 3 has been posted independently by Kevin Soto Palacios and Soumava Chakraborty; Solution 4 is by Amit Itagi.

106. A Cyclic Inequality in Three Variables XXIV

Prove that, for $a, b, c > 0$,

$$\sum_{cycl} \frac{a^2b^2(1 + a^2)(1 + b^2)}{(1 + a)(1 + b)} \geq 4(3 - 2\sqrt{2})abc(a + b + c).$$

Proposed by Daniel Sitaru

Lemma

For $x > 0$,

$$\frac{1 + x^2}{1 + x} \geq 2(\sqrt{2} - 1)$$

Proof by algebra (by Daniel Sitaru).

$$\left(x - (\sqrt{2} - 1)\right)^2 \geq 0 \Leftrightarrow$$

$$x^2 - 2(\sqrt{2} - 1)(\sqrt{2} - 1)^2 \geq 0 \Leftrightarrow x^2 - 2(2\sqrt{2} - 1)(3 - 2\sqrt{2}) \geq 0 \Leftrightarrow$$

$$1 + x^2 \geq 2\sqrt{2} + 2x\sqrt{2} - 2 - 2x \Leftrightarrow 1 + x^2 \geq (1 + x)(2\sqrt{2} - 2) \Leftrightarrow$$

$$\frac{1 + x^2}{1 + x} \geq 2(\sqrt{2} - 1).$$

□

Proof by calculus (by Alexander Bogomony).

If $(x) = \frac{1+x^2}{1+x}, f'(x) = \frac{x^2+2x-1}{(1+x)^2}$, with the only positive zero $x = \sqrt{2}-1$, where $f(\sqrt{2}-1) = 2(\sqrt{2}-1)$. Furthere, $f''(x) = \frac{4}{(1+x)^3} \geq 0$, implying

$$f(x) \geq 2(\sqrt{2}-1)$$

□

Solution(by Daniel Sitaru).

Be Lemma,

$$\frac{(1+a^2)b^2}{1+a} \geq 2b^2(\sqrt{2}-1) \text{ and}$$

$$\frac{(1+b^2)a^2}{1+b} \geq 2a^2(\sqrt{2}-1)$$

so that, given the sequence $(2,2,0)$ ***majorizes*** $(2,1,1)$,

$$\frac{a^2b^2(1+a^2)(1+b^2)}{(1+a)(1+b)} \geq 4(\sqrt{2}-1)a^2b^2 \textit{ and, subsequently,}$$

$$\sum_{cycl} \frac{a^2b^2(1+a^2)(1+b^2)}{(1+a)(1+b)} \geq 4(\sqrt{2}-1)^2 \sum_{cycl} a^2b^2$$

$$\overbrace{\geq}^{Muirhead} 4(2+1-2\sqrt{2}) \sum_{cycl} a^2bc = 4(3-2\sqrt{2})abc(a+b+c)$$

Inequality is attained for $a = b = c = \sqrt{2}-1$.

□

Acknowledgment (by Alexander Bogomolny)

Dan Sitaru has kindly posted the problem from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath page*** and later communicated the above proof in a LaTeX file.

107. A Partly Cyclic Inequality in Four Variables

Prove that, for $x \leq y \leq -2 \leq z \leq t$,

$$\sum_{cycl} xe^x \geq (x+y+2)e^{x+y+2} + (z+t-2)\sqrt[3]{e^{z+t-2}}$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

Let $f(x) = xe^x, f''(x) = (x+2)e^x$. Thus for $x \geq -2, f$ is convex so that, by ***Jensen's inequality***,

$$\frac{f(z)+f(t)+f(-2)}{3} \geq f\Big(\frac{z+t-2}{3}\Big).$$

i.e.,

$$f(z)+f(t)+f(-2) \geq 3f\Big(\frac{z+t-2}{3}\Big) \qquad (1)$$

For $x \leq -2, f(x)$ is concave. $x+y = -2+(x+y+2)$. Thus, by *Karamata's inequality,*

$$f(x)+f(y)+f(-2) \geq f(-2)+f(x+y+2) \qquad (2)$$

Adding up (1) and (2) gives

$$f(z)+f(t)+f(-2)+f(x)+f(y)\geq f(-2)+f(x+y+2)+3f\Big(\frac{z+t-2}{3}\Big),$$

i.e., $\sum_{cycl} f(t) \geq f(x+y+2)+3f\left(\frac{z+t-2}{3}\right)$ or more explicitly,

$$ze^z+te^t+xe^x+ye^y\geq (x+y+2)e^{x+y+2}+3\frac{z+t-2}{3}\sqrt[3]{e^{z+t-2}}.$$

and, finally,

$$ze^z+te^t+xe^x+ye^y\geq (x+y+2)e^{x+y+2}+(z+t-2)\sqrt[3]{e^{z+t-2}}.$$

□

Acknowledgment (by Alexander Bogomolny)
Dan Sitaru has kindly posted the problem from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath page*** and later communicated the above solution in a LaTeX file.

108. A Problem From a Mongolian Olympiad for Grade 11

Prove that, for $a,b,c>0$, subject to $a^2+b^2+c^2=3$,

$$\frac{a}{3a+2b^3}+\frac{b}{2b+2c^3}+\frac{b}{2b+2c^3}+\frac{c}{3c+2a^3}\leq\frac{1}{5}\Big(\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\Big)$$

Solution (by Daniel Sitaru).
By the ***AM-GM inequality***,

$$3a+2b^3=(a^2+b^2+c^2)a+2b^3$$

$$=a^3+ab^2+ac^2+b^3+b^3\geq 5(a^5b^8c^2)^{\frac{1}{5}},$$

and similar for the other two fractions, so that, by the ***Rearrangement inequality***,

$$\sum_{cycl}\frac{a}{3a+2b^3}\leq\sum_{cycl}\frac{a}{5(a^5b^8c^2)^{\frac{1}{5}}}$$

$$=\sum_{cycl}\frac{1}{5(b^8c^2)^{\frac{1}{5}}}\leq\sum_{cycl}\frac{1}{5(b^8b^2)^{\frac{1}{5}}}=\sum_{cycl}\frac{1}{5(a^{10})^{\frac{1}{5}}}=\frac{1}{5}\sum_{cycl}\frac{1}{a^2}.$$

□

Acknowledgment (by Alexander Bogomolny)
Dan Sitaru has kindly posted at the ***CutTheKnotMath page*** the above problem from the ***Mongolian Mathematical Olympiad***, Grade 11.

109. An Acyclic Inequality in Three Variables

Assuming $a,b,c>0$, prove that

$$\frac{(a^2-bc)^2+(b^2-ca)^2+(c^2-ab)^2}{a^2+b^2+c^2+ab+bc+ca}\geq 3(a-b)(b-c).$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

$$\left(\sum_{cycl} a^2 - \sum_{cycl} ab\right)\left(\sum_{cycl} a^2 + \sum_{cycl} ab\right) = \left(\sum_{cycl} a^2\right)^2 - \left(\sum_{cycl} ab\right)^2$$
$$= a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2 -$$
$$-a^2b^2 - a^2c^2 - b^2c^2 - 2abc(a+b+c)$$
$$= (a^4 - 2a^2bc + b^2c^2) + (b^4 - 2b^2ac + a^2c^2)$$
$$+(c^4 - 2c^2ab + b^2a^2) = \sum_{cycl}(a^2 - bc)^2$$

It follows that

$$\frac{(a^2-bc)^2 + (b^2-ca)^2 + (c^2-ab)^2}{a^2+b^2+c^2+ab+bc+ca}$$
$$= \frac{(\sum_{cycl} a^2 - \sum_{cycl} ab)(\sum_{cycl} a^2 + \sum_{cycl} ab)}{\sum_{cycl} a^2 + \sum_{cycl} ab} = \sum_{cycl} a^2 - \sum_{cycl} ab$$

Suffice it to prove that

$$\sum_{cycl} a^2 - \sum_{cycl} ab \geq 3(a-b)(b-c) \Leftrightarrow$$
$$a^2 + b^2 + c^2 - ab - bc - ca \geq 3ab - 3ac - 3b^2 + 3bc \Leftrightarrow$$
$$a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ca \geq 0 \Leftrightarrow (a - 2b + c)^2 \geq 0$$

Equality is attained for $b = \frac{a+c}{2}$. □

Remark (by Alexander Bogomolny)

It may be worth noting that the inequality at hand is not cyclic, and, in this sense, the appearance of the cyclic sums may lead the reader to think otherwise. The fact is that the three variables do not occur in the inequality in a symmetric manner, so that an argument that relies on th ***"WLOG"*** reasoning may not be valid. For example, if we assume – apparently WLOG – that $a \geq c \geq b$, then the proof becomes immediate because

$$\frac{(a^2-bc)^2 + (b^2-ca)^2 + (c^2-ab)^2}{a^2+b^2+c^2+ab+bc+ca} \geq 0 \geq 3(a-b)(b-c).$$

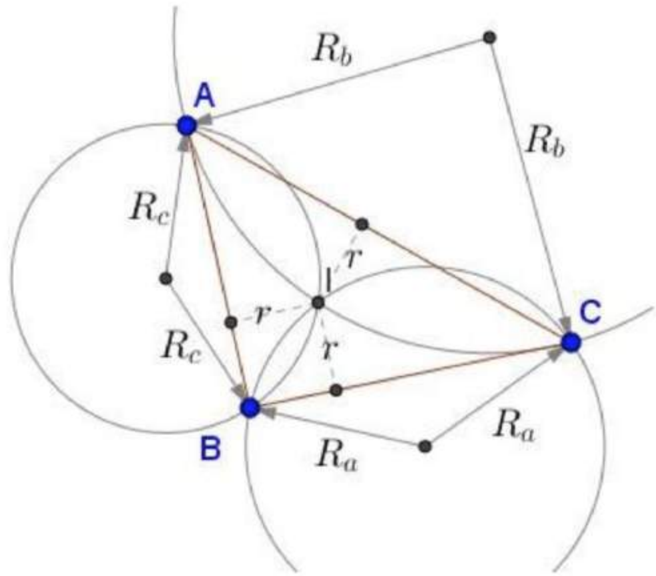
However, such a proof is faulty.

Acknowledgment (by Alexander Bogomolny)

The problem above has been posted on the ***CutTheKnotMath page*** and the solution above communicated to me by Daniel Sitaru. Originally, the problem has been published at the ***Romanian Mathematical Magazine.***

110. An Identity with Inradius and Circumradii

If I is the incenter of ΔABC, with r, R the inradius and circumradius, and R_a, R_b, R_c the circumradii of triangles IBC, ICA, IAB.



Prove that $R_a \cdot R_b \cdot R_c = 2R^2 r$

Proposed by Mehmet Şahin

Solution 1 (by Daniel Sitaru).

In ***every triangle*** with side lengths a, b, c area S and circumradius R, $abc = 4RS$. So, in ΔIAB,

$$R_c = \frac{AI \cdot BI \cdot AB}{4[\Delta IAB]} = \frac{\frac{r}{\sin\frac{A}{2}} \cdot \frac{r}{\sin\frac{B}{2}} \cdot c}{4\frac{rc}{2}} = \frac{r}{2\sin\frac{A}{2} \cdot \sin\frac{B}{2}}$$

etc. ***So that***

$$\prod_{cycl} R_a = \frac{r^3}{8\prod_{cycl}\sin^2\frac{A}{2}} = \frac{r^3}{8\prod_{cycl}\frac{(s-b)(s-c)}{bc}}$$
$$= \frac{r^3 a^2 b^2 c^2 s^2}{8S^4} = \frac{16S^2 r^3 s^2}{8S^4} = \frac{16 r^3 s^2}{8S^2} = 2R^2 r$$

□

Solution 2 (by Marian Dincă).

$$BC = 2R_a \sin\angle BIC = 2R_a \sin\left(\pi - \frac{B+C}{2}\right) = 2R_a \sin\frac{B+C}{2}$$

On the other hand,

$$BC = 2R\sin A = 2R\sin(A+B) = 4R\sin\frac{B+C}{2}\cos\frac{B+C}{2}$$

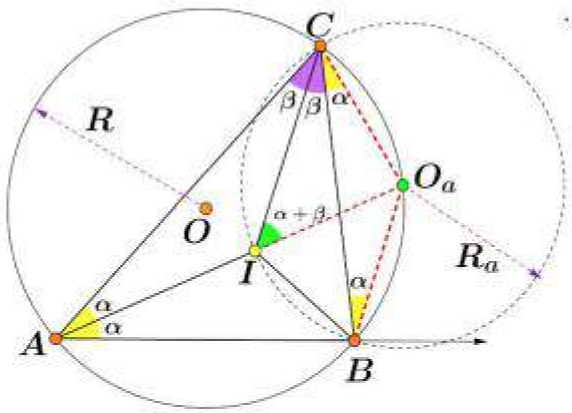
It follows that $R_a = 2R\sin\frac{A}{2}$, etc. ***We conclude that***

$$R_a \cdot R_b \cdot R_c = 8R^3 \sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2} = 2R^2 r.$$

□

Solution 3 (by Miquel Ochoa Sanchez).

If O_a is the circumcenter of ΔIBC then $O_aB = O_aI = O_aC = R_a$.



In $\Delta AO_aB, O_aB = 2R\sin\alpha$, implying $R_a = 2R\sin\frac{A}{2}$. Similarly, $R_b = 2R\sin\frac{B}{2}$ and $R_c = 2R\sin\frac{C}{2}$.

$$R_a \cdot R_b \cdot R_c = 2R^2\Big(4R\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\Big) = 2R^2 r.$$

Hence, $R_a \cdot R_b \cdot R_c = 2R^2 r$. □

Acknowledgment (by Alexander Bogomolny)

The problem by Mehmet Şahin (Turkey) has been posted to the site of the ***Romanian Mathematical Magazine*** and communicated to me by Daniel Sitaru. Solution 1 is by Daniel Sitaru (Romania); Solution 2 is by Marian Dincă (Romania); Solution 3 is by Miguel Ochoa Sanchez (Peru).

111. An Inequality In Triangle and Without II

In $\Delta ABC, a, b, c$ are the side lengths and R, r are the circumradius and inradius, respectively. Prove that:

$$\sqrt{\sqrt{\frac{a}{c}}+\sqrt{\frac{b}{c}}}+\sqrt{\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{a}}}+\sqrt{\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{b}}} \geq \frac{6r\sqrt{2}}{R}$$

Proposed by Daniel Sitaru

Remark (by Alexander Bogomolny)

All solvers have observed that, due to ***Euler's inequality*** $R \geq 2r$, $\frac{6r\sqrt{2}}{R} \leq 3\sqrt{2}$, and went to prove

$$\sqrt{\sqrt{\frac{a}{c}}+\sqrt{\frac{b}{c}}}+\sqrt{\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{a}}}+\sqrt{\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{b}}} \geq 3\sqrt{2}$$

Thus, this is the inequality that is proved below without additional comments. All solutions employ the ***AM-GM inequality***.

Solution 1 (by Seyran Ibrahimov).

Set $\frac{a}{c} = x^2, \frac{b}{c} = y^2, \frac{b}{a} = z^2$. The required inequality becomes $\sqrt{x+y} + \sqrt{z + \frac{1}{x}} + \sqrt{\frac{1}{y} + \frac{1}{z}} \geq 3\sqrt{2}$. We have

$$\sqrt{x+y} + \sqrt{z + \frac{1}{x}} + \sqrt{\frac{1}{y} + \frac{1}{z}} \geq 3\sqrt[6]{(x+y)\left(z + \frac{1}{x}\right)\left(\frac{1}{y} + \frac{1}{z}\right)}$$

$$\geq 3\sqrt[6]{2\sqrt{xy} \cdot 2\frac{\sqrt{z}}{\sqrt{x}} \cdot \frac{2}{\sqrt{yz}}} = 3\sqrt{2}.$$

□

Solution 2 (by Daniel Sitaru).

First off, for all $x, y, z > 0$,

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 = x + y + z + 2(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})$$

$$\geq x + y + z + 2 \cdot 3 \cdot \sqrt[3]{\sqrt{xy}\sqrt{yz}\sqrt{zx}} = x + y + z + 6\sqrt[3]{xyz}$$

Thus,

(1) $(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \geq x + y + z + 6\sqrt[3]{xyz}$. Set $x = \dfrac{u+v}{w}, y = \dfrac{u+w}{u}, z = \dfrac{w+u}{v}$

Then (1) gives

$$\left(\sum_{cycl} \frac{u+v}{w}\right)^2 \geq \sum_{cycl} \frac{u+v}{w} + 6\sqrt[3]{\prod_{cycl} \frac{u+v}{w}}$$

$$\geq \sum_{cycl} \frac{u+v}{w} + 6\sqrt[6]{\frac{2\sqrt{uv} \cdot 2\sqrt{vw} \cdot 2\sqrt{wu}}{uvw}}$$

$$= \sum_{cycl} \frac{u+v}{w} + 6\sqrt[3]{8} = \frac{(u+v)(v+w)(w+u)}{uvw} - 2 + 12$$

$$\geq \frac{2\sqrt{uv} \cdot 2\sqrt{vw} \cdot 2\sqrt{wu}}{uvw} + 10 = 8 + 10 = 18.$$

It follows that

$$\sqrt{\frac{u+v}{w}} + \sqrt{\frac{v+w}{u}} + \sqrt{\frac{w+u}{v}} \geq 3\sqrt{2}. \quad (2)$$

With $u = \sqrt{a}, v = \sqrt{b}, w = \sqrt{c}$ (2) becomes the required inequality. □

Solution 3 (by Soumava Chakraborty).

$$LHS \geq 3\sqrt[3]{\sqrt{\frac{\sqrt{a} + \sqrt{b}}{\sqrt{c}}} \cdot \sqrt{\frac{\sqrt{b} + \sqrt{c}}{\sqrt{a}}} \cdot \sqrt{\frac{\sqrt{c} + \sqrt{a}}{\sqrt{b}}}}$$

$$= 3\sqrt[6]{\frac{(x+y)(y+z)(z+x)}{xyz}}, \text{ where } x = \sqrt{a}, \text{ etc.}$$

$$\geq 3\sqrt[6]{\frac{2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{zx}}{xyz}} = 3\sqrt[6]{8} = 3\sqrt{2}$$

□

Acknowledgment (by Alexander Bogomolny)
This is a problem form the ***Romanian Mathematics Magazine***; it was kindly communicated to me by Daniel Sitaru, along with his solution (Solution 2). Solution 1 is by Seyran Ibrahimov; Solution 3 is by Soumava Chakraborty.

112. An Inequality in Triangle, Mostly with the Medians

Prove that in any ΔABC,

$$\prod_{cycl}(5m_a+3m_b)(3m_a+5m_b) < 64\prod_{cycl}(2s+a)^2.$$

where $2s = a+b+c$.

Proposed by Daniel Sitaru

Proof 1 (by Soumava Chakraborty).

By the ***AM-GM inequality***,

$$\sqrt{(5m_a+3m_b)(3m_a+5m_b)} \le \frac{8(m_a+m_b)}{2}, \text{ so that}$$

$$(5m_a+3m_b)(3m_a+5m_b) \le 16(m_a+m_b)^2. \text{ Now, obviously,}$$

$$m_a < \frac{b+c}{2}, m_b < \frac{c+a}{2}, m_c < \frac{a+b}{2}, \text{ implying}$$

$$16(m_a+m_b)^2 < 4(a+b+2c)^2 = 4(2s+c)^2. \text{ It follows that}$$

$$(5m_a+3m_b)(3m_a+5m_b) \le 4(2s+c)^2.$$

We similarly obtain

$$(5m_b+3m_c)(3m_b+5m_c) < 4(2s+a)^2 \text{ and}$$

$$(5m_c+3m_a)(3m_c+5m_a) < 4(2s+b)$$

The product of the three gives the required inequality.

□

Proof 2 (by Soumitra Mandal).

First off, $2s+a = (b+a)+(c+a) > c+b \ge 2\sqrt{bc}$ *so that*

$$\prod_{cycl}(2s+a)^2 > \prod_{cycl}(2\sqrt{bc})^2 = 4^3(abc)^2.$$

On the other hand,

$$(5m_a+3m_b)(3m_a+5m_b) = 15(m_a^2+m_b^2)+34m_am_b$$
$$\le \frac{15}{4}(4c^2+a^2+b^2)+\frac{34}{2}(2c^2+ab),$$

because $m_a^2 = \dfrac{2b^2+2c^2-a^2}{4}, m_b^2 = \dfrac{2a^2+2c^2-b^2}{4}$ *and* $m_am_b \le \dfrac{2c^2+ab}{4}$

We need to prove that

$$32c^2+\frac{15}{4}(a^2+b^2)+\frac{17}{2}ab < 4(2s+c)^2$$

This is equivalent to $16c^2+\dfrac{ab}{2} < \dfrac{a^2+b^2}{4}+16c(a+b)$, *which is true because* $c < a+b$ *and* $2ab < a^2+b^2$. *We only need to take the product of this and the two analogous inequalities to obtain the result.*

□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted at the ***CutTheKnotMath page*** the above problem of his that was published in the ***Romanian Mathematical Magazine***. Proof 1 is by Soumava Chakraborty; Proof 2 is by Soumitra Mandal.

113. An Inequality in Triangle, with Integrals

For $a, b, c \in \left[0, \frac{\pi}{2}\right], a + b + c = \pi$, *prove that*

$$4 \sum_{cycl} \sin^2 \frac{a}{2} + \pi \sum_{cycl} \int_0^a \cos(\sin x) dx \geq \pi^2.$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

By ***Kober's inequality***, $\cos x \geq 1 - \frac{2x}{\pi}$. It follows that $\cos(\sin x) \geq 1 - \frac{2 \sin x}{\pi}$ such that, say

$$\int_0^a \cos(\sin x) dx \geq \int_0^a \left(1 - \frac{2 \sin x}{\pi}\right) dx = a + \frac{2}{\pi} \cos x \Big|_0^a = a + \frac{2}{\pi}(\cos a - 1)$$

$$= a + \frac{2}{\pi}\left(1 - 2\sin^2 \frac{a}{2} - 1\right) = a = \frac{4}{\pi} \sin^2 \frac{a}{2}$$

Thus, altogether,

$$4 \sum_{cycl} \sin^2 \frac{a}{2} + \pi \sum_{cycl} \int_0^a \cos(\sin x) dx$$

$$\geq 4 \sum_{cycl} \sin^2 \frac{a}{2} + \sum_{cycl} \left(\pi a - 4 \sin^2 \frac{a}{2}\right) = \pi \sum_{cycl} a = \pi^2$$

□

Refinement (by Leonard Giugiuc)

For $a, b, c \in \left[0, \frac{\pi}{2}\right], a + b + c = \pi$, *prove that*

$$\frac{\pi}{3} \sum_{cycl} \sin^2 \frac{a}{2} + \sum_{cycl} \int_0^a \cos(\sin x) dx \geq \pi$$

It's known that $\sum_{cycl} \sin^2 \frac{a}{2} \geq \frac{3}{4}$ *and* $\cos t \geq 1 - \frac{t^2}{2}, t > 0$. *From the latter,* $\cos(\sin x) \geq 1 - \frac{\sin^2 x}{2}$, *for* $x \in \left[0, \frac{\pi}{2}\right]$.

Suffice it to show that $\sum_{cycl} \int_0^a \left(1 - \frac{\sin^2 x}{2}\right) dx \geq \frac{3\pi}{4}$. *But*

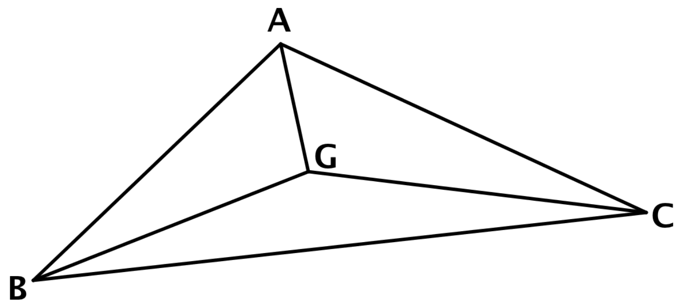
$$\sum_{cycl} \int_0^a \left(1 - \frac{\sin^2 x}{2}\right) dx = \pi - \sum_{cycl} \int_0^a \left(\frac{\sin^2 x}{2}\right) dx$$

$$= \pi + \sum_{cycl} \left(-\frac{a}{4} + \frac{\sin 2a}{8}\right) \geq \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Acknowledgment (by Daniel Sitaru)
This is Daniel Sitaru's problem for the ***Romanian Mathematical Magazine***. Solution is by Daniel Sitaru. Leo Giugiuc came up with an essential refinement above.

114. An Inequality in Triangle, with Sides and Medians

Given ΔABC, with centroid G; side lengths a, b, c and the medians m_a, m_b, m_c.



Prove that

$$16\sum\left(\frac{m_a}{m_c}+\frac{m_b}{m_c}\right)^4 > 81\left(\left(\frac{a}{m_a}\right)^4+\left(\frac{b}{m_b}\right)^4+\left(\frac{c}{m_c}\right)^4\right).$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

We have a self – explanatory sequence of steps:

$$GB + GC > BC$$

$$GB + GC > a \Rightarrow \frac{GB+GC}{GA} > \frac{a}{GA}$$

$$\left(\frac{GB+GC}{GA}\right)^4 > \left(\frac{a}{GA}\right)^4$$

$$\left(\frac{GB}{GA}+\frac{GC}{GA}\right)^4 > \left(\frac{a}{\frac{2}{3}m_a}\right)^4$$

$$\left(\frac{\frac{2}{3}m_b}{\frac{2}{3}m_a}+\frac{\frac{2}{3}m_c}{\frac{2}{3}m_a}\right)^4 > \frac{81}{16}\left(\frac{a}{m_a}\right)^4$$

$$\left(\frac{m_b}{m_a}+\frac{m_c}{m_c}\right)^4 > \frac{81}{16}\left(\frac{a}{m_a}\right)^4$$

$$16\left(\frac{m_b}{m_a}+\frac{m_c}{m_a}\right)^4 > 81\left(\frac{a}{m_a}\right)^4$$

$$16\sum\left(\frac{m_a}{m_c}+\frac{m_b}{m_c}\right)^4 > 81\left(\left(\frac{a}{m_a}\right)^4+\left(\frac{b}{m_b}\right)^4+\left(\frac{c}{m_c}\right)^4\right)$$

□

Acknowledgment (by Alexander Bogomolny)
The problem form the ***Romanian Mathematical Magazine*** has been kindly posted by Dan Sitaru at the ***CutTheKnotMath page***. Dan also has communicated his solution in a latex file.

115. An Inequality in Triangle, with Sides and Sums

For any ΔABC, prove that

$$\frac{a(2s-a)}{4(s-a)}+\frac{a(2s-b)}{4(s-b)}+\frac{a(2s-c)}{4(s-c)}\geq a+b+c.$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

Rewrite the inequality as

$$\sum_{cycl}\frac{a(b+c)}{2(b+c-a)}\geq\sum_{cycl}a.$$

Using ***Bergstrom's inequality***,

$$\sum_{cycl}\frac{a(b+c)}{b+c-a}=\sum_{cycl}\left[\frac{a^2}{b+c-a}+a\right]$$

$$\geq\frac{(\sum_{cycl}a)^2}{\sum_{cycl}(b+c-a)}+\sum_{cycl}a=\frac{(\sum_{cycl}a)^2}{\sum_{cycl}a}+\sum_{cycl}a=2\sum_{cycl}a$$

This is the required inequality. □

Acknowledgment (by Alexander Bogomolny)

The problem from the ***Romanian Mathematical Magazine*** has been kindly posted by Daniel Sitaru at the ***CutTheKnotMath page.***

116. An Inequality in Triangle, with Sines II

For any ΔABC, prove that

$$\left(\sum_{cycl}\frac{\sin A}{\sin B}\right)\left(\sum_{cycl}\frac{\sin A}{\sin^2 B}\right)\left(\sum_{cycl}\frac{\sin A}{\sin^3 B}\right)\geq 24\sqrt{3}$$

Proposed by Daniel Sitaru - Romania

Solution 1 (by Kevin Soto Palacios).

In an triangle $\sin A,\sin B,\sin C>0$. ***In addition***, $\sin A\sin B\sin C\leq\frac{3\sqrt{3}}{8}$. Thus applying first *Hölder's inequality* and, subsequently, the ***AM-GM inequality***,

$$\left(\sum_{cycl}\frac{\sin A}{\sin B}\right)\left(\sum_{cycl}\frac{\sin A}{\sin^2 B}\right)\left(\sum_{cycl}\frac{\sin A}{\sin^3 B}\right)\geq\left(\sum_{cycl}\frac{\sin A}{\sin^2 B}\right)^3$$

$$\geq\frac{27}{\sin A\sin B\sin C}\geq 27\cdot\frac{8}{3\sqrt{3}}=24\sqrt{3}.$$

Equality holds only for equilateral triangles. □

Solution 2 (by Myagmarsuren Yadamsuren).

In a triangle $\sin A, \sin B, \sin C > 0$. ***In addition,***

$$3\sqrt{3} \geq 8\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8} = \frac{abc}{R^3}.$$

Thus,

$$24\sqrt{3} \leq 8R^3\Big(\frac{3}{\sqrt[3]{abc}} \cdot \frac{3}{\sqrt[3]{(abc)^2}} \cdot 3\Big)$$

$$= 8R^3 \cdot 3\left(\sqrt[3]{\frac{a}{b^2} \cdot \frac{b}{c^2} \cdot \frac{c}{a^2}}\right) \cdot 3\left(\sqrt[3]{\frac{a}{b^3} \cdot \frac{b}{c^3} \cdot \frac{c}{a^3}}\right) \cdot 3\left(\sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}}\right)$$

$$\leq 8R^3\left(\sum_{cycl}\frac{a}{b}\right)\left(\sum_{cycl}\frac{a}{b^2}\right)\left(\sum_{cycl}\frac{a}{b^3}\right)$$

$$= \left(\sum_{cycl}\frac{\frac{a}{(2R)}}{\frac{b}{(2R)}}\right)\left(\sum_{cycl}\frac{\frac{a}{(2R)}}{\frac{b}{(2R)^2}}\right)\left(\sum_{cycl}\frac{\frac{a}{(2R)}}{\frac{b}{(2R)^3}}\right)$$

$$= \left(\sum_{cycl}\frac{\sin A}{\sin B}\right)\left(\sum_{cycl}\frac{\sin A}{\sin^2 B}\right)\left(\sum_{cycl}\frac{\sin A}{\sin^3 B}\right)$$

□

Solution 3 (by Soumava Chakraborty).

With the AM-GM inequality,

$$RHS \leq (3^3)\sqrt[3]{1}\left(\sqrt[3]{\frac{\prod_{cycl}\sin A}{\prod_{cycl}\sin^2 B}}\right)\left(\sqrt[3]{\frac{\prod_{cycl}\sin A}{\prod_{cycl}\sin^3 B}}\right)$$

$$= \frac{27}{\prod_{cycl}\sin A} \leq \frac{27 \cdot 8}{3\sqrt{3}} = 24\sqrt{3}$$

□

Solution 4 (by Leonard Giugiuc).

Basically, it's all about ***rearrangements:***

$$\sum_{cycl}\frac{\sin A}{\sin B} \geq \sum_{cycl}\frac{\sin A}{\sin B} = 3,$$

$$\sum_{cycl}\frac{\sin A}{\sin^2 B} \geq \sum_{cycl}\frac{\sin A}{\sin^2 A} = \sum_{cycl}\frac{1}{\sin A},$$

$$\geq \frac{3}{\sin\left(\frac{A+B+C)}{3}\right)}, \textit{ by Jensen's inequality,}$$

$$= \frac{3}{\sin\left(\frac{180^\circ}{3}\right)} = 2\sqrt{3}$$

$$\sum_{cycl}\frac{\sin A}{\sin^3 B} \geq \sum_{cycl}\frac{\sin A}{\sin^3 A} = \sum_{cycl}\frac{1}{\sin^2 A} = 3 + \sum_{cycl}\cot^2 A \geq 4$$

□

Acknowledgment (by Alexander Bogomolny)
The problem form the ***Romanian Mathematical Magazine*** has been kindly posted by Daniel Sitaru at the ***CutTheKnotMath page***. Solution 1 is by Kevin Soto Palacios; Solution 2 is by Myagmarsuren Yadamsuren; Solution 3 is by Soumava Chakraborty; Solution 4 is by Leo Giugiuc.

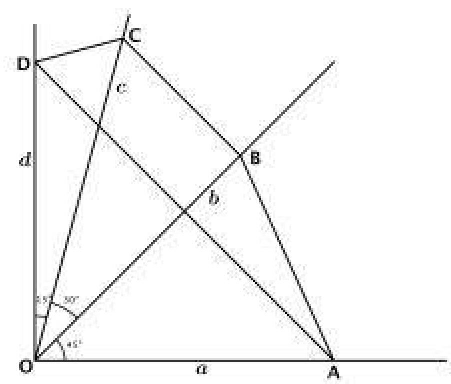
117. An Inequality Not in Triangle

Prove that for $a, b, c, d \geq 0$,

$$\sqrt{a^2+b^2-ab\sqrt{2}}+\sqrt{b^2+c^2-bc\sqrt{3}}+\sqrt{c^2+d^2-\frac{cd(\sqrt{6}+\sqrt{2})}{2}} \geq \sqrt{a^2+d^2}.$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).



With the reference to the above diagram, $OA = a, OB = b, OC = c, OD = d$.
By the ***Law of Cosines***,

$$AB^2 = a^2b^2 - 2ab\cos 45^\circ = a^2+b^2-ab\sqrt{2},$$
$$BC^2 = b^2+c^2-2bc\cos 30^\circ = b^2+c^2-bc\sqrt{3}$$
$$CD^2 = c^2+d^2-2cd\cos 15^\circ = c^2+d^2-\frac{cd(\sqrt{6}+\sqrt{2})}{2},$$
$$AD^2 - a^2+d^2, AB+BC+CD \geq AD$$

are the required inequality follows. □

Acknowledgment (by Alexander Bogomolny)
This is a problem and solution from the ***Romanian Mathematical Magazine*** kindly communicated to me by Daniel Sitaru.

118. An Inequality with Areas, Norms, and Complex Numbers

Prove that for $a, b, c, d \in \mathbb{R}$, with $a^2+b^2 \neq 0$ and $c^2+d^2 \neq 0$, one has

$$\frac{(ad-bc)\Big(3(a^2+b^2)(c^2+d^2)-4(ad-bc)^2\Big)}{\Big((a^2+b^2)(c^2+d^2)\Big)^{\frac{3}{2}}} \leq 1$$

Proposed by Daniel Sitaru

Solution 1(by Alexander Bogomolny).

Recollect that

$$(*) \qquad \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

and consider two points $A=(a,b)$ and $B=(c,d)$ in the plane with the origin $O=(0,0)$. The are $[\Delta ABO]$ can be expressed in two ways, viz.,

$$2[\Delta ABO] = |ad-bc|, \text{ and}$$

$$2[\Delta ABO] = \sqrt{a^2+b^2}\sqrt{c^2+d^2}\sin\alpha,$$

where α is the angle at vertex O of the triangle. It follows that

$$\sin\alpha \frac{|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

By noticing that $|\sin 3\alpha| \leq 1$ and substituting into (*), we obtain

$$1 \geq 3\frac{|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}} - 4\left(\frac{|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}}\right)^3$$
$$= \frac{|ad-bc|\Big(3(a^2+b^2)(c^2+d^2)-4(ad-bc)^2\Big)}{\Big((a^2+b^2)(c^2+d^2)\Big)^{\frac{3}{2}}}$$

which is the required inequality. □

Solution 2 (by Ravi Prakash).

Let $z_1 = a+ib, z_2 = d+ic$. Then $z_1z_2 = (ad-bc)+i(ac+bd)$, $ad-bc = \frac{1}{2}(z_1z_2 - \overline{z_1z_2}), ac+bd = \frac{1}{2}(z_1z_2+\overline{z_1z_2})$. Also, $|z_1|^2 = a^2+b^2$ and $|z_2|^2 = c^2+d^2$. Now,

$$3(a^2+b^2)(c^2+d^2)-4(ad-bc)^2 = 3z_1\overline{z_1}z_2\overline{z_2} - \frac{4}{4}(z_1z_2+\overline{z_1z_2})^2$$
$$= -[z_1^2z_2^2 + \overline{z_1}^2\overline{z_2}^2 + 2z_1z_2\overline{z_1z_2} - 3z_1z_2\overline{z_1z_2}]$$
$$= -[z_1^2z_2^2 + \overline{z_1}^2\overline{z_2}^2 - z_1z_2\overline{z_1z_2}]$$

Set

$$Num = -\frac{1}{2}(z_1z_2+\overline{z_1z_2})[z_1^2z_2^2+\overline{z_1}^2\overline{z_2}^2 - z_1z_2\overline{z_1z_2}$$
$$= -\frac{1}{2}[(z_1z_2)^3 - (\overline{z_1z_2})^3]$$

so that

$$|Num| \leq \frac{1}{2}|(z_1z_2)^3 - (\overline{z_1z_2})^3| \leq \frac{1}{2}[|z_1z_2|^3 + |\overline{z_1z_2}|^3]$$
$$= |z_1z_2|^3 = \Big((a^2+b^2)(c^2+d^2)\Big)^{\frac{3}{2}} = Den,$$

$$\text{implying } \frac{Num}{Den} \leq \frac{|Num|}{Den} \leq 1.$$

□

Solution 3 (by Ravi Prakash).

Set $z_1 = a + ib = r_1(\cos\theta + i\sin\theta)$ and $z_2 = d + ic = r_2(\cos\phi + i\sin\phi)$.

$$z_1 z_2 = (ad - bc) + i(ac + bd)$$

$$= r_1 r_2[\cos(\theta + \phi) + i\sin(\theta + \phi)].$$

Also, $|z_1| = r_1$ and $|z_2| = r_2$. Now,

$$\frac{(ad-bc)\Big(3(a^2+b^2)(c^2+d^2) - 4(ad-bc)^2\Big)}{\Big((a^2+b^2)(c^2+d^2)\Big)^{\frac{3}{2}}}$$

$$= \frac{r_1 r_2 \cos(\theta+\phi)[3r_1^2 r_2^2 - 4r_1^2 r_2^2 \cos^2(\theta+\phi)]}{r_1^3 r_2 r^3}$$

$$= 3\cos(\theta+\phi) - 4\cos^3(\theta+\phi) = -\cos 3(\theta+\phi) \le 1$$

□

Solution 4(by Soumitra Mandal).

Let $(a^2+b^2)(c^2+d^2) = x^2, x > 0$. We need to prove that

$$\frac{(ad-bc)(3x^2 - 4(ad-bc)^2)}{x^3} \le 1 \Leftrightarrow$$

$$x^3 - 3x^2(ad-bc) + 4(ad-bc)^3 \ge 0 \Leftrightarrow$$

$$x^3 + (ad-bc)^3 - 3(ad-bc)[x^2 - (ad-bc)^2] \ge 0 \Leftrightarrow$$

$$(x + ad - bc)(x^2 - x(ad-bc) + (ad-bc)^2)$$

$$(x + ad - bc)[x^2 - 4x(ad-bc) + 4(ad-bc)^2] \Leftrightarrow$$

$$-3(ad-bc)[x^2 - (ad-bc)^2] \ge 0 \Leftrightarrow$$

$$(x + ad - bc)(x - 2(ad-bc))^2 \ge 0,$$

which is true, for $x \ge bc - ad$ since $(ac+bd)^2 \ge 0$, for $a, b, c, d \in \mathbb{R}$. □

Solution 5 (by Soumava Chakraborty).

From $(ac+bd)^2 + (ad-bc)^2 = (a^2+b^2)(c^2+d^2)$,

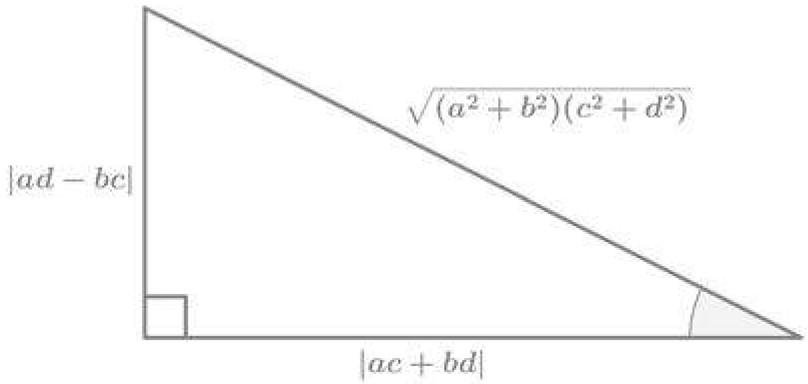
$$3(a^2+b^2)(c^2+d^2) - 4(ad-bc)^2$$

$$= 3[(ac+bd)^2 + (ad-bc)^2] - 4(ad-bc)^2]$$

$$\overbrace{=}^{(1)} 3(ac+bd)^2 - (ad-bc)^2$$

$$= 3|ac+bd|^2 - |ad-bc|^2$$

Case 1: $ac+bd\neq 0, ad-bc\neq 0$



From the diagram, $|ac+bd| = p\cos\theta$ (2) and $|ad-bc| = p\sin\theta, (3)$, where $p=\sqrt{(a^2+b^2)(c^2+d^2)}$. It follows that

$$LHS = \frac{(ad-bc)(3p^2\cos^2\theta - p^2\sin^2\theta)}{p^3}, \text{ using } (1),(2),(3)$$

$$\overbrace{=}^{(4)} \frac{(ad-bc)(3\cos^2\theta-\sin^2\theta)}{p}.$$

Now, according as $ad-bc\geq 0$ or $ad-bc\leq 0, ad-bc=\pm|ad-bc|$, (5). In any event, $\dfrac{|d-bc|}{p}=\sin\theta$. Such that, using (4) and (5),

$$\begin{aligned}LHS &= \pm\sin\theta(3\cos^2\theta-\sin^2\theta)\\ &= \pm\sin\theta\Big(3(1-\sin^2\theta)-\sin^2\theta\Big)\\ &= \pm(3\sin\theta-4\sin^3\theta)\\ &= \pm\sin 3\theta.\end{aligned}$$

Since $|\sin\theta|\leq 1, |LHS|\leq 1$.

Case 2: $ad-bc=0$

Then $ac+bd\neq 0$ and $LHS=0\leq 1$

Case 3: $ac+bd=0$

Then $ad-bc=0$ and $LHS=\dfrac{-(ad-bc)}{|ad-bc|}=\pm 1\leq 1$

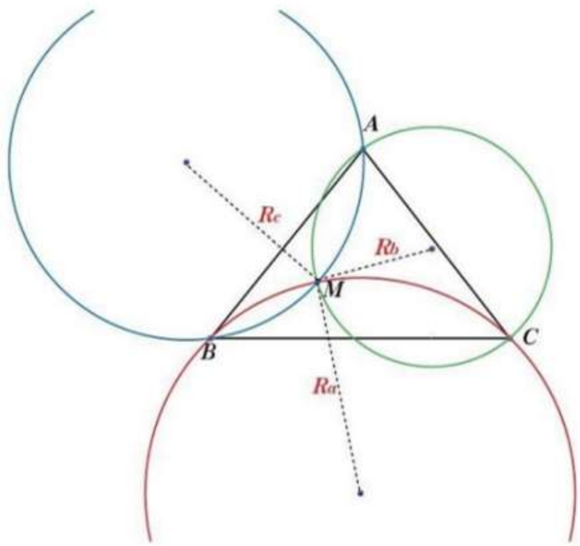
□

Acknowledgment (by Alexander Bogomolny)

This is Problem SP060 from the ***2017 Spring issue of the Romanian Mathematical Magazine***, proposed by Daniel Sitaru. Solution 2 and 3 are by Ravi Prakash, Solution 4 is by Soumitra Mandal; Solution 5 is by Soumava Chakraborty (all India).

119. An Inequality with Circumradii And Distances to the Vertices

Let M be an interior point of ΔABC. Denote by R_a, R_b and R_c the circumradii of the triangles MBC, MCA and MAB, respectively.



Prove that

$$\frac{MB \cdot MC}{R_a} + \frac{MC \cdot MA}{R_b} + \frac{MA \cdot MB}{R_c} \leq MA + MB + MC$$

Proposed by Leonard Giugiuc, Abdilkadir Altintas

Solution 1(by proposers).

Denote $MA = x, MB = y, MC = z, \angle BMC = 2\alpha, \angle AMC = 2\beta, \angle AMB = 2\gamma$

Clearly, $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ and $\alpha + \beta + \gamma = \pi$.

From the ***Law of Cosines*** in $\Delta MBC, BC = \sqrt{y^2 - 2yz\cos 2\alpha + z^2}$. Byt the ***Law of Sines*** in $\Delta MBC, R_a = \dfrac{\sqrt{y^2 - 2yz\cos\alpha + z^2}}{2\sin 2\alpha}$. As $y^2 + z^2 \geq 2yz$,

$$R_a = \frac{\sqrt{y^2 - 2yz\cos 2\alpha + z^2}}{2\sin 2\alpha} \geq \frac{\sqrt{2yz - 2yz\cos 2\alpha}}{2\sin 2\alpha}$$
$$= \frac{2\sqrt{yz}\sin\alpha}{2\sin 2\alpha} = \frac{\sqrt{yz}}{2\cos\alpha}$$

such that $\dfrac{1}{R_a} \leq \dfrac{2\cos\alpha}{\sqrt{yz}}$.We have

$$\frac{MB \cdot MC}{R_a} = \frac{yz}{R_a} \leq \frac{2yz\cos\alpha}{\sqrt{yz}} = 2\sqrt{yz}\cos\alpha.$$

Similarly,

$$\frac{MC \cdot MA}{R_b} \leq 2\sqrt{zx}\cos\beta$$

$$\frac{MA \cdot MB}{R_c} \leq 2\sqrt{xy}\cos\gamma$$

Thus,

$$\frac{MB \cdot MC}{R_a} + \frac{MC \cdot MA}{R_b} + \frac{MA \cdot MB}{R_c} \leq 2\sqrt{yz}\cos\alpha + 2\sqrt{zx}\cos\beta + 2\sqrt{xy}\cos\gamma$$

Suffice it to show that

$$2\sqrt{yz}\cos\alpha + 2\sqrt{zx}\cos\beta + 2\sqrt{xy}\cos\gamma \leq x + y + z$$

But, according to the famous ***Wolstenholme's inequality***,

For real x, y, z and $\alpha + \beta + \gamma = \pi$,

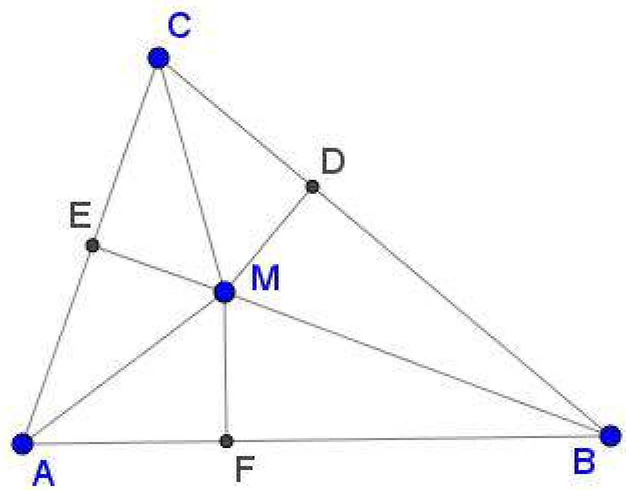
$$yz\cos\alpha + zx\cos\beta + xy\cos\gamma \leq x^2 + y^2 + z^2$$

Replacing here x, y, z with $\sqrt{x}, \sqrt{y}, \sqrt{z}$ completes the proof.

□

Solution 2 (by Daniel Sitaru).

Based on the following configuration and ***the formula*** $abc = 4RS$,



$$\frac{MB \cdot MC}{R_a} = \frac{MB \cdot MC}{MB \cdot MC \cdot MC \cdot \frac{1}{4[\Delta MBC]}} = 4 \cdot \frac{[\Delta MBC]}{BC} = 2MD,$$

such that, by similarity,

$$\frac{MB \cdot MC}{R_a} + \frac{MC \cdot MA}{R_b} + \frac{MB \cdot MB}{R_c} = 2(MD + ME + MF) \leq MA + MB + MC$$

by the ***Erdös - Mordell inequality***. □

Aknowledgment (by Alexander Bogomolny)

The problem due to Leo Giugiuc and Kadir Altintas has been posted by Leo Giugiuc at the ***CutTheKnotMath facebook page***, with their solution (Solution 1) communicated privately. Solution 2 is by Daniel Sitaru.

120. An inequality with Cosines and a Sine

Prove that in acute ΔABC

$$\cos A + 4\cos B + 4\sin\frac{C}{2} \le 9\cos\frac{\pi + B - C}{3}$$

Proposed by Daniel Sitaru

Solution 1(by Leonard Giugiuc).

On the internal $\left[0, \frac{\pi}{2}\right]$, function $\cos x$ is concave, such that, by ***Jensen's inequality***,

$$\cos A + \cos B \le 2\cos\frac{A+B}{2} = 2\sin\frac{C}{2}$$

Hence suffice it to prove that $\cos B + 2\cos\frac{A+B}{2} \le 3\cos\frac{\pi+B-C}{3}$. We again apply Jensen's inequality:

$$\cos B + 2\cos\frac{A+B}{2} = \cos A + \cos\frac{A+B}{2} + \cos\frac{A+B}{2}$$

$$\le 3\cos\left(\frac{B + \frac{A+B}{2} + \frac{A+B}{2}}{3}\right) = 3\cos\frac{A+2B}{3}$$

and note that, since $A + B + C = \pi$, $\frac{A+2B}{3} = \frac{\pi+B-C}{3}$ □

Solution 2 (by Alexander Bogomolny).

Observe that $\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$ and make use of Jensen's inequality as above:

$$\cos A + 4\cos B + 4\sin\frac{C}{2} = \cos A + 4\cos B + 4\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\le 9\cos\left(\frac{A + 4B + 4\left(\frac{\pi}{2} - \frac{C}{2}\right)}{3}\right) = 9\cos\frac{2\pi + A + 4B - 2C}{9}$$

$$= 9\cos\frac{3\pi + 3B - 3C}{9} = 9\cos\frac{\pi + B - C}{3}.$$

□

Aknowledgment (by Alexander Bogomolny)

The problem (from the Romanian Mathematical Magazine) has been posted by Dan Sitaru at the ***CutTheKnotMath page***, and commented on by Leo Giugiuc with his solution (Solution 1). Solution 2 may seem as a slight modification of Solution 1.

121. An Inequality with Cyclic Sums And Products

Prove that if $a, b, c, d, e > 0$ are pairwise distinct, then

$$\sum_{cycl}\frac{a^2}{(b+c+d+e)(a-b)(a-c)(a-d)(a-e)} < \frac{(a+b+c+d+e)^2}{1024abcde}$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).
Let's split the left – hand side into partial fractions:

$$\frac{x^2}{\prod_{cyc}(x-a)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d} + \frac{E}{x-e}$$

Then

$$\frac{x^2}{\prod_{k\neq a(x-k)}} = A + \frac{B(x-a)}{x-b} + \frac{C(x-a)}{x-c} + \frac{D(x-a)}{x-d} + \frac{E(x-a)}{x-e}.$$

This shows that $= \frac{a^2}{(a-b)(a-c)(a-d)(a-e)}$. Similar expressions can be found for B, C, D and E. It follows that

$$\sum_{cycl} \frac{A}{x-a} = \sum_{cycl} \frac{a^2}{(x-a)(a-b)(a-c)(a-d)(a-e)}$$

$$= \frac{x^2}{(x-a)(x-b)(x-c)(x-d)(x-e)}.$$

Replacing x with $a+b+c+d+e$ and using the ***AM-GM inequality***, we get

$$\sum_{cycl} \frac{a^2}{(b+c+d+e)\prod_{k\neq a}(a-k)} = \frac{(a+b+c+d+e)}{\prod_{cycl}[\sum_{k\neq a} k]}$$

$$< \frac{(a+b+c+d+e)^2}{\prod_{cycl}[4\sqrt[4]{\prod_{k\neq a} k}]} = \frac{(a+b+c+d+e)^2}{4^4 abcde}$$

□

Aknowledgment (by Alexander Bogomolny)
The problem has been kindly posted by Dan Sitaru at the ***CutTheKnotMath page***; his solution in a latex file came via email.

122. An Inequality with Inradius and Circumradii

If I is the incenter of ΔABC, with r, R the inradius and circumradius, and R_a, R_b, R_c the circumradii of triangles IBC, ICA, IAB.

$$(R_a + R_b + R_c)\Big(\frac{R_a}{R_b R_c} + \frac{R_b}{R_c R_a} + \frac{R_c}{R_a R_b}\Big) \geq 12 - \frac{6r}{R}$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).
In every triangle with the side lengths a, b, c are S, and circumradius R, we have $abc = 4RS$. So, in ΔIAB,

$$R_c = \frac{AI\cdot BI\cdot AB}{4[\Delta IAB]} = \frac{\frac{r}{\sin\frac{A}{2}}\cdot\frac{r}{\sin\frac{B}{2}}\cdot a}{4\cdot\frac{ar}{2}} = \frac{r}{2\sin\frac{A}{2}\sin\frac{B}{2}},$$

Therefore, $R_c = \frac{r}{2\sin\frac{A}{2}\sin\frac{B}{2}}$ and, similarly, $R_b = \frac{r}{2\sin\frac{C}{2}\sin\frac{A}{2}}$ and $R_a = \frac{r}{2\sin\frac{B}{2}\sin\frac{C}{2}}$.
Thus, using ***Bergstrom's inequality,***

$$\sum_{cycl} R_a = \frac{r}{2}\sum_{cycl}\frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}} \geq \frac{r}{2}\cdot\frac{(1+1+1)^2}{\sum_{cycl}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\geq \frac{r}{2}\cdot\frac{9}{\frac{1}{3}\left(\sum_{cycl}\sin\frac{A}{2}\right)^2}=\frac{r}{2}\cdot\frac{27}{\left(\frac{3}{2}\right)^2}=\frac{r}{2}\cdot\frac{4}{9}\cdot 27$$
$$= 6r.$$

It follows that

(1) $$\sum_{cycl} R_a = 6r.$$

Further,

$$\sum_{cycl}\frac{R_a}{R_bR_c}=\sum_{cycl}\frac{\frac{r}{2\sin\frac{B}{2}\sin\frac{C}{2}}}{\sin^2\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{2}{r}\sum_{cycl}\sin^2\frac{A}{2}$$
$$=\frac{1}{r}\sum_{cycl}(1-\cos A)=\frac{3}{r}=\frac{1}{r}\sum_{cycl}\cos A$$
$$=\frac{3}{r}-\frac{1}{r}\left(1+\frac{r}{R}\right)=\frac{3}{r}-\frac{1}{r}-\frac{1}{R}=\frac{2}{r}-\frac{1}{R},$$

implying

(2) $$\sum_{cycl}\frac{R_a}{R_bR_c}=\frac{2R-r}{Rr}$$

Multiplying (1) and (2), we get

$$\left(\sum_{cycl}R_a\right)\left(\sum_{cycl}\frac{R_a}{R_bR_c}\right)\geq(6R)\left(\frac{2R-r}{Rr}\right)=6\cdot\frac{2R-r}{R}=12-\frac{6r}{R}$$

□

Acknowledgment (by Alexander Bogomolny)
This is Dan Sitaru's problem from the ***Romanian Mathematical Magazine***. Solution is by Dan Sitaru.

123. An Inequality with Integrals and Radicals

Let $f:[0,1]\to(0,\infty)$ *be a continuous function such that*

$$\left(\int_0^1\sqrt[3]{f(x)}dx\right)\left(\int_0^1\sqrt[5]{f(x)}dx\right)\left(\int_0^1\sqrt[7]{f(x)}dx\right)\leq 1$$

Proposed by Daniel Sitaru

Solution 1(by Chris Kyriazis).

By the ***AM-GM inequality***,

$$\sqrt[3]{f(x)}dx=\sqrt[3]{f(x)\cdot 1\cdot 1}\leq\frac{f(x)+1+1}{3}=\frac{f(x)+2}{3},$$

so that

(1) $$\int_0^1\sqrt[3]{f(x)}dx\leq\frac{1}{3}\int_0^1 f(x)dx+\frac{2}{3}\int_0^1 dx=1$$

Also

$$\sqrt[5]{f(x)}dx=\sqrt[5]{f(x)\cdot 1\cdot 1}\leq\frac{f(x)+1+1+1+1}{5}=\frac{f(x)+4}{5},$$

so that

$$\int_0^1 \sqrt[5]{f(x)}dx \le \frac{1}{5}\int_0^1 f(x)dx + \frac{4}{5}\int_0^1 dx = 1 \tag{2}$$

Similarly,

$$\int_0^1 \sqrt[7]{f(x)}dx \le \frac{1}{7}f(x)dx + \frac{6}{7}\int_0^1 dx = 1 \tag{3}$$

By multiplying (1), (2); (3):

$$\left(\int_0^1 \sqrt[3]{f(x)}dx\right)\left(\int_0^1 \sqrt[5]{f(x)}dx\right)\left(\int_0^1 \sqrt[7]{f(x)}dx\right) \le 1$$

□

Solution 2 (by Amit Itagi).

Function x^n is convex for $x \ge 0, n$ an integer so that, by the integral form of ***Jensen's inequality***

$$\left(\int_0^1 \sqrt[n]{f(x)}dx\right)^n \le \int_0^1 \left[\sqrt[n]{f(x)}\right]^n dx = \int_0^1 f(x)dx = 1$$

It follows that each of the three integrals on the left is not greater than 1, and so is their product. □

Solution 3 (by Nassim Nicholas Taleb).

By the L^p - norm inequality, in its general form, for any L^1 function f and for $0 < p \le q$,

$$\left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \ge \left(\int_a^b |f(x)|^q dx\right)^{\frac{1}{q}}$$

Here apply $p = \frac{1}{n_j}, q = 1, f(x) \ge 0$, for all $n_j > 1$:

$$\left(\int_a^b f(x)^{\frac{1}{n_j}} dx\right)^{n_j} \le 1.$$

Thus,

$$\prod_j \left(\int_a^b f(x)^{\frac{1}{n_j}} dx\right)^{n_j} \le 1.$$

□

Solution 4 (by Andrea Aquaviva).

Using *Hölder's inequality, with* $p = n, q = \frac{n}{n-1}, \frac{1}{p} + \frac{1}{q} = 1$,

$$\int_0^1 g^h dx \le \left(\int_0^1 h^p dx\right)^{\frac{1}{p}} \left(\int_0^1 g^q dx\right)^{\frac{1}{q}}, h = \sqrt[n]{f}, g \equiv 1:$$

$$\int_0^1 \sqrt[n]{f(x)}dx = \int_0^1 \sqrt[n]{f(x)} \cdot 1dx$$

$$\le \left(\int_0^1 \left(\sqrt[n]{f(x)}\right)^n dx\right)^{\frac{1}{n}} \left(\int_0^1 1^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} = \left(\int_0^1 f(x)dx\right)^{\frac{1}{n}} = 1.$$

□

Aknowledgment (by Alexander Bogomolny)
The problem (from the ***Romanian Mathematical Magazine***) has been posted by Dan Sitaru at the ***CutTheKnot Math page***. Dan later communicated by email his solution on a LaTeX file. The solution has been obtained independently by Chirs Kyriazis. Solution 2 is by Amit Itagi, Solution 3 is by N. N. Taleb; Solution 4 is by Andrea Acquaviva.

124. An Inequality with Powers of Six

Prove that in any ΔABC,

$$a^6+b^6+c^6 \geq 8r^2 s\sum_{cycl}\frac{a^5}{b^2-bc+c^2},$$

where $=\dfrac{a+b+c}{2}$, *the semiperimeter of* $\Delta ABC, r$ *its inradius.*

Proposed by Daniel Sitaru

Proof (by Soumitra Mandal , Seyran Ibrahimov).
From $(a-b)^2 \geq 0, a^2-ab+b^2 \geq ab$. So, using ***Euler's inequality*** $R \geq 2r$, $abc = 4RS$ and $S = rs$,

$$RHS = 8r^2 s\sum_{cycl}\frac{a^5}{b^2-bc+c^2} \leq 8r^2 s\sum_{cycl}\frac{a^5}{bc}$$

$$\leq \frac{8r^2 s}{abc}\left(\sum_{cycl}a^6\right) = \frac{8r^2 s}{4RS}\left(\sum_{cycl}a^6\right) = \sum_{cycl}a^6 = LHS.$$

□

Acknowledgment (by Alexander Bogomolny)
Dan Sitaru has kindly posted at the ***CutTheKnotMath page*** the above problem from his book Math Accent, with a proof by Soumitra Mandal (India); Seyran Ibrahimov (Azerbaijan) submitted the same proof independently.

125. An Inequality with Tangents and Cotangents

Prove that in ΔABC the following relation holds

$$\prod_{cycl}\Big(\tan\frac{A}{2}\tan\frac{B}{2}+\cot\frac{A}{2}\cot\frac{B}{2}\Big) \geq \frac{1000}{27}$$

Proposed by Daniel Sitaru

Solution 1 (by Leonard Giugiuc).
Set $\tan\frac{A}{2}\tan\frac{B}{2} = z, \tan\frac{B}{2}\tan\frac{C}{2} = x, \tan\frac{C}{2}\tan\frac{A}{2} = y$. We have to prove that

$$\Big(x+\frac{1}{x}\Big)\Big(y+\frac{1}{y}\Big)\Big(z+\frac{1}{z}\Big) \geq \frac{1000}{27},$$

subject to $x+y+z = 1$.
Consider function $f:(0,1)\to\mathbb{R}$, defined by $f(t) = \ln\Big(t+\frac{1}{t}\Big)$. We find $f'(t) = \frac{2t}{t^2+1}-\frac{1}{t} < 0, f''(0) = \frac{2(1-t)^2}{(t^2+1)^2}+\frac{1}{t^2} > 0$, for $t \in (0,1)$, implying that the function is convex and decreasing so that, via ***Jensen's inequality,***

$$f(x)+f(y)+f(z) \geq 3f\Big(\frac{x+y+z}{3}\Big) = 3f\Big(\frac{1}{3}\Big)$$

i.e.

$$\ln\Big(x+\frac{1}{x}\Big)+\ln\Big(y+\frac{1}{y}\Big)+\ln\Big(z+\frac{1}{z}\Big)\geq 3\ln\Big(3+\frac{1}{3}\Big)=\ln\Big(\frac{10}{3}\Big),$$

implying the required inequality. □

Solution 2 (by Leonard Giugiuc).

With the same change of variables, the problem reduces to

$$\Big(x+\frac{1}{x}\Big)\Big(y+\frac{1}{y}\Big)\Big(z+\frac{1}{z}\Big)\geq\frac{1000}{27},$$

subject to $x+y+z=1$.

With *Hölder's inequality*, we get

$$\Big(x+\frac{1}{x}\Big)\Big(y+\frac{1}{y}\Big)\Big(z+\frac{1}{z}\Big)\geq\Big(\sqrt[3]{xyz}+\frac{1}{\sqrt[3]{xyz}}\Big)^3.$$

By the ***AM-GM inequality***, $\sqrt[3]{xyz}\leq\frac{1}{3}$, and, since the function $f(t)=t+\frac{1}{t}$ is decreasing for $t\in(0,1)$,

$$\Big(\sqrt[3]{xyz}+\frac{1}{\sqrt[3]{xyz}}\Big)^3\geq\Big(3+\frac{1}{3}\Big)^3=\frac{1000}{27},$$

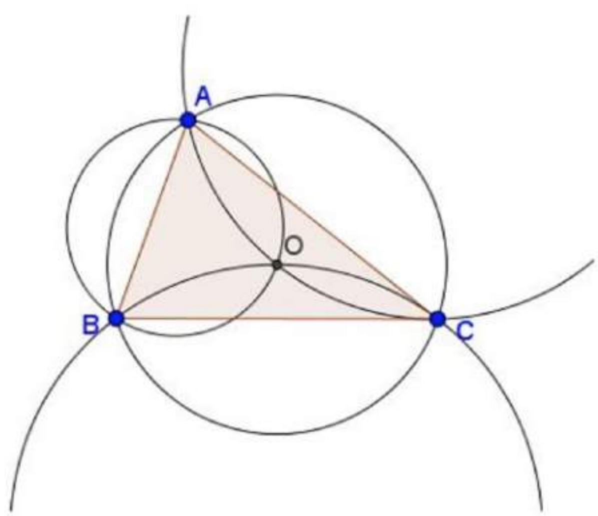
which completes the proof. □

Acknowledgment (by Alexander Bogomolny)

I am grateful to Dan Sitaru for communicating to me his problem and its two solutions by Leo Giugiuc.

126. An Inequality with a Variety of Circumradii

Let R_a, R_b, R_c be the circumradius of $\Delta BOC, \Delta AOC$ respectively, ΔAOB, where O is the circumcenter of an ***acute*** ΔABC.



Prove that

$$\frac{R_a^2}{R_b}+\frac{R_b^2}{R_c}+\frac{R_c^2}{R_a}\geq 3R.$$

Proposed by Daniel Sitaru

Solution 1 (by Mehmet Şahin).

It's well known that $R_a = \frac{R}{2\cdot\cos A}$ and $\cos A \cdot \cos B \cdot \cos C \leq \frac{1}{8}$. Thus,

$$R_a R_b R_c = \frac{R^3}{8 \cdot \cos A \cdot \cos B \cdot \cos C} \geq R^3,$$

$$\frac{R_a^2}{R_b} + \frac{R_b^2}{R_c} + \frac{R_c^2}{R_a} \geq \frac{(R_a + R_b + R_c)^2}{R_a + R_b + R_c} = R_a + R_b + R_c,$$

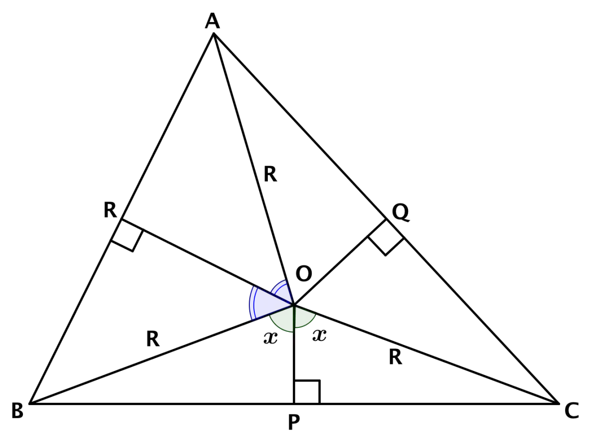
$$R_a + R_b + R_c \geq 3 \cdot \sqrt[3]{R_a R_b R_c} = 3 \cdot \sqrt[3]{R^3}, \textit{ by the AM-GM inequality}$$

$$R_a + R_b + R_c \geq 3R$$

as desired.

□

Solution 2 (by Soumava Chakraborty).



By ***the property*** of central and inscribed angels subtended by the same arc, $\angle BOC = 2A$, thus, in the above diagram, $x = A$. In $\Delta OBP, OP = R\cos x$, implying $OP = R\cos A$. It follows that

$$R_a = \frac{OB \cdot OC \cdot a}{4[\Delta OBC]} = \frac{aR^2}{4 \cdot \frac{1}{2}a \cdot OP} = \frac{R^2}{2R\cos A} = \frac{R}{2\cos A}$$

Similarly, $R_b = \frac{R}{2\cos B}$ and $R_c = \frac{R}{2\cos C}$. Therefore

$$LHS = \frac{R}{2}\left(\frac{\cos B}{\cos^2 A} + \frac{\cos C}{\cos^2 B} + \frac{\cos A}{\cos^2 C}\right)$$

$$\geq \frac{3R}{2}\sqrt[3]{\frac{1}{\cos A \cos B \cos C}}, \text{ due to the AM-GM inequality}$$

$$\geq 3R \text{ because } \prod_{cyc} \cos A \leq \frac{1}{8}$$

□

Solution 3 (by George Apostolopoulos).

It is well – known that

$$R_a = \frac{R}{2\cos A}, R_b = \frac{R}{2\cos B}, R_c = \frac{R}{2\cos C}$$

and

$$\cos A \cdot \cos B \cdot \cos C \leq \frac{1}{8}$$

So by ***AM-GM inequality***, we have

$$\frac{R_a^2}{R_b}+\frac{R_b^2}{R_c}+\frac{R_c^2}{R_a}\ge 3\sqrt[3]{\frac{R_a^2R_b^2R_c^2}{R_bR_cR_a}}$$

$$=3\sqrt[3]{R_aR_bR_c}=3\sqrt[3]{\frac{R^3}{8\cos A\cdot\cos B\cdot\cos C}}$$

$$\ge 3\sqrt[3]{\frac{R^3}{1}}=3R.$$

Equality holds when ΔABC is equilateral. □

Solution 4 (by Daniel Sitaru).

$$[\Delta BOC]=\frac{1}{2}OA\cdot OB\cdot\sin(\widehat{BOC})=\frac{1}{2}R^2\sin 2A$$

$$R_a=\frac{OB\cdot OC\cdot BC}{4S[BOC]}=\frac{R\cdot R\cdot a}{4\cdot\frac{1}{2}R^2\sin 2A}$$

$$=\frac{a}{2\sin 2A}=\frac{2R\sin A}{2\cdot 2\sin A\cos A}=\frac{R}{2\cos A}$$

$$\frac{R_a^2}{R_b}+\frac{R_b^2}{R_c}+\frac{R_c^2}{R_a}\overbrace{\ge}^{AM-GM}3\sqrt[3]{\frac{R_a^2}{R_b}\cdot\frac{R_b^2}{R_c}\cdot\frac{R_c^2}{R_a}}$$

$$=3\sqrt[3]{R_aR_bR_c}=3\sqrt[3]{\frac{R^3}{8\cos A\cos B\cos C}}\ge 3\sqrt[3]{\frac{R^3}{8\cdot\frac{1}{8}}}=3R.$$

□

Acknowledgment (by Alexander Bogomolny)
This is Daniel Sitaru's problem from the ***Romanian Mathematical Magazine***. Solution 1 by Mehmet Sahin (Turkey); Solution 2 is by Soumava Chakraborty (India); Solution 3 is by George Apostolopoulos (Greece); Solution 4 is by Daniel Sitaru (Romania) who kindly provided a tex file with all the solutions.

127. An Inequality with Just Two Variable VII

Prove that, for $x,y\ge 0$

$$(x^3+y^3)^3(x^2-xy+y^2)\ge x^2y^2\sqrt{xy}(x^2+y^2)^3.$$

Proposed by Daniel Sitaru

Solution 1(by Kevin Soto Palacios).

1. $x^2-xy+y^2\ge\frac{1}{2}(x^2+y^2)\ge\frac{1}{4}(x+y)^2$. Also $x^2+y^2\ge 2xy$.
2. For $x,y\ge 0, x+y\ge 2\sqrt{xy}$.

The given inequality is equivalent to

$$(x+y)^3(x^2-xy+y^2)^3(x^2-xy+y^2)\ge x^2y^2\sqrt{xy}(x^2+y^2)^3$$

Thus,

$$(x+y)^3(x^2-xy+y^2)^3\ge\frac{1}{4}(x+y)^4(x+y)\ge 8x^2y^2\sqrt{xy},$$

$$(x^2-xy+y^2)^3\ge\frac{1}{8}(x^2+y^2)^3$$

Multiplying the two gives the require inequality. □

Solution 2 (by Seyran Ibrahimov).

By ***Chebyshev's inequality***,

$$x^3+y^3 \geq \frac{1}{2}(x+y)(x^2+y^2)$$

By the ***AM-GM inequality***,

$$x^2-xy+y^2 \geq xy.$$

Thus

$$(x^3+y^3)^3(x^2-xy+y^2) \geq \frac{xy}{8}(x+y)^2(x^2+y^2)^3 \geq RHS$$

Because,

$$(x+y)^3 \geq (2\sqrt{xy})^3 = 8xy\sqrt{xy}$$

□

Solution 3 (by Myagmarsuren Yadamsuren).

Multiply the require inequality by $(x+y)$ which reduce it to:

$$(x^3+y^3)^4 \geq x^2y^2\sqrt{xy}(x^2+y^2)^3(x+y).$$

We have

$$(x^3+y^3)^4 \geq \Big(\frac{1}{2}(x+y)(x^2+y^2)\Big)^4$$

$$= \Big(\frac{x+y}{2}\Big)^2 \cdot \Big(\frac{x^2+y^2}{2}\Big) \cdot \frac{(x+y)^2}{2} \cdot (x^2+y^2)^3$$

$$\geq (xy)^2 \cdot \frac{x+y}{2} \cdot (x+y) \cdot (x^2+y^2)^3 \geq (xy)^2\sqrt{xy}(x^2+y^2)^3(x+y).$$

□

Solution 4 (by Ravi Prakash).

If $x=0$ or $y=0$, there is nothing to prove. Assume $x, y > 0$. Set $x = r\cos\theta$, $y = r\sin\theta$. The required inequality reduces to

$$r^{11}(\cos^3\theta+\sin^3\theta)^3(\cos^2\theta-\cos\theta\sin\theta+\sin^2\theta) \geq r^{11}(\cos\theta\sin\theta)^{\frac{5}{2}}$$

or, $(\cos\theta+\sin\theta)^3(1-\cos\theta\sin\theta)^4 \geq (\cos\theta\sin\theta)^{\frac{5}{2}}$.
By the AM-GM inequality, $\cos\theta+\sin\theta \geq 2\sqrt{\cos\theta\sin\theta}$. Also, since $2\cos\theta\sin\theta = \sin 2\theta < 1, 1-\cos\theta\sin\theta \geq \cos\theta\sin\theta$. In addition,

$$1-\cos\theta\sin\theta = 1-\frac{1}{2}[1-(\cos\theta-\sin\theta)^2] = \frac{1}{2}[1+(\cos\theta-\sin\theta)^2]$$

It follows that

$$LHS \geq 8(\cos\theta\sin\theta)^{\frac{3}{2}}(\cos\theta\sin\theta) \times \frac{1}{8}[1+(\cos\theta-\sin\theta)^2]^3$$

$$= (\cos\theta\sin\theta)^{\frac{5}{2}}[1+(\cos\theta-\sin\theta)^2]^3 \geq (\cos\theta\sin\theta)^{\frac{5}{2}} = RHS.$$

□

Solution 5 (by Alexander Bogomolny).

From $\sqrt[3]{\frac{x^3+y^3}{2}} \geq \sqrt{\frac{x^2+y^2}{2}}$, we have

$$2(x^3+y^3)^2 \geq (x^2+y^2)^3.$$

Also, $x^3+y^3 \geq 2\sqrt{x^3y^3}$ and $x^2-xy+y^2 \geq xy$. Multiplying all three gives the required inequality.

□

Solution 6 (by Alexander Bogomolny).

The require inequality is equivalent to

$$(x^3 + x^3)^4 \geq (xy)^{\frac{5}{2}}(x^2 + y^2)^3(x + y).$$

Now,

$$2(x^3 + y^3)^2 \geq (x^2 + y^2)^3$$

$$x^3 + y^3 \geq 2(xy)^{\frac{3}{2}}$$

$$x^3 + y^3 \geq \frac{1}{2}(x^2 + y^2)(x + y) \geq \frac{1}{2} \cdot 2\sqrt{x^2y^2}(x + y) = xy(x + y)$$

The product of the three yields the required inequality. □

Acknowledgment (by Alexander Bogomolny)

This problem from his book "Algebraic Phenomenon" has been kindly posted at the ***CutTheKnotMath page*** by Daniel Sitaru. Solution 1 is by Kevin Soto Palacios; Solution 2 is by Seyran Ibrahimov; Solution 3 by Myagmarsuren Yadamsuren; Solution 4 by Ravi Prakash.

128. An Inequality with Just Two Variable VIII

If $a, b \in [0, 2]$ *then*

$$\frac{a^2}{b+2} + \frac{b^3}{a+2} + (2 - a)b^2 \leq 12.$$

Proposed by Daniel Sitaru - Romania

Solution 1 (by Redwane El Mellass).

Let $f(a, b) = \frac{a^2}{b+2} + \frac{b^3}{a+2} + (2 - a)b^2$. Then $f(0, 0) = 0, f(a > 0, 0) = \frac{a^2}{2} \leq 12$; $f(0, b > 0) = \frac{b^3}{2} + 2b^2 \leq 12$. Suppose $a, b > 0$:

1. If $b \leq 1$,

$$f(a, b) \leq \frac{a^2}{a} + \frac{b^2}{b} + (2 - a) = b^2 + 2 \leq 12.$$

2. If $b > 1$,

$$f(a, b) - f(0, b) = \frac{a^2}{b+2} + \frac{b^3}{a+2} + (2 - a)b^2 - \left(\frac{b^3}{2} + 2b^2\right)$$

$$= \frac{a^2}{b+2} - ab^2 - \frac{ab^3}{2(a+2)} = a\left(\frac{a}{b+2} - b^2 - \frac{b^3}{2(a+2)}\right)$$

$$\leq a\left(1 - b^2 - \frac{b^3}{2(a+2)}\right) < 0,$$

$$\text{implying that } f(a, b) < f(0, b) = \frac{b^3}{2} = 2b^2 \leq 12.$$

Finally, $f(a, b) \leq 12$, with equality if and only if $a = 0$ and $b = 2$.

□

Solution 2 (by Richdad Phuc).

We have

$$\frac{a^2}{b+2}+\frac{b^3}{a+2}+(2-a)b^2 \le 12 \Leftrightarrow$$
$$\frac{a^2}{b+2}-4+2b^2-8+\frac{a^2}{b+2}-ab^2 \le 0 \Leftrightarrow$$
$$\frac{(b-2)(b^2+2b+4)-4a}{a^2}+2(b-2)(b+2)+\frac{a^2}{b+2}-ab^2 \le 0 \Leftrightarrow$$
$$\frac{(b-2)(b^2+2b+4)}{a+2}+2(b-2)(b+2)+a\Big(\frac{a}{b+2}-b^2-\frac{4}{a+2}\Big) \le 0,$$

which is true because

$$\frac{a}{b+2}-b^2-\frac{4}{a+2} \le \frac{a}{2}-\frac{4}{2+2}-b^2 \le 0,$$

because $a,b \in [0,2]$. Equality holds if and only if $a=0$ and $b=2$. □

Solution 3 (by Amit Itagi).

$$\frac{a^2}{b+2}+\frac{b^3}{a+2}+(2-a)b^2 \le \frac{2a}{b+2}+4+2b(2-a)$$

For a fixed value of b, the right hand side is a linear function of with slope $\frac{2}{(b+2)}-2b$. The slope is non-negative when $b \in [0,\sqrt{2}-1]$ and negative when $b \in (\sqrt{2}-1,2]$.

In the regime where the slope is non-negative, RHS attains maximum value of $\frac{4}{(b+2)}+4$ at $a=2$ (the largerst value of a). This function of b, in turn, attains a maximum value of 6 at $b=0$.

In the regime where the slope is negative, RHS attains maximum value of $4+4b$ at $a=0$ (the smallest value of a). Clearly, this function takes a maximum value of 12 when $b=2$ (the largest value of b).

Thus, $LHS \le 12$. □

Acknowledgment (by Alexander Bogomolny)

This problem from his book "Algebraic Phenomenon" has been kindly posted at the ***CutTheKnotMath page*** by Dan Sitaru. Solution 1 is by Redwane El Mellass; Solution 2 is by Richdad Phuc; Solution 3 is by Amit Itagi.

129. An Inequality with Just Two Variables III

Prove that, for positive a,b,

$$\frac{a}{b\sqrt{2}}+\frac{b\sqrt{2}}{a}+2\Big(\frac{\sqrt{a^2+b^2}}{b}+\frac{b}{a^2+b^2}\Big) \ge \frac{9\sqrt{2}}{2}.$$

Proposed by Daniel Sitaru

Solution 1(by Soumava Chakraborty).

In a right triangle with sides $a,b,\sqrt{a^2+b^2}$.

Let θ be the acute angle opposite a:

$$a=\sqrt{a^2+b^2}\sin\theta; a:b=\sqrt{a^2+b^2}\cos\theta.\ \text{Then}$$

$$LHS = \frac{1}{\sqrt{2}}\tan\theta + \sqrt{2}\cot\theta + 2\cos\theta + 2\sec\theta = \overbrace{=}^{def} f(\theta)$$

Solving

$$f'(\theta) = \frac{1}{\sqrt{2}\cos^2\theta} - \sqrt{2}\frac{1}{\sin^2\theta} - 2\sin\theta + \frac{2\sin\theta}{\cos^2\theta} = 0.$$

we get successively

$$\frac{\sin^2\theta - 2\cos^2\theta}{\sqrt{2}\cos^2\theta\sin^2\theta} - 2\sin\theta\frac{\cos^2\theta - 1}{\cos\theta} = 0,$$
$$\frac{3\sin^2\theta - 2}{\sqrt{2}\cos^2\theta\sin^2\theta} + 2\frac{\sin^3\theta}{\cos\theta} = 0,$$
$$\frac{2 - 3\sin^2\theta}{\sqrt{2}\cos^2\theta\sin^2\theta} = 2\frac{\sin^3\theta}{\cos\theta},$$
$$2 - 3\sin^2\theta = 2\sqrt{2}\sin^5\theta,$$
$$4 + 9\sin^4\theta - 12\sin^2\theta = 8\sin^{10}\theta.$$

With $t = \sin^2\theta > 0$,

$$8t^2 - 9t^4 + 12t - 4 = 0,$$
$$(2t-1)(4t^2 + 2t^3 + (t-2)^2) = 0,$$
$$t = \frac{1}{2}, \sin\theta = \frac{\sqrt{2}}{2}, \theta = \frac{\pi}{4}.$$

Now,

$$f''(\theta) = \sqrt{2}\sec^2\theta\tan\theta + 2\sqrt{2}\frac{\cot\theta}{\sin^2\theta} - 2\cos\theta + 2\sec^3\theta + 2\tan^2\theta\sec\theta.$$

$f''\left(\frac{\pi}{4}\right) = 11\sqrt{2} > 0$, implying that $\theta = \frac{\pi}{4}$ is a minimum ant that f never attends a maximum on $\left(0, \frac{\pi}{2}\right)$. Hence,

$$f(\theta) \geq f\left(\frac{\pi}{2}\right) = 4\sqrt{2} + \frac{1}{\sqrt{2}} = \frac{9\sqrt{2}}{2}.$$

□

Solution 2 (by Seyran Ibrahimov).

Note that $a\sin x + b\cos x \leq \sqrt{a^2+b^2}$ so that

$$\frac{a\sin x}{\sqrt{a^2+b^2}} + \frac{b\cos x}{\sqrt{a^2+b^2}} \leq 1,$$

with equality only when $a = \sqrt{a^2+b^2}\sin x, b = \sqrt{a^2+b^2}\cos x, x \in \left(0, \frac{\pi}{2}\right)$.
The given inequality is equivalent to

$$f(x) = \frac{1}{\sqrt{2}}\tan x + \sqrt{2}\cot x + \frac{2}{\cos x} + 2\cos x \geq \frac{9\sqrt{2}}{2}.$$

where $f(x) = 0$

$$f'(x) = \frac{1}{\sqrt{2}\cos^2 x} + \frac{\sqrt{2}}{\sin^2 x} - \frac{2\sin x}{\cos^2 x} - 2\sin x = 0,$$

or, $(2\sqrt{2}\sin^3 x - 1)(\cos^2 x + 1)(2\sqrt{2}\sin^3 x - 1) = 0$ implies

$$x = \frac{\pi}{4} \cdot f\left(\frac{\pi}{4}\right) = \frac{9\sqrt{2}}{2}.$$

□

Solution 3 (by Su Tanaya).

Using the ***AM-GM inequality***,

$$3\cdot\frac{\frac{a}{b\sqrt{2}}+\frac{b}{\sqrt{a^2+b^2}}+\frac{b}{\sqrt{a^2+b^2}}}{3}+3\cdot\frac{\frac{b\sqrt{2}}{a}+\frac{\sqrt{a^2+b^2}}{b}+\frac{\sqrt{a^2+b^2}}{b}}{3}$$

$$=3\Big(\frac{ab}{\sqrt{2}(a^2+b^2)}\Big)^{\frac{1}{3}}+6\Big(\frac{\sqrt{2}(a^2+b^2)}{8ab}\Big)^{\frac{1}{3}}$$

$$=3\cdot 3\cdot\frac{\Big(\frac{ab}{\sqrt{2}(a^2+b^2)}\Big)^{\frac{1}{3}}+\Big(\frac{\sqrt{2}(a^2+b^2)}{8ab}\Big)^{\frac{1}{3}}+\Big(\frac{\sqrt{2}(a^2+b^2)}{8ab}\Big)^{\frac{1}{3}}}{3}$$

$$\geq 9\cdot\left[\Big(\frac{ab}{\sqrt{2}(a^2+b^2)}\Big)^{\frac{1}{3}}\Big(\frac{\sqrt{2}(a^2+b^2)}{8ab}\Big)^{\frac{2}{3}}\right]^{\frac{1}{3}}=9\cdot\left[\Big(\frac{a^2+b^2}{ab}\Big)^{\frac{1}{3}}\cdot(\sqrt{2})^{\frac{2}{3}-\frac{1}{3}-4}\right]^{\frac{1}{3}}$$

$$\geq 9\cdot\left[2^{\frac{1}{3}}\cdot(\sqrt{2})^{\frac{1}{3}-4}\right]^{\frac{1}{3}}\ \text{(because } \frac{a^2+b^2}{2}\geq ab)=9\cdot\left[(\sqrt{2})^{\frac{2}{3}+\frac{1}{3}-4}\right]^{\frac{1}{3}}=9\cdot(\sqrt{2})^{-1}$$

$$=\frac{9\sqrt{2}}{2}.$$

□

Acknowledgment (by Alexander Bogomolny)

The problem above has been kindly posted to the ***CutTheKnotMath page*** by Dan Sitaru, along with several solutions. Solutions 1 is by Soumava Chakraborty; Solution 2 by Seyran Ibrahimov; Solution 3 is by Su Tanaya.

130. An Inequality with Just Two Variables V

Prove that, for $a,b\in\Big(\frac{1}{e},1\Big)$,

$$\left(\ln\Big(\frac{1}{2a}+\frac{1}{2b}\Big)\right)^{a+b}\geq\Big(\ln\frac{1}{a}\Big)^{b}\Big(\ln\frac{1}{b}\Big)^{a}.$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

Define function $f:\big(\frac{1}{e},1\big)\to\mathbb{R}$ by $\ln\big(\ln\frac{1}{x}\big)$. Then

$$f'(x)=\frac{1}{x\ln x}$$

$$f''(x)=-\frac{\ln x+1}{(x\ln 2)^2}<0,\forall x\in\Big(\frac{1}{e},1\Big).$$

Hence, f is concave on $\big(\frac{1}{e},1\big)$. By ***Jensen's inequality***,

$$f\Big(\frac{b}{a+b}\cdot a+\frac{a}{a+b}\cdot b\Big)\geq\frac{b}{a+b}f(a)+\frac{a}{a+b}f(a),$$

i.e., $f\big(\frac{2ab}{a+b}\big)\geq\frac{bf(a)}{a+b}+\frac{af(b)}{a+b}$. To continue,

$$\ln\Big(\ln\frac{a+b}{2ab}\Big)\geq\frac{b}{a+b}\ln\Big(\ln\frac{1}{a}\Big)+\frac{a}{a+b}\ln\Big(\ln\frac{1}{b}\Big),$$

$$\ln\left(\ln\left(\frac{1}{2a}+\frac{1}{2b}\right)\right)\geq \ln\left(\left(\ln\frac{1}{a}\right)^{\frac{b}{a+b}}\cdot\left(\ln\frac{1}{b}\right)^{\frac{a}{a+b}}\right)$$

which is

$$\ln\left(\frac{1}{2a}+\frac{1}{2b}\right)\geq\left(\left(\ln\frac{1}{a}\right)\cdot\left(\ln\frac{1}{b}\right)\right)^{\frac{1}{a+b}}$$

and, finally,

$$\left(\ln\left(\frac{1}{2a}+\frac{1}{2b}\right)\right)^{a+b}\geq\left(\ln\frac{1}{a}\right)^{b}\cdot\left(\ln\frac{1}{b}\right)^{a}$$

□

Solution 2 (by Nassim Nicholas Taleb).

Let $x=\frac{1}{a}$ and $y=\frac{1}{b}, x,y\in(1,e)$. We need to show that

$$\ln\frac{x+y}{2}\geq \ln(x)^{\frac{x}{x+y}}\ln(y)^{\frac{y}{x+y}}$$

Let $y=x+\epsilon$, where $0<\epsilon<e-x$. We need to show that

$$\ln\frac{2x+\epsilon}{2}\geq \ln(x)^{\frac{x}{2x+\epsilon}}\ln(x+\epsilon)^{\frac{x+\epsilon}{2x+\epsilon}}.$$

Expanding up to orders of $\epsilon^2, LHS=\ln+\frac{\epsilon}{2x}-\frac{\epsilon^2}{8x^2}; RHS=\ln x+\frac{\epsilon}{2x}-\frac{\epsilon^2}{8x^2\ln x}$.
So, since $\ln a<1$, we have $LHS\geq RHS$.
We can refine further with the higher orders of $O(\epsilon^4)$, by taking even orders.
This is a more engineering oriented but functional approach. □

Acknowledgment (by Alexander Bogomolny)

The problem above has been kindly posted to the ***CutTheKnotMath page*** and several other forums (in particular at the ***Romanian Mathematical Magazine***) by Daniel Sitaru. After a length of time that it had not gathered any solution, Daniel has communicated his solution by private mail. Solution 2 is by N.N. Taleb.

131. An Inequality with One Tangent and Six Sines

Given an acute ΔABC. Prove that

$$\frac{\tan A}{\sin B+5\sin C}+\frac{\tan B}{\sin C+5\sin A}+\frac{\tan C}{\sin A+5\sin B}>\frac{1}{2}$$

Proposed by Daniel Sitaru

Solution (by Soumava Chakraborty).

Note, that, for $\alpha\in\left(0,\frac{\pi}{2}\right), \tan\alpha>\alpha$ and $\sin\alpha<\alpha$. Thus, suffice it to prove that

$$\frac{A}{B+5C}+\frac{B}{C+5A}+\frac{C}{A+5B}\geq\frac{1}{2}$$

As in proof of Nesbitt's inequality, the above is equivalent to

$$\frac{A^2}{A(B+5C)}+\frac{B^2}{B(C+5A)}+\frac{C^2}{C(A+5B)}\geq\frac{1}{2}$$

By ***Bergstrom's inequality***,

$$\sum_{cycl} \frac{A^2}{A(B+5C)} \geq \frac{(\sum_{cycl} A)^2}{6\sum_{cycl} AB} = \frac{\sum_{cycl} A^2 + 2\sum_{cycl} AB}{6\sum_{cycl} AB} \geq \frac{3\sum_{cycl} AB}{6\sum_{cycl} AB} \geq \frac{1}{2}$$

□

Refinement (by Marian Dincă)

Given an acute ΔABC. *Prove that*

$$\frac{\tan A}{\sin B + 5\sin C} + \frac{\tan B}{\sin C + 5\sin A} + \frac{\tan C}{\sin A + 5\sin B} \geq 1$$

Indeed, using the AM-GM inequality, we obtain

$$\sum_{cycl} \frac{\tan A}{\sin B + 5\sin C} \geq 3\sqrt[3]{\prod_{cycl} \frac{\tan A}{\sin B + 5\sin C}} = 3\frac{\sqrt[3]{\tan A \tan B \tan C}}{\sqrt[3]{\prod_{cycl}(\sin B + 5\sin C)}}$$

However, $\prod_{cycl} \tan A = \sum_{cycl} \tan A \geq 3\sqrt[3]{\prod_{cycl} \tan A}$, implying $\prod_{cycl} \tan A \geq 3\sqrt{3}$. To continue,

$$\sqrt[3]{\prod_{cycl}(\sin B + 5\sin C)} \leq \frac{\sum_{cycl}(\sin B + 5\sin C)}{3} = 2\sum_{cycl} \sin A \leq 3\sqrt{3}.$$

Combining the latest results,

$$\sum_{cycl} \frac{\tan A}{\sin B + 5\sin C} \geq 3\frac{\sqrt{3}}{3\sqrt{3}} = 1.$$

Equality is attained for $A = B = C = \frac{\pi}{3}$.

Acknowledgment (by Alexander Bogomolny)

The problem (from the ***Romanian Mathematical Magazine***) has been posted by Dan Sitaru at the ***CutTheKnotMath page***, Dan later communicated a solution by Soumava Chakraborty. Marian Dinca came up with the refinement of the original inequality.

132. An Inequality with Sides, Altitudes, Angle Bisectors and Medians

Given ΔABC, *with* $c \leq b \leq a$. *Prove that*

$$\left(\frac{h_b}{m_a} + \frac{h_c}{m_b} + \frac{h_a}{m_c}\right)\left(\frac{h_b}{l_a} + \frac{h_c}{l_b} + \frac{h_a}{l_c}\right) \geq \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)^2$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

Note that the condition $c \leq b \leq a$ implies $(a-c)(b-c)(a-b) \geq 0$. Using that we prove that

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

The two are equivalent as follows from the sequence below:

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \Leftrightarrow$$

$$b^2c + c^2a + a^2b \geq a^2c + b^2a + c^2b \Leftrightarrow$$
$$b^2(c-a) + ca(c-a) - b(c^2 - a^2) \geq 0 \Leftrightarrow$$
$$(c-a)(b^2 + ca - bc - ba) \geq 0 \Leftrightarrow$$
$$(c-a)\Big(b(b-c) + a(c-b)\Big) \geq 0 \Leftrightarrow (c-a)(b-c)(a-b) \geq 0.$$

To continue, since, say $h_a \leq l_a \leq m_a$,

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{h_b}{h_a} + \frac{h_c}{h_b} + \frac{h_a}{h_c}$$
$$\geq \frac{h_b}{l_a} + \frac{h_c}{l_b} + \frac{h_a}{l_c} \geq \frac{h_b}{m_a} + \frac{h_c}{m_b} + \frac{h_a}{m_c}$$

Thus we have

$$\frac{h_b}{m_a} + \frac{h_c}{m_b} + \frac{h_a}{m_c} \leq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

and also

$$\frac{h_b}{l_a} + \frac{h_c}{l_b} + \frac{h_a}{l_c} \leq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

Taking the product of the two gives

$$\Big(\frac{h_b}{m_a} + \frac{h_c}{m_b} + \frac{h_a}{m_c}\Big)\Big(\frac{h_b}{l_a} + \frac{h_c}{l_b} + \frac{h_a}{l_c}\Big) \leq \Big(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\Big)^2$$

as required. □

Aknowledgment (by Alexander Bogomolny)

The problem (from the ***Romanian Mathematical Magazine***) has been kindly posted by Dan Sitaru at the ***CutTheKnotMath facebook page***, Dan later emailed me his solution in a LatTex file.

133. An Inequality with Sin, Cos, Tan, Cot, and Some

Prove that, in acute ΔABC, the following inequality holds:

$$2S^2 \sum_{cycl} (\sin A + \cos A + \tan A + \cot A) > 81\pi R^4 \prod_{cycl} \cos A,$$

where $S = [\Delta ABC]$, the area and R the circumradius of ΔABC.

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

First for $x \in \Big(0, \frac{\pi}{2}\Big)$ we prove that:

$$\sin x + \tan x > 2x$$

Let be $f : \Big(0, \frac{\pi}{2}\Big) \to \mathbb{R}, f(x) = \sin x + \tan x - 2x$. We have $f'(x) = \cos x + \tan^2 x - 1$ and $f''(c) = \frac{\sin x}{\cos^3 x}(2 - \cos^2 x) > 0$. It follows that

$$f'(x) > \lim_{x \to 0^+} f'(x) = 0$$

such that $f'(x) > 0, x \in \Big(0, \frac{\pi}{2}\Big)$, implying

$$f(x) > \lim_{x \to 0^+} f(x) = 0,$$

so that $f(x) > 0, x \in \left(0, \frac{\pi}{2}\right)$. Hence,

$$\sin x + \tan x > 2x. \tag{1}$$

Replace in (1) x with $\frac{\pi}{2} - x$:

$$\cos x + \cot x > 2\left(\frac{\pi}{2} - x\right) \tag{2}$$

By adding (1) and (2):

$$\sin x + \tan x + \cos x + \cot x > \pi. \tag{3}$$

For $x = A; x = B; c = C$ in (3) and adding up:

$$\sum_{cycl}(\sin A + \cos A + \tan A + \cot A) > 3\pi. \tag{4}$$

By the ***AM-GM inequality***, $a + b + c \geq \sqrt[3]{abc}$ such that $8\left(\frac{a+b+c}{2}\right)^3 \geq 27abc$, i.e., $8s^3 \geq 28abc$. ***Further*** $8s^3 \geq 27 \cdot 4RS = 27 \cdot 4Rrs$, or, $27 \cdot Rr \leq 2s^2$ and, finally

$$9Rr \leq \frac{2s^2}{3}. \tag{5}$$

Now for a few facts, concerning the orthic triangle of ΔABC: the side length of the orthic triangle are:

$$a' = a\cos A, b' = b\cos B, c' = c\cos c$$

The inradius: $r' = 2R\cos A\cos B\cos C$,
The circumradius: $R' = \frac{R}{2}$,
The semiperimeter: $s' = \frac{s}{R}$.
Now, we apply (1) the orthic triangle:

$$9R'r' \leq \frac{2s^2}{3},$$

$$9 \cdot \frac{R}{2} \cdot 2R\cos A\cos B\cos C \leq \frac{2}{3} \cdot \frac{S^2}{R^2},$$

$$\cos A\cos B\cos C \leq \frac{2S^2}{27R^4}, \frac{1}{\prod_{cycl}\cos A} \geq \frac{27R^4}{2S^2}, \frac{2S^2}{\prod_{cycl}\cos A} \geq 27R^4,$$

implying $2S^2 \geq 27R^4 \prod_{cycl}\cos A$. Multiplying (4), (5):

$$2S^2\sum_{cycl}(\sin A + \cos A + \tan A + \cot A) > 81\pi R^4\prod_{cycl}\cos A.$$

□

Acknowledgment (by Alexander Bogomolny)
I am grateful do Dan Sitaru for posting this problem from his book "Math Accent" at the ***CutTheKnotMath page*** and later supplying a LaTeX file with its solution.

134. An Inequality with Tangents and Sides

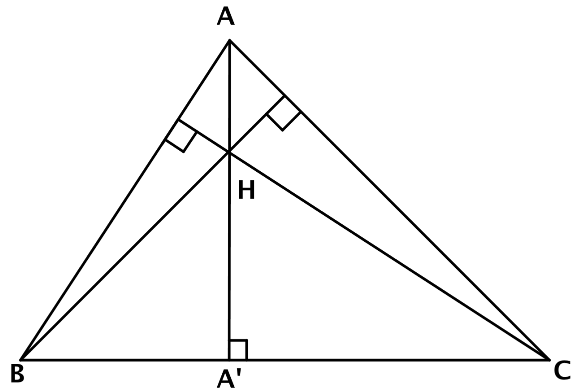
Prove that in any acute ΔABC the following relationship holds:

$$\frac{a^2}{\tan B + \tan C} + \frac{b^2}{\tan C + \tan A} + \frac{c^2}{\tan A + \tan B} \leq sR.$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

$$\tan C = \frac{BA'}{HA'}, \tan B = \frac{A'C}{HA'}.$$



$$\tan B + \tan C = \frac{BA' + A'C}{HA'} = \frac{a}{HA'} \Rightarrow HA' = \frac{a}{\tan B + \tan C}$$

$$S[BHC] = \frac{HA' \cdot BC}{2} = \frac{1}{2} \cdot \frac{a^2}{\tan B + \tan C}$$

$$S[AHC] = \frac{b^2}{2(\tan A + \tan C)}$$

$$S[AHB] = \frac{c^2}{2(\tan A + \tan B)}$$

$$S[ABC] = S[AHB] + S[AHC] + S[AHB]$$

$$S = \frac{1}{2} \sum_{cycl} \frac{a^2}{\tan B + \tan C}$$

$$\sum_{cycl} \frac{a^2}{\tan B + \tan C} = 2S = 2rs \overbrace{\leq}^{Euler} sR.$$

□

Acknowledgment (by Alexander Bogomolny)
The problem (from the ***Romanian Mathematical Magazine***) has been posted by Dan Sitaru at the ***CutTheKnotMath***, Dan later has kindly communicated his solution in a LaTex file.

135. Another Integral Inequality from the RMM

Set

$$\Omega(a) = \int_{-\frac{1}{a}}^{\frac{1}{a}} (2x^6 + 2x^4 + 3) \arccos(ax) dx, a \geq 1.$$

Then

$$\Omega(a) \leq \frac{129\pi}{35a}$$

Proposed by Daniel Sitaru

Proof (by Ravi Prakash).

For $a \geq 1$,

$$\Omega(a) = \int_{-\frac{1}{a}}^{\frac{1}{a}} (2x^6 + 2x^4 + 3)\arccos(ax)dx$$

$$= \int_{-\frac{1}{a}}^{\frac{1}{a}} (2x^6 + 2x^4 + 3)\Big(\frac{\pi}{2} - \arcsin(ax)\Big)dx = \frac{\pi}{2}I_1 - I_2,$$

where

$$I_1 = \int_{-\frac{1}{a}}^{\frac{1}{a}} (2x^6 + 2x^4 + 3)dx = 2\Big[\frac{2}{7}x^7 + \frac{2}{5}x^5 + 3x\Big]_0^{\frac{1}{a}}$$

$$= 2\Big[\frac{2}{7a^7} + \frac{2}{5a^5} + \frac{3}{a}\Big] \leq \frac{2}{35}[10 + 14 + 105]\frac{1}{a} = \frac{2 \cdot 129}{35a},$$

because $a^7 \geq a^5 \geq a$. On the other hand, $I_2 = 0$, being an integral of an odd function. So, finally,

$$\Omega(a) = \frac{\pi}{2}I_1 - I_2 \leq \frac{\pi}{2} \cdot \frac{2 \cdot 129}{35a} = \frac{129\pi}{35a}.$$

□

Acknowledgment (by Alexander Bogomolny)

The problem form the ***Romanian Mathematical Magazine*** has been kindly posted at CutTheKnotMath page by Dan Sitaru, along with the solution by Ravi Prakash (India).

136. Another Problem from the 2016 Danubius Contest

Let $a, b, c > 0$ satisfy $ab + bc + ca = 3$. Prove that

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$

Proposed by Leonard Giugiuc , Daniel Sitaru

Solution 1 (by proposers).

The inequality is equivalent to

$$12 + 4\sum_{cycl} a^2 + \sum_{cycl} a^2b^2 \leq 8 + 4\sum_{cycl} a^2 + 2\sum_{cycl} a^2b^2 + a^2b^2c^2,$$

or,

$$4 \leq a^2b^2 + b^2c^2 + c^2a^2 + a^2b^2c^2$$

let's denote $bc = x, ca = y$, and $ab = z$. Then $x, y, z > 0, x + y + z = 3$ and $xyz = a^2b^2c^2$. We have to show that

$$x^2 + y^2 + z^2 + xyz \geq 4.$$

We can homogenize the inequality:

$$4\Big(\frac{x+y+z}{3}\Big)^3 \leq (x^2 + y^2 + z^2)\Big(\frac{x+y+z}{3}\Big) + xyz,$$

reducing it to $5S_3 - 3s + 3xyz$, where $s_3 = x^3 + y^3 + z^3$ and

$$s = \sum_{cycl} xy(x + y).$$

From ***Schur's inequality***, $S_3 - s + 3xyz \geq 0$ and, from the well – known inequality $u^3 + v^3 \geq uv(u + v), 4S_3 - 2s \geq 0$. Adding the two up gives the required inequality.

□

Solution 2 (by Marian Daniel Vasile).

$\sum_{cycl} \frac{1}{a^2+2} \le 1$ is equivalent $\sum_{cycl} \frac{2}{a^2+2} \le 2$ and, further, to

$\sum_{cycl}\left(1-\frac{a^2}{a^2+2}\right) \le 2$ which is $\sum_{cycl} \frac{a^2}{a^2+2} \ge 1$. To prove this we can use

Bergstrom's inequality:

$$\sum_{cycl} \frac{a^2}{a^2+2} \ge \frac{(a+b+c)^2}{a^2+b^2+c^2+6}$$

$$= \frac{(a^2+b^2+c^2)+2(ab+bc+ca)}{a^2+b^2+c^2+6} = \frac{(a^2+b^2+c^2)+2\cdot 3}{a^2+b^2+c^2+6} = 1.$$

□

Solution 3 (by Vasile Cîrtoaje).

Let's denote $bc = x, ca = y$, and $ab = z$. Then $x, y, z > 0, x + y + z = 3$. We have to show that

$$x^2 + y^2 + z^2 + xyz \ge 4$$

Assuming $x = \min\{x, y, z\}, x \le 1$, and we have

$$x^2 + y^2 + z^2 + xyz - 4 = x^2 + (y+z)^2 + yz(x-2) - 4$$

$$\ge x^2 + (y+z)^2 + \frac{1}{4}(y+z)^2(x-2) - 4$$

$$\ge x^2 + \frac{x+2}{4}(y+z)^2 - 4 = x^2 + x + 24(3-x)^2 - 4$$

$$= \frac{1}{4}(x-1)^2(x+2) \ge 0.$$

Equality occurs for $a = b = c = 1$.

□

Solution 4 (by Srinivas Vemuri).

$$c = \frac{3-ab}{a+b}$$

$$RHS - LHS = \frac{c^2+1}{c^2+2} - \frac{a^2+b^2+4}{(a^2+2)(b^2+2)}$$

$$= \frac{(3-ab)^2+(a+b)^2}{(3-ab)^2+2(a+b)^2} - \frac{a^2+b^2+4}{(a^2+2)(b^2+2)}$$

Denominators being positive, we'll focus on the numerators:

$$(2a^2+2b^2+4)\Big((3-ab)^2+(a+b)^2\Big) - (a^2+b^2+4)\Big((3-ab)^2+2(a+b)^2\Big) +$$

$$+a^2b^2\Big((3-ab)^2+(a+b)^2\Big)$$

$$= (a^2+b^2)(3-ab)^2 - 4(a^2+b^2) - 8ab + a^2b^2(a^2+b^2-4ab+9+a^2b^2)$$

$$= (a^2+b^2)(a^2b^2-6ab+9-4+a^2b^2) - ab(a^3b^3-4a^2b^2+9ab-8)$$

$$= (a^2+b^2)(2a^2b^2-6ab+4) + ab(a^3b^3-4a^2b^2+9ab-6) + (a-b)^2$$

$$= (a^2+b^2)(ab-1)(ab-2) + ab(ab-1)(a^2b^2-3ab+6) + (a-b)^2$$

$$\ge 2\cdot 2ab(ab-1) + ab(ab-1)(a^2b^2-3ab+6) + (a-b)^2$$

$$= ab(ab-1)(a^2b^2+ab-2)+(a-b)^2$$

$$= ab(ab-1)(ab-1)(ab+2)+(a-b)^2$$

$$= ab(ab-1)^2(ab+2)+(a-b)^2 \geq 0.$$

$\square$

Solution 5 (by Amit Itagi).

We make the same substitution as in Solution 1 and 3 to obtain the problem of proving

$$xyz + x^2 + y^2 + z^2 - 4 \geq 0,$$

provided $x + y + z = 3$ and $x, y, z > 0$. Let $x = k + m, y = k - m$, $z = 3 - x - y = 3 - 2k$. The positivity of x, y, and z implies

$$\frac{3}{2} > k > 0 \text{ and } k > m > -k.$$

We have

$$f(k,m) := (k+m)(k-m)(3-2k) + (k+m)^2 + (k-m)^2 + (3-2k)^2 - 4$$

$$= [(5-2k)(k-1)^2] + [(2k-1)m^2].$$

Note: The first term (the first square braket) is non − negative over the allowed range of k, becoming 0 at $k = 1$. The sign of the second term depends on k.

Let us consider two cases:

Case 1: $\frac{1}{2} > k > 0$. In this case, the second term is negative an for a fixed value of k is monotonically decreasing function of m^2. Moreover, $m^2 < k^2$ implies

$$f > (5-2k)(k-1)^2 + (2k-1)k^2 = \frac{(4k-3)^2}{2} + \frac{1}{2} > \frac{1}{2}$$

Case 2: $\frac{3}{2} > k \geq \frac{1}{2}$. In this case, the second term is non − negative and takes value 0 when $k = \frac{1}{2}$ or $m = 0$. Thus, f takes minimum value of 0 when $k = 1$ and $m = 0$. $\square$

Solution 6 (by Andrea Aquaviva).

From $ab+bc+ca = 3, a^2+b^2+c^2 \geq 3$, by the AM-GM inequality. If we use 3D polar coordinates, then the problem reduces to finding the maximum of

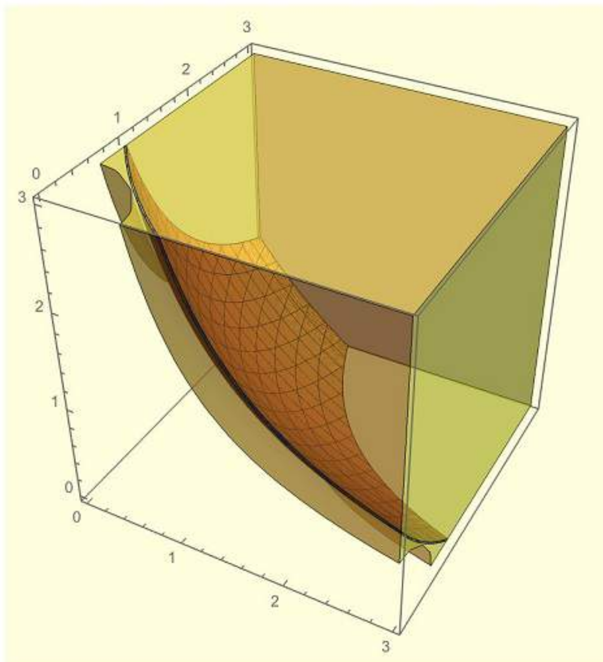
$$\frac{1}{\rho\sin^2\theta\cos^2\varphi} + \frac{1}{\rho^2\sin^2\theta\sin^2\varphi} + \frac{1}{\rho^2\cos^2\theta},$$

where, as we just observed $\rho^2 \geq 3$. The above function decreases as ρ grows o that it achieves its maximum on the sphere $\rho^2 = 3$ - for the minimal value of ρ. Further, from $ab + bc + ca = 3$ and $a^2 + b^2 + c^2 = 3$ it follows that $(a+b+c)^2 = 9$, i.e. $a+b+c = 3$, the tangent plane to the sphere $a^2+b^2+c^2$ at the point $(1,1,1)$. $\square$

Illustration (by Nassim Nicholas Taleb)

Let $a, b, c > 0$ *satisfy* $ab + bc + ca = 3$. *Prove that*

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$



Showing the boundary in darker color where the constraint
$a\,b + b\,c + c\,a = 3$
holds (for values of a, b, and c) and in
lighter ones the area where the inequality holds.

Acknowledgment (by Alexander Bogomolny)

Leo Giugiuc has kindly posted at the ***CutTheKnotMath page*** the above problem which he coauthored with Daniel Sitaru. The problem has been included at 2016 Danubius contest.

Solution 1 is by authors; Solution 2 is by Marian Daniel Vasile; Solution 3 is by Vasile Cîrtoaje; Solution 4 is by Srinivas Vemuri; Solution 5 is by Amit Itagi; Solution 6 is by Andrea Acquaviva. The illustration is by N.N. Taleb.

137. Area Inequality in Three Triangles

Prove that in three acute triangles $A_1B_1C_1, A_2B_2C_2, A_3B_3C_3$,

$$2(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2 < \sum_{cycl} a_1^2 + \sum_{cycl} a_2^2 + \sum_{cycl} a_3^2$$

Proposed by Daniel Sitaru

Solution 1 (by Kevin Soto Palacios).

First, recollect that

$$\sum_{cycl} a^2 \geq 4\sqrt{3}S.$$

By the ***Cauchy – Schwarz inequality***,

$$(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_2})^2 \leq (S_1 + S_2 + S_3)(1 + 1 + 1).$$

From here,

$$\sum_{cycl} a_1^2 + \sum_{cycl} a_2^2 + \sum_{cycl} a_3^2 \geq 4\sqrt{3}(S_1 + S_2 + S_3)$$

$$\geq 4\sqrt{3}\frac{1}{3}(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2 > 2(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2$$

□

Solution 2 (by Soumitra Mandal).

By the ***Cauchy – Schwarz inequality***,

$$\left(\sum_{k=1}^{3} \sqrt{S_k}\right)^2 \leq 3\sum_{k=1}^{3} S_k.$$

Also, $abc \geq 8(s-a)(s-b)(s-c)$, where $2s = a + b + c$. It follows that $\sqrt{abc(a+b+c)} \geq 4S$. By the AM-GM inequality, $xy + yx + yz \geq \sqrt{3xyz(x+y+z)}$. Combining everything and using the ***Rearrangement inequality***,

$$\sum_{k=1}^{3} \leq \frac{1}{4\sqrt{3}}\left(\sum_{k=1}^{3}(a_k b_k + b_k c_k + c_k a_k)\right) \leq \frac{1}{4\sqrt{3}}\left(\sum_{k=1}^{3}(a_k^2 + b_k^2 + c_k^2)\right),$$

implying

$$2(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2 \geq 6\left(\sum_{k=1}^{3} S_k\right)$$

$$\leq \frac{\sqrt{3}}{2}\left(\sum_{k=1}^{3}(a_k^2 + b_k^2 + c_k^2)\right) < \sum_{cycl} a_1^2 + \sum_{cycl} a_2^2 + \sum_{cycl} a_3^2.$$

□

Solution 3 (by Soumava Chakraborty).

By the Cacuchy – Schwarz inequality, $2(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2 \leq 6(S_1 + S_2 + S_3)$. We shall prove that in any ΔABC,

$$\sum_{cycl} ab \geq 4\sqrt{3}S.$$

The latter is equivalent to $s^2 + r(4R + r) \geq 4\sqrt{3}S$. But

$$s^2 + r(4R + r) \geq s(3\sqrt{3}r) + r(s\sqrt{3}),$$

because $s \geq 3\sqrt{3}r$ and $4R + r \geq s\sqrt{3}$. Thus, $s^2 + r(4R + r) \geq 4\sqrt{3}rs = 4\sqrt{3}S$, with a conclusion that

$$6(S_1 + S_2 + S_3) \leq 4\sqrt{3}(S_1 + S_2 + S_3)$$

$$\leq \sum_{cycl}(a_1 b_1 + a_2 b_2 + a_3 b_3) \leq \sum_{cycl}(a_1^2 + a_2^2 + a_3^2)$$

Please note that we proved a stronger inequality than required:

$$(2\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3})^2<\sum_{cycl}(a_1b_1+a_2b_2+a_3b_3).$$

□

Solution 4 (by Myagmarsuren Yadamsuren).

$S=sr$ and $\left(\frac{s}{3\sqrt{3}}\right)\geq r$, i.e., $S\leq\frac{s^2}{3\sqrt{3}}$. With the ***AM-QM inequality***,

$$S\leq\frac{1}{4\sqrt{3}}\sum_{cycl}a^2.$$

$$2\left(\sum_{k=1}^{3}\sqrt{S_k}\right)^2\leq\frac{2}{4\sqrt{3}}\left[\sqrt{\sum_{cycl}a_1^2}+\sqrt{\sum_{cycl}a_2^2}+\sqrt{3}\sum_{cycl}a_3^2\right]$$

$$\leq\frac{1}{2\sqrt{3}}\left(3\sum_{cycl}a_1^2+3\sum_{cycl}a_2^2+3\sum_{cycl}a_3^2\right)$$

$$=\frac{\sqrt{3}}{2}\sum_{cycl}(a_1^2+a_2^2+a_3^2)<\sum_{cycl}(a_1^2+a_2^2+a_3^2).$$

□

Remark (by Alexander Bogomolny)

The required inequality is rather weak. Each of the available proofs established a stronger inequality:

Prove that in three acute triangles $A_1B_1C_1, A_2B_2C_2, A_3B_3C_3$,

$$3(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3})^2\leq\sqrt{3}\left(\sum_{cycl}a_1^2+\sum_{cycl}a_2^2+\sum_{cycl}a_3^2\right).$$

Solution 3 provided a further refinement:

Prove that in three acute triangles $A_1B_1C_1, A_2B_2C_2, A_3B_3C_3$,

$$4(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3})^2\leq\sqrt{3}\left(\sum_{cycl}a_1b_1+\sum_{cycl}a_2b_2+\sum_{cycl}a_3b_3\right).$$

Equality is achieved for three equal equilateral triangles.

Acknowledgment (by Alexander Bogomolny)

The inequality ***(Romanian Mathematical Magazine)*** has been kindly posted at the ***CutTheKnotMath page*** by Daniel Sitaru. Solution 1 is by Kevin Soto Palacios (Peru); Solution 2 is by Soumitra Mandal (India); Solution 3 is by Soumava Chakraborty (India); Solution 4 is by Myagmarsuren Yadamsuren (Mongolia).

138. Cyclic Inequality with Square Roots

Prove that, for $x,y,z\geq0$,

$$2\sqrt{2}\sum_{cycl}xy\geq\sqrt{2xyz}\sum_{cycl}\sqrt{x}+\sum_{cycl}\sqrt{x^2z^2}+y^2z^2$$

Proposed by Daniel Sitaru

Solution 1 (by Kevin Soto Palacios).

Set $x = a^2, y = b^2, z = c^2$. We choose $a, b, c \geq 0$. The given inequality is equivalent to

$$2\sqrt{2}(a^2b^2 + b^2c^2 + c^2a^2) \geq \sqrt{2a^2b^2}z(a + b + c) + \sum_{cycl} a^2\sqrt{c^4 + b^4}$$

Let's prove $c^4 + b^4 \leq 2(b^2 - bc + c^2)^2$. This is equivalent to

$$c^4 + b^4 \leq 2(b^2 + c^2)^2 + 2b^2c^2 - 4bc(b^2 + c^2)$$

and, it turn, to

$$(b^4 + c^4 + 2b^2c^2) + 4b^2c^2 - 4bc(b^2 + c^2)$$
$$= (b^2 + c^2)^2 - 4bc(b^2 + c^2) + 4b^2c^2 = (b - c)^4 \geq 0.$$

So, we have

$$\sum_{cycl} a^2\sqrt{c^4 + b^4} \leq \sum_{cycl} \sqrt{2}a^2(b^2 - bc + c^2)$$
$$= 2\sqrt{2}(a^2b^2 + b^2c^2 + c^2a^2) - \sqrt{2}abc(a + b + c) \Leftrightarrow$$
$$2\sqrt{2}(a^2b^2 + b^2c^2 + c^2a^2) \geq \sqrt{2}abc(a + b + c) + \sum_{cycl} a^2\sqrt{c^4 + b^4}$$

□

Solution 2 (by Myagmarsuren Yadamsuren).

$$2\sqrt{2}\sum_{cycl} xy = \sqrt{2}\sum_{cycl}(xy + zx) = \sqrt{2}\sum_{cycl} x(y + z) = \sum_{cycl} x\sqrt{2(y + z)^2}$$
$$= \sum_{cycl} x\sqrt{2(x^2 + 2xy + y^2)}$$
$$= \sum_{cycl} x\sqrt{(1^2 + 1^2)\left[\left(\sqrt{y^2 + z^2}\right)^2 + \left(\sqrt{2yz}\right)^2\right]}$$
$$\geq \sum_{cycl} x(\sqrt{y^2 + z^2} + \sqrt{2yz}), \text{ by the Cauchy - Schwarz inequality}$$
$$= \sum_{cycl} \sqrt{x^2y^2 + z^2x^2} + \sum_{cycl} \sqrt{2xyz}\sqrt{x}$$
$$= \sqrt{2xyz}(\sqrt{x} + \sqrt{y} + \sqrt{z}) + \sum_{cycl} \sqrt{x^2y^2 + y^2z^2}$$

□

Acknowledgement (by Alexander Bogomolny)

Dan Sitaru has kindly posted this problem form the ***Romanian Mathematical Magazine***, with two solutions, at the ***CutTheKnotMath page***. Solution 1 is by Kevin Soto Palacios; Solution 2 is by Myamagsuren Yadmasuren.

139. Dan Sitaru's Cyclic Inequality in Three Variables with Constraints

Prove that, for $a, b, c > 0$, subject to $a^2 + b^2 + c^2 = 26(a + b + c)$,

$$\frac{1}{\sqrt{a + b^2}} + \frac{1}{\sqrt{b + c^2}} + \frac{1}{c + a^2} \geq \frac{1}{\sqrt{a + b + c}}$$

Proposed by Daniel Sitaru

Solution 1 (by Marjan Milanovic).

Since $y = \frac{1}{\sqrt{x}}$ *is a convex function,*

$$\sum_{cycl} \frac{1}{\sqrt{a+b^2}} \geq 3\left(\frac{\sum_{cycl} a + \sum_{cycl} a^2}{3}\right)^{-\frac{1}{2}} = 3\left(\frac{27(a+b+c)}{3}\right)^{-\frac{1}{2}}$$

$$= (a+b+c)^{-\frac{1}{2}}$$

□

Solution 2 (by Alexander Bogomolny).

By ***Bergstrom's inequality,***

$$LHS \geq \frac{9}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}}. \tag{1}$$

Now, using $\left(\sum_{cycl} x\right)^2 \leq 3\sum_{cycl} x^2,$

$$\left(\sum_{cycl} \sqrt{a+b^2}\right)^2 \leq 3\left(\sum_{cycl} a + \sum_{cycl} a^2\right) = 81\left(\sum_{cycl} a\right)$$

So to continue from (1),

$$LHS \geq \frac{9}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}} \geq \frac{9}{9\sqrt{a+b+c}} = \frac{1}{\sqrt{a+b+c}}$$

□

Solution 3 (by Daniel Sitaru).

By ***Hölder's inequality***,

$$\left(\sum_{cycl} \frac{1}{\sqrt{a+b^2}}\right)\left(\sum_{cycl} \frac{1}{\sqrt{a+b^2}}\right)\left(\sum_{cycl} (a+b^2)\right)$$

$$\geq \left(\sum_{cycl} \frac{1}{\sqrt[6]{a+b^2}} \cdot \frac{1}{\sqrt[6]{a+b^2}} \cdot \sqrt[3]{a+b^2}\right)^3 = 27.$$

In other words,

$$\left(\sum_{cycl} \frac{1}{\sqrt{a+b^2}}\right)^2 \left(\sum_{cycl} a + \sum_{cycl} a^2\right) \geq 27,$$

implying,

$$\left(\sum_{cycl} \frac{1}{\sqrt{a+b^2}}\right)^2 \geq \frac{27}{\sum_{cycl} a + 26\sum_{cycl} a} = \frac{27}{27\sum_{cycl} a} = \frac{1}{\sum_{cycl} a}$$

This equivalent to the required inequality. □

Solution 4 (by Shivam Sharma).

Applying the ***AM-HM inequality***,

$$\sum_{cycl}\frac{1}{\sqrt{a+b^2}} \geq \frac{9}{\sqrt{(1+1+1)(a+b+c+a^2+b^2+c^2)}}$$

$$= \frac{9}{\sqrt{3\Big(27(a+b+c)\Big)}} = \frac{9}{\sqrt{9\cdot 9(a+b+c)}} = \frac{1}{a+b+c}$$

□

Acknowledgment (by Alexander Bogomolny)

Dan Sitaru has kindly posted the ***CutTheKnotMath*** page the above problem from his book "Algebraic Phenomenon". Solution 1 is by Maki Milanovic; Solution 3 is by Dan Sitaru; Solution 4 is by Shivam Sharma.

140. Gireaux's Theorem

Theorem

If a continuous function of several variables is defined on a hyperbrick and is convex in each of the variables, it attains its maximum at one of the corners. More formally:

Assume $I_k = [a_k, b_k] \subset \mathbb{R}, k = \overline{1,n}$ and
$f : I_1 \times I_2 \times \ldots \times I_n \to \mathbb{R}$ is a continuous
function convex separately in each of the
variables in the domain of definition. Then it
attains its maximum at point $C = (c_1, \ldots, c_n)$
where $c_k \in \{a_k, b_k\}, k \in \overline{1,n}$

The statement of the theorem is a specification of a theorem of Weierstrass (the ***Extreme Values Theorem***) that states that a continuous function defined on a compact set attains its extremes in a set. Assume now that the function is convex in each variables (i.e., as a function of one argument, with other arguments fixed.) A continuous function of one variable, convex on a closed interval, attains its maximum at one of the endpoints of the interval. This means that the maximum of the given function is attained at either, say, $a \times I_2 \times \ldots \times I_n$ or $b_1 \times I_2 \times \ldots \times I_n$, which reduces the dimension of the search for the maximum by 1. Doing this recursively proves the statement.

References:

1. Israel Meireles Chrisostomo, *Trigonometria Pura e Aplicações e um pouco além: problemas de Olimpíadas, 3 de Julho, 2015*

USA 1980

Prove that, for $a, b, c \in [0,1]$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

The function $f(a,b,c) = \sum_{cycl}\frac{a}{b+c+1} + \prod_{cycl}(1-a)$ is a convex in each of the three variables a, b, c so that f takes its maximum value in one of either vertices of the cube $0 \leq a \leq 1, 0 \leq b \leq 1, 0 \leq c \leq 1$. Since $f(a, b, c)$ takes value 1 in each of these points, the required inequality is proven.

References:

1. M. S. Klamkin, *USA Mathematical Olympiads 1972 – 1986, MAA, 1988*

Dan Sitaru I

Prove that, for $a, b, c, d \in [0, 2]$,

$$\frac{9a}{1+bcd} + \frac{9b}{1+cda} + \frac{9c}{1+dab} + \frac{9d}{1+abc} + 9e^{abcd} \leq 8 + 9e^{16}.$$

$$f : [0,2]^4 \to \mathbb{R}, f(a, b, c, d) = 9\sum \frac{a}{1+bcd} + 9e^{abcd}.$$

$$f'_a = \frac{9}{1+bcd} - \frac{9bcd}{(1+cda)^2} - \frac{9cdb}{(1+dab)^2} - \frac{9dbc}{(1+abc)^2} + 9bcde^{abcd},$$

$$f''_{aa} = \frac{18bc^2d^2}{(1+cda)^3} + \frac{18cd^2b^2}{(1+dab)^3} + \frac{18db^2c^2}{(1+abcd)^3} + 9b^2c^2d^2e^{abcd} > 0.$$

f strictly convex in variable and, similarly, in the rest of the variables.f defined on a compact set $[0, 2]^4$, hence, by Gireaux's theorem f attains it maximum at the vertices of the hypercube $[0, 1]^4$. It is easy to check that the maximum is attained for $f(2, 2, 2, 2) = 4 \cdot \frac{18}{1+8} + 9e^{16} = 8 + 9e^{16}$, thus proving the inequality.

Dan Sitaru II

Prove that, for $x, y, z \in [0, 1]$

$$\frac{x}{y+z+2016} + \frac{y^2}{z+x+2016} + \frac{z^3}{x+y+2016} + (1-x)(1-y)(1-z) \leq 1.$$

$$f : [0, 2]^3 \to \mathbb{R},$$

$$f(x, y, z) = \frac{x}{y+z+2016} + \frac{y^2}{z+x+2016} + \frac{z^3}{x+y+2016} + (1-x)(1-y)(1-z)$$

We easily check that

$$f'_x x = \frac{2y^2}{(x+z+2016)^3} + \frac{2z^3}{(x+y+2016)^3} > 0.$$

f strictly convex in variable a and, similarly, in the rest of the variables. f defined on a compact set $[0, 2]^4$, hence, by Gireaux's theorem f attains it maximum at the vertices of the hypercube $[0, 1]^3$. It is easy to check that the maximum is attained for $f(0, 0, 0) = 1$, thus proving the inequality.

Second proof (Leo Giugiuc).

From $x, y, z \in [0,1]$ it follows that $\frac{x}{y+z+2016} \le \frac{x}{3}$; $\frac{y^2}{z+x+2016} \le \frac{y}{3}$, $\frac{z^3}{x+y+2016} \le \frac{z}{3}$. Thus suffice it to prove that

$$f(x,y,z) = \frac{x+y+z}{3} + (1-x)(1-y)(1-z) \le 1.$$

Let $a = 1-x, b = 1-y, c = 1-z$. The inequality to prove becomes

$$\frac{1-a+1-b+1-c}{3} + abc \le 1,$$

or, $abc \le \frac{a+b+c}{3}$, which is true because, for $a, b, c \in [0,1], abc \le \sqrt[3]{abc}$ and by the AM-GM inequality. □

Third proof (by Alexander Bogomolny).

Observe that $f(x,y,z)$ defined in the seconds proof is linear, hence convex, in each of its arguments. Gireaux's theorem applies. $f(0,0,0) = 1$,
$f(0,0,1) = \frac{1}{3}, f(0,1,1) = \frac{2}{3}, f(1,1,1) = \frac{1}{3} = 1.$ □

Aknowledgment (by Alexander Bogomolny)

I am indebted to Dan Sitaru for supplying the references and the examples.

141. Hung Viet's Inequality III

Prove that, for all real $a, b, c \ge 0$,

$$\left(\sum_{cycl} a^4\right)\left(\sum_{cycl} ab^3\right) \ge \left(\sum_{cycl} a^3b\right)\left(\sum_{cycl} a^2b^2\right)$$

Proposed by Hung Nguyen Viet

Solution (by Kevin Soto Palacios).

Observe that by *Hölder's inequality*

$$\left(\sum_{cycl} a^4\right)\left(\sum_{cycl} a^2b^2\right) \ge \left(\sum_{cycl} a^3b\right)^2,$$

$$\left(\sum_{cycl} ab^3\right)\left(\sum_{cycl} a^3b\right) \ge \left(\sum_{cycl} a^2b^2\right)^2.$$

The product of the above two is exactly the required inequality. □

Acknowledgment (by Alexander Bogomolny)

The inequality – by Nguyen Viet Hung – has been published at ***Spring issue of the Romanian Mathematical Magazine***. This is Problem SP048. I reproduce here the charming solution by Kevin Soto Palacios. Additional solutions can be found at the link.

142. Inequality with Constraint XV and XVI

If a, b, c are positive numbers such that $a+b+c=3$, then

(1) $$\frac{a^2}{\sqrt{b^2+4}}+\frac{b^2}{\sqrt{c^2+4}}+\frac{c^2}{\sqrt{a^2+4}}>\frac{3}{5}$$

Proposed by Henry Ricardo

(2) $$\frac{a^2}{\sqrt{b^2+4}}+\frac{b^2}{\sqrt{c^2+4}}+\frac{c^2}{\sqrt{a^4+4}}>\frac{3}{5}$$

Proposed by Daniel Sitaru

Problem 1, Solution 1 (by Henry Ricardo).

WLOG,we may assume that $a \geq b \geq c$. It follows that $a^2 \geq b^2 \geq c^2$ and $\frac{1}{\sqrt{a^2+4}} \leq \frac{1}{b^2+4} \leq \frac{1}{c^2+4}$. Now the ***Rearrangement inequality*** gives us

$$\sum_{cycl}\frac{a^2}{\sqrt{b^2+4}} \geq \sum_{cycl}\frac{a^2}{\sqrt{a^2+4}}$$

It can be seen graphically (and proved with some tedious algebra/analysis) that the curve given by $y=\frac{x^2}{\sqrt{x^2+4}}$ lies on or above the tangent line to the curve at $x=1, y=\frac{9\sqrt{5}}{25}(x-1)+\frac{\sqrt{5}}{5}$, on the interval $(0,3)$. Thus we have

$$\sum_{cycl}\frac{a^2}{\sqrt{b^2+4}} \geq \sum_{cycl}\Big(\frac{9\sqrt{5}}{25}(a-1)+\frac{\sqrt{5}}{5}\Big)$$

$$=\frac{9\sqrt{5}}{25}\sum_{cycl}-\frac{12\sqrt{5}}{25}=\frac{27\sqrt{5}}{25}-\frac{12\sqrt{5}}{25}=\frac{3\sqrt{5}}{5}>\frac{3}{5}.$$

□

Problem 1, Solution 2 (by Alexander Bogomolny).

Using ***Bergström inequality*** and the obvious $x+2>\sqrt{x^2+4}$,

$$\sum_{cycl}\frac{a^2}{\sqrt{b^2+4}} \geq \frac{(a+b+c)^2}{\sum_{cycl}\sqrt{a^2+4}}>\frac{(a+b+c)^2}{\sum_{cycl}(a+2)}=\frac{9}{9}=1$$

□

Problem 2, Solution (by Anish Ray).

As above, using Bergström inequality, the obvious $x^2+2>\sqrt{x^4+4}$ and

$$(x+y+z)^2 \geq x^2+y^2+z^2,$$

$$\sum_{cycl}\frac{a^2}{\sqrt{b^4+4}} \geq \frac{(a+b+c)^2}{\sum_{cycl}\sqrt{a^4+4}}$$

$$>\frac{9}{\sum_{cycl}(a^2+2)} \geq \frac{9}{(a+b+c)^2+6}=\frac{9}{15}=\frac{3}{5}.$$

□

Acknowledgment (by Alexander Bogomolny)

The ***problem (Problem 1) with a solution*** (Problem 1, Solution 1) has been posted to the Romanian Mathematical Magazine by Henry Ricardo. A somewhat more complicated (but similar) ***problem*** has been posted by Dan Sitaru, with credits to Anish Ray.

143. Inequality with Constraint from Dan Sitaru's Math Phenomenon

If $a \ge b \ge c > 0$ and $a+b+c=10$, then

$$b+2a+20 \ge 2\sum_{cycl}\frac{a^2+ab+b^2}{a+b} \ge b+2c+20.$$

Proposed by Daniel Sitaru

Solution 1 (by Soumitra Mandal).

Note that

$$2\sum_{cycl}\frac{a^2+ab+b^2}{a+b} \ge 2\sum_{cycl}\frac{\frac{3}{4}(a+b)^2}{a+b} = 2\sum_{cycl}\frac{3}{4}(a+b)$$

$$= 2\cdot\frac{3}{2}(a+b+c) = 3(a+b+c)$$

$$\ge (a+b+c)+2(a+b+c) \ge b+2c+20,$$

because $a \ge c$. This proves the right inequality. The left inequality is equivalent to

$$4a+3b+2c \ge \sum_{cycl}\frac{a^2+ab+b^2}{a+b}$$

which, in turn is equivalent to

$$\left(2a+b-\frac{2(a^2+ab+b^2)}{a+b}\right)+\left(2b+c-\frac{2(b^2+bc+c^2)}{b+c}\right)+$$

$$+\left(2a+c-\frac{2(a^2+ac+c^2)}{a+c}\right)\ge 0,$$

or,

$$\frac{b(a-b)}{a+b}+\frac{b(b-c)}{b+c}+\frac{c(a-c)}{c+a}\ge 0,$$

which is true because $a \ge b \ge c$. □

Solution 2 (by Alexander Bogomolny).

The left inequality is equivalent to

$$4a+3b+2c \ge 2\sum_{cycl}\frac{a^2+ab+b^2}{a+b}$$

$$= 2\sum_{cycl}\frac{a^2+2ab+b^2}{a+b} - 2\sum_{cycl}\frac{ab}{a+b}$$

$$= 2\sum_{cycl}(a+b) - 2\sum_{cycl}\frac{ab}{a+b} = 4(a+b+c) - 2\sum_{cycl}\frac{ab}{a+b}$$

which can be rewritten as

$$2\sum_{cycl}\frac{ab}{a+b} \ge b+2c.$$

Now note that

$$\frac{2ab}{a+b}+\frac{2bc}{b+c} \ge \frac{2ab}{2a}+\frac{2bc}{2b} = b+c.$$

Also $\frac{2ca}{c+a} \geq c$ is equivalent to $2ca \geq c^2 + ca$, or $a \geq c$, which is true. Now, adding this to

$$\frac{2ab}{a+b} + \frac{2bc}{b+c} \geq b + c$$

completes the proof of the left inequality. For the right inequality, observe that, as we just showed,

$$2\sum_{cycl} \frac{ab}{a+b} \geq b + 2c.$$

Thus suffice it to prove that

$$2\sum_{cycl} \frac{a^2+b^2}{a+b} \geq 20.$$

This is indeed so due to ***Bergström's inequality***:

$$2\sum_{cycl} \frac{a^2+b^2}{a+b} = 2\sum_{cycl} \frac{a^2}{a+b} + 2\sum_{cycl} \frac{b^2}{a+b}$$

$$\geq 2\frac{(a+b+c)^2}{2(a+b+c)} + 2\frac{(a+b+c)^2}{2(a+b+c)} = 2(a+b+c) = 20.$$

□

Solution 3 (by Nassim Nicholas Taleb).

Let $= 2\sum_{cycl} \frac{a^2+ab+b^2}{a+b}$. We reexpress:

$$f = 2\sum_{cycl} \left(\frac{(a+b)^2 - ab}{a+b} \right) = 2\left(20 - \sum_{cycl} \frac{ab}{a+b} \right)$$

Let us further establish that from the assumtions $a \geq b \geq c > 0$ and $a + b + c = 10$, it is necessary that $a \geq \frac{10}{3}$ and $c \leq \frac{10}{3}$. We have the right side inequality:

$$b + 2c + 20 \leq 20 + \left(1 - \frac{10}{3}\right) + \frac{10}{3} < 30.$$

We also have $f \geq 30$, for $\frac{ab}{a+b} \leq \frac{1}{4}(a+b)$ from which

$$\sum_{cycl} \frac{ab}{a+b} \leq \frac{1}{2}(a+b+c) = 5$$

For the left inequality, we have

$$\frac{ab}{a+b} + \frac{cb}{b+c} + \frac{ca}{c+a} \geq \frac{ab}{2a} + \frac{bc}{2b} + \frac{ca}{2a} = \frac{b}{2} + c,$$

so $f \leq 40 - b - 2c$. The left side inequality becomes
$b + 2a + 20 \geq 40 - b - 2c$, i.e., $2(a+b+c) \geq 20$ which is true. □

Acknowledgment (by Alexander Bogomolny)

The above problem, originally from his book Math Phenomenon, has been posted by Dan Sitaru at the ***CutTheKnot Math***. Solution 1 is by Soumitra Mandal; Solution 3 is by N. N. Taleb.

144. Problem 4 From the 2016 Pan – African Math Olympiad

Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\sum_{cycl} \frac{1}{(x+1)^2 + y^2 + 1} \leq \frac{1}{2}$$

Solution (by Daniel Sitaru).

We'll first use the ***AM-GM inequality:***

$$(x+1)^2 + y^2 + 1 = (x^2 + y^2) + 2x + 2 \geq 2(xy + x + 1).$$

The other two summands are modified appropriately. Introduce a, b, c via $x = \frac{b}{a}, y = \frac{c}{b}, z = \frac{a}{c}$. Note that $xy = \frac{c}{a}$. It follows that

$$\sum_{cycl} \frac{1}{(x+1)^2 + y^2 + 1} \leq \sum_{cycl} \frac{1}{2(xy + x + 1)}$$

$$= \frac{1}{2} \frac{1}{\frac{c}{a} + \frac{b}{a} + 1} = \frac{1}{2} \sum_{cycl} \frac{a}{a+b+c} = \frac{1}{2} \cdot \frac{a+b+c}{a+b+c} = \frac{1}{2}$$

□

Acknowledgment (by Alexander Bogomolny)

Dan Sitaru has kindly posted at the ***CutTheKnotMath page*** the above problem from the 24th ***Pan – African Mathematical Olympiad***, along with his solution. Dan had also remarked that the problem was created in 2006 by Cristinel Mortici from Romania.

145. Simple Inequality with a Variety of Solutions

Prove that, for $x, y, z > 1$,

$$\sum_{cycl} \left(\frac{\ln x}{\ln y \ln z} + \frac{\ln y}{\ln z \ln x} \right) \geq \frac{18}{\ln(xyz)}.$$

Proposed by Daniel Sitaru - Romania

Solution 1(by Kevin Soto Palacios).

Let $a = \ln x, b = \ln y, c = \ln z$. The inequality becomes

$$\left(\frac{a}{bc} + \frac{b}{ca} \right) + \left(\frac{b}{ca} + \frac{c}{ab} \right) + \left(\frac{c}{ab} + \frac{a}{bc} \right) \geq \frac{18}{a+b+c},$$

or, equivalently,

$$\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \right)(a+b+c) \geq 9.$$

This is seen to be true by first applying the ***Cauchy - Schwarz*** and then ***AM-GM inequality***:

$$\left(\sum_{cycl} \frac{a}{bc} \right)(a+b+c) \geq \frac{(a+b+c)^2}{3abc} \cdot (a+b+c)$$

$$= \frac{(a+b+c)^3}{3abc} \geq 9.$$

□

Solution 2 (by Nirapada Pal).

Let $a = \ln x, b = \ln y, c = \ln z$. The inequality reduces to

$$\left(\frac{a}{bc} + \frac{b}{ca}\right) + \left(\frac{b}{ca} + \frac{c}{ab}\right) + \left(\frac{c}{ab} + \frac{a}{bc}\right) \geq \frac{18}{a+b+c},$$

$$LHS = \sum_{cycl}\left(\frac{a}{bc} + \frac{b}{ca}\right) = \sum_{cycl}\frac{a^2+b^2}{ab} \geq \sum_{cycl}\frac{2ab}{abc} = 2\sum_{cycl}\frac{1}{a}$$

$$\overbrace{\geq}^{AM-HM} 2 \cdot \frac{9}{a+b+c} = \frac{18}{a+b+c}.$$

□

Solution 3(by Daniel Sitaru).

Let $a = \ln x, b = \ln y, c = \ln z$. The inequality reduces to

$$\left(\frac{a}{bc} + \frac{b}{ca}\right) + \left(\frac{b}{ca} + \frac{c}{ab}\right) + \left(\frac{c}{ab} + \frac{a}{bc}\right) \geq \frac{18}{a+b+c},$$

$$LHS = \sum_{cycl}\left(\frac{a}{bc} + \frac{b}{ca}\right) = 2\sum_{cycl}\frac{a}{bc}$$

$$\geq \frac{2}{abc}\sum_{cycl}a^2 \geq \frac{2}{abc}\sum_{cycl}ab = 2\sum_{cycl}\frac{1}{a} \overbrace{\geq}^{AM-HM} 2 \cdot \frac{9}{a+b+c}$$

$$= \frac{18}{a+b+c}$$

□

Solution 4 (Nikolaos Skoutaris).

Let $a = \ln x, b = \ln y, c = \ln z$. The inequality reduces to

$$\left(\frac{a}{bc} + \frac{b}{ca}\right) + \left(\frac{b}{ca} + \frac{c}{ab}\right) + \left(\frac{c}{ab} + \frac{a}{bc}\right) \geq \frac{18}{a+b+c}$$

$$LHS = \sum_{cycl}\left(\frac{a}{bc} + \frac{b}{ca}\right) = \sum_{cycl}\frac{a^2+b^2}{abc}$$

$$\geq \sum_{cycl}\frac{2ab}{abc} = 2\sum_{cycl}\frac{1}{a} = 2 \cdot 3 \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \overbrace{\geq}^{AM-HM} 6 \cdot \frac{3}{\frac{1}{\frac{1}{a}} + \frac{1}{\frac{1}{b}} + \frac{1}{\frac{1}{c}}}$$

$$= \frac{18}{a+b+c}$$

□

Solution 5 (by Nguyen Than Nho).

$$x, y, z > 1 \Rightarrow \ln x, \ln y, \ln z > 0$$

$$LHS = \sum_{cycl}\left(\frac{\ln x}{\ln y \ln z} + \frac{\ln y}{\ln z \ln x}\right) \overbrace{\geq}^{AM-GM} 2\left(\frac{1}{\ln x} + \frac{1}{\ln y} + \frac{1}{\ln z}\right)$$

$$\overbrace{\geq}^{Cauchy-Schwarz} 2 \cdot \frac{(1+1+1)^2}{\ln x + \ln y + \ln z} = \frac{18}{\ln(xyz)}.$$

□

Solution 6 (by Soumava Chakraborty , Geanina Tudose).

Let $a = \ln x, b = \ln y, c = \ln z$. Then inequality reduces to

$$\left(\frac{a}{bc} + \frac{b}{ca}\right) + \left(\frac{b}{ca} + \frac{c}{ab}\right) + \left(\frac{c}{ab} + \frac{a}{bc}\right) \geq \frac{18}{a+b+c}$$

or, equivalently,

$$\frac{1}{abc} \sum_{cycl} a^2 \geq \frac{9}{a+b+c},$$

or,

$$\sum_{cycl} a^2 \cdot \sum_{cycl} a \geq 9.$$

By the AM-GM inequality

$$\sum_{cycl} a^2 \geq 3\sqrt[3]{a^2b^2c^2}$$

$$\sum_{cycl} a \geq 3\sqrt[3]{abc}$$

the product of which is $\sum_{cycl} a^2 \cdot \sum_{cycl} a \geq 9abc$, which is equivalent to the required inequality. □

Solution 7 (by Uche Eliezer Okeke).

Let $a = \ln x, b = \ln y, c = \ln z$. The inequality reduces to

$$\left(\frac{a}{bc} + \frac{b}{ca}\right) + \left(\frac{b}{ca} + \frac{c}{ab}\right) + \left(\frac{c}{ab} + \frac{a}{bc}\right) \geq \frac{18}{a+b+c},$$

or, equivalently,

$$\frac{1}{abc} \sum_{cycl} a^2 \geq \frac{9}{a+b+c}$$

$$LHS = \frac{1}{abc} \sum_{cycl} a^2 \geq \frac{2}{abc} \cdot \frac{(\sum_{cycl} a)^2}{3}$$

$$\overbrace{\geq}^{AM-GM} \frac{2}{3} \cdot \frac{(\sum_{cycl} a)^2}{(\sum_{cycl})^3} \cdot \frac{27}{1} = \frac{18}{\sum_{cycl} a}$$

□

Solution 8 (by Seyran Ibrahimov).

Let $a = \ln x, b = \ln y, c = \ln z$. The inequality reduces to

$$\left(\frac{a}{bc} + \frac{b}{ca}\right) + \left(\frac{b}{ca} + \frac{c}{ab}\right) + \left(\frac{c}{ab} + \frac{a}{bc}\right) \geq \frac{18}{a+b+c},$$

$$LHS = \sum_{cycl} \frac{a}{bc} \overbrace{\geq}^{Chebyshev} \frac{1}{3}(a+b+c)\left(\sum_{cycl} \frac{1}{bc}\right)$$

$$\overbrace{\geq}^{Cauchy-Schwarz} \frac{3(a+b+c)}{ab+bc+ca} \geq \frac{9}{a+b+c}$$

because $(a+b+c)^2 \geq 3(ab+bc+ca)$ which follows from

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

□

Solution 9 (by Myagmarsuren Yadamsuren).

Let $\ln x = t_1, \ln y = t_2, \ln z = t_3$. The inequality reduces to

$$LHS = \sum_{cycl} \frac{t_1}{t_2 t_3} = \sum_{cycl} \frac{1}{t_3}\left(\frac{t_1}{t_2} + \frac{t_2}{t_1}\right)$$

$$\underbrace{\geq}^{AM-GM} 2 \cdot \sum_{cycl} \frac{1}{t^3} \underbrace{\geq}^{Bergstrom} 2 \cdot \frac{9}{\sum_{cycl} t_1} = \frac{18}{t_1 + t_2 + t_3}$$

□

Solution 10 (by Alexander Bogomolny).

Let $a = \ln x, b = \ln y, c = \ln z$. The inequality reduces to

$$\sum_{cycl} \frac{a}{bc} \underbrace{\geq}^{AM-GM} 3\sqrt[3]{\frac{1}{abc}} \underbrace{\geq}^{AM-GM} \frac{9}{a+b+c}$$

□

Acknowledgment (by Alexander Bogomolny)

Dan Sitaru has kindly posted the problem form the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page*** and latter commented with several proofs. Solution 1 is by Kevin Soto Palacios; Solution 2 is by Nirapada Pal; Solution 3 is by Dan Sitaru; Solution 4 is by Nikolaos Skoutaris; Solution 5 is by Nguyen Thanh Nho; Solution 6 is by Soumava Chakraborty, Geanina Tudose came up with the same solution; Solution 7 is by Eliezer Okeke; Solution 8 is by Seyran Ibrahimov; Solution 9 is by Myagmarsuren Yadamsuren;

146. Sitaru – Schweitzer Inequality

Below is a slightly modified versions of Dan Sitaru's statement from his book "Math Phenomenon". The problem represents an integral analog and a generalization of the well known Schweitzer's Inequality derived next.

Let $f : [a, b] \to [m, M]$, with $m > 0$, be an Riemann integrable function such that $\frac{1}{f(x)}$ is also Riemann integrable. Then

$$\left(\int_a^b f(x)dx\right)\left(\int_a^b \frac{1}{f(x)}dx\right) \leq \frac{(m+M)^2}{4mM}(b-a)^2$$

Remark(by Alexander Bogomolny)

Note that, by the *Cauchy criterion for integrability*, if function $f : [a, b] \to [m, M]$, with $m > 0$ is Riemann integrable, then the function $\frac{1}{f} : [a, b] \to \left[\frac{1}{M}, \frac{1}{m}\right]$ is also Riemann integrable.

Solution 1 (by Ravi Prakash).

It follows from the premises that $\Big(f(x) - m\Big)\Big(M - f(x)\Big) \geq 0$, implying $f^2(x) + mM \leq (m+M)f(x)$, or

$$f(x) + \frac{mM}{f(x)} \leq m + M,$$

so that

$$\int_a^b f(x)dx + mM\int_a^b \frac{dx}{f(x)} \le (m+M)\int_a^b dx = (m+M)(b-a)$$

Let $J = mM\int_a^b \frac{dx}{f(x)}$ and $H = \int_a^b f(x)dx$. We have $J + H \le (m+M)(b-a)$, implying $J^2 + JH \le (m+M)(b-a)J$, i.e.,

$$JH \le \frac{1}{4}(m+M)^2(b-a)^2 - \left[\frac{1}{2}(m+M)(b-a) - J\right]^2$$
$$\le \frac{1}{4}(m+M)^2(b-a)^2$$

This is the required inequality. □

Solution 2 (by Soumitra Mandal).

We proceed as above to obtain

$$\int_a^b f(x)dx + mM\int_a^b \frac{dx}{f(x)} \le (m+M)\int_a^b dx = (m+M)(b-a).$$

By the AM-GM inequality,

$$(m+M)(b-a) \ge 2\sqrt{mM\Big(\int_a^b f(x)dx\Big)\Big(\int_a^b \frac{dx}{f(x)}\Big)}.$$

Squaring gives the required inequality. □

Schweitzer's Inequality

For $0 < m < M$, *and* $x_k \in [m, M]$, *for* $k \in \overline{1,n}$,

$$\left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n \frac{1}{x_k}\right) \le \frac{(m+M)^2}{4mM} n^2.$$

Proof of Schweitzer's Inequality (by Alexander Bogomolny).

Given sequence $\{x_k\} \subset [m, M]$, we define a function piece − wise: set, for $x \in [k, k+1]$, $f(x) = x_k, k \in \overline{1,n}$. Then, with $a = 1$ and $b = n+1$, $f : [a, b] \to [m, M]$, and it remains to observe that,

$$b - a = n$$
$$\int_a^b f(x)dx = \sum_{k=1}^n x_k$$
$$\int_a^b \frac{dx}{f(x)} = \sum_{k=1}^n \frac{1}{x_k}.$$

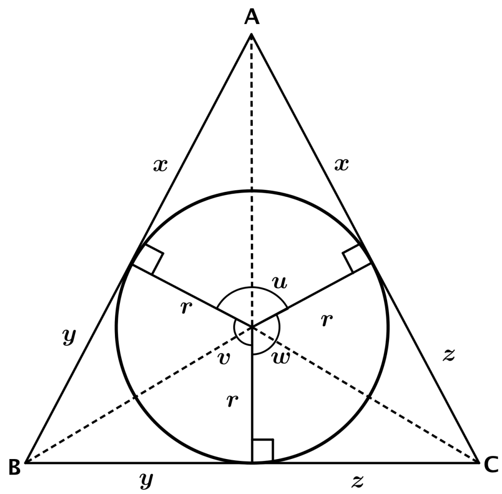
Note that two additional proofs appear ***elsewhere***. □

Acknowledgment (by Alexander Bogomolny)

The problem has been kindly posted by Dan Sitaru at ***CutTheKnotMath facebook page***. Solution 1 is by Ravi Prakash; Solution 2 is by Soumitra Mandal.

147. The Beauty of Fractions

On a circle of radius r, three points are chosen so that the circle is divided into three arcs in the ratios $u : v : w$. At the division points, tangents are drawn to the circle:



Prove that the area of the triangle formed by the tangents equals

$$S = \pm r^2 \tan\frac{\pi u}{u+v+w} \cdot \tan\frac{\pi v}{u+v+w} \cdot \tan\frac{\pi w}{u+v+w}.$$

Proposed by Francisco Javier Garcia Capitan

Solution (by Daniel Sitaru).

Set $AB = x+y, BC = y+z, AC = z+x$. Scale the arc ratios to satisfy $u+v+w = 2\pi$. As we can see, $x = r\tan\frac{u}{2}, y = r\tan\frac{v}{2}, z = r\tan\frac{w}{2}$.

Heron's formula reduces to $S = \sqrt{xyz(x+y+z)}$, so we have

$$\begin{aligned} S &= \sqrt{xyz(x+y+z)} \\ &= \sqrt{r^4\left(\sum_{cycl}\tan\frac{u}{2}\right)\prod_{cycl}\tan\frac{u}{2}} \\ &= r^2\sqrt{\left(\prod_{cycl}\tan\frac{u}{2}\right)^2} = \pm r^2\prod_{cycl}\tan\frac{u}{2} \\ &= \pm r^2\tan\frac{\pi u}{u+v+w}\cdot\tan\frac{\pi v}{u+v+w}\cdot\tan\frac{\pi w}{u+v+w}, \end{aligned}$$

where we applied ***an identity*** valid for $\alpha+\beta+\gamma = \pi$.

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\cdot\tan\beta\cdot\tan\gamma$$

□

Acknowledgment (by Alexander Bogomolny)

The problem by Francisco Javier García Capitán has been kindly communicated to me by Dan Sitaru, along with his solution.

148. Twin Inequalities in Four Variables: Twin 1

Assume $a, b, c, d > 0$. *Prove that*

$$(ac+bd)^2 \le (b\sqrt[5]{ab^4} + d\sqrt[5]{cd^4})(a\sqrt[5]{a^4b} + c\sqrt[5]{c^4d})$$

Proposed by Daniel Sitaru

Solution 1 (by Kevin Soto Palacios , Chris Kyriazis).

Recollect the ***Cauchy − Schwarz inequality***:

For real numbers $x_1, x_2, \ldots x_n$ *and* $y_1, y_2, \ldots, y_n$ *and integer* $n \ge 1$,

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \le \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right),$$

with equality when the vectors $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *are proportional.*

Take $x_1 = \sqrt{b}\sqrt[10]{ab^4}, y_1 = \sqrt{a}\sqrt[10]{a^4b}, x_2 = \sqrt{2}\sqrt[10]{cd^4}, y_2 = \sqrt{c}\sqrt[10]{c^2d}$.

Then the above becomes

$$(ac+bd)^2 \le (b\sqrt[5]{ab^4} + d\sqrt[5]{cd^4})(a\sqrt[5]{a^4b} + c\sqrt[5]{c^4d}),$$

with equality when $\dfrac{\sqrt{b}\sqrt[10]{ab^4}}{\sqrt{a}\sqrt[10]{a^4b}} = \dfrac{\sqrt{d}\sqrt[10]{cd^4}}{\sqrt{c}\sqrt[10]{c^4d}}$, *i.e., for* $bc = ad$.

□

Solution 2 (by Seyran Ibrahimov).

Let $a = x^5, b = y^5, c = z^5, d = t^5$. Then the required inequality becomes

$$(x^5y^5 + z^5t^5)^2 \le (y^9x + t^9z)(x^9y + z^9t)$$

which is, when expanded and simplified, reduces to

$$xty^9z^9 + zyx^9t^9 \ge 2x^5y^5z^5t^5$$

which is true by the ***AM-GM inequality***. □

Acknowledgment (by Alexander Bogomolny)

Dan Sitaru has kindly posted this problem from the ***Romanian Mathematical Magazine*** at ***the CutTheKnotMath page***, along with three solutions. Solution 1 is by Kevin Soto Palacios; Chris Kyriazis independently submitted a similar solution; Solution 2 is by Seyran Ibrahimov.

149. Twin Inequalities in Four Variables: Twin 2

Assume $a, b, c, d > 0$. *Prove that*

$$(a\sqrt[3]{a^2b} + c\sqrt[3]{c^2d})(b\sqrt[3]{ab^2} + d\sqrt[3]{cd^2}) \le (a^2+c^2)(b^2+c^2)$$

Proposed by Daniel Sitaru

Solution(by Soumava Chakraborty).

After expanding and simplifying, the required inequality becomes

$$a^{\frac{5}{3}}d^{\frac{5}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + b^{\frac{5}{3}}c^{\frac{5}{3}}a^{\frac{1}{3}}d^{\frac{1}{3}} \le a^2d^2 + b^2c^2.$$

Let $x = a^{\frac{1}{3}}, y = b^{\frac{1}{3}}, u = c^{\frac{1}{3}}, v = d^{\frac{1}{3}}$. The new variables are all positive. We need to prove that

$$F = x^5v^5yu + y^5u^5xv - x^6v^6 - y^6y^6 \le 0$$

We have

$$F = x^5v^5yu + y^5u^5xv - x^6v^6 - y^6u^6$$

$$= (yu - xv)\Big((xv)^5 - (yu)^5\Big)$$
$$= -(yu + xv)^2\left(\sum_{k=0}^{4}(xv)^k(yu)^{4-k}\right) \le 0,$$

with equality when $yu = xv$, i.e., $a^{\frac{1}{3}}d^{\frac{1}{3}} = b^{\frac{1}{3}}c^{\frac{1}{3}}$, or $ad = bc$. □

Acknowledgment (by Alexander Bogomolny)

Dan Sitaru has kindly posted this problem from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath page***, along with three solutions. The above solution is by Soumava Chakraborty; Ravi Prakash and Seyran Ibrahimov have independently submitted two solutions along the same lines.

150. An Inequality with Three Points

Let O, I, G be the circumcenter, the incenter, and the centroid of an acute ΔABC. Prove that

$$\sum_{P\in\{O,I,G\}}\sum_{cycl}\left(\frac{[\Delta APB]}{[\Delta ABC]} + \frac{[\Delta ABC]}{[\Delta APB]}\right)^2 \ge 100$$

where $[F]$denotes the area of shape F.

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

Let P be a point in the interior of ΔABC. Then, obviously,

$$\frac{[\Delta APB] + [\Delta BPC] + [\Delta CPA]}{[\Delta ABC]} = 1$$

Thus, if we introduce $x = \frac{[\Delta APB]}{[\Delta ABC]}, y = \frac{[\Delta BPC]}{[\Delta ABC]}, z = \frac{[\Delta CPA]}{[\Delta ABC]}$, then $x, y, z > 0$ and $x + y + z = 1$.

Define function $f : (0, \infty) \to \mathbb{R}$ as $f(x) = \left(x + \frac{1}{x}\right)^2$. Then

$$f'(x) = 2\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x^2}\right) = 2x - \frac{2}{x^3}$$

And $f''(x) = 2 + \frac{6}{x^4} \ge 0$, making function f convex and ***Jensen's inequality*** applicable:

$$f(x) + f(y) + f(z) \ge 3f\left(\frac{x+y+z}{3}\right)$$
$$= 3f\left(\frac{1}{3}\right) = 3\left(3 + \frac{1}{3}\right)^2 = \frac{100}{3}$$

Thus,

$$\sum_{P\in\{OIG\}}\sum_{cycl}\left(\frac{[\Delta APB]}{[\Delta ABC]} + \frac{[\Delta ABC]}{[\Delta APB]}\right)^2 \ge \sum_{P\in\{OIG\}}\frac{100}{3} = 100.$$

□

Remark (by Alexander Bogomolny)

The restriction to acute triangles would have been superfluous, if it were not for one of the points involved being the circumcenter O which lies in the exterior of obtuse triangles. This makes at least one of the variables x, y, z negative. If the problem stipulates that all three selected points are located in the interior of the triangle, the inequality become true for every triangle. For example, assume ΔRST is the ***Morley triangle*** of ΔABC. Then

$$\sum_{P\in\{OIG\}}\sum_{cycl}\left(\frac{[\Delta APB]}{[\Delta ABC]}+\frac{[\Delta ABC]}{[\Delta APB]}\right)^2\geq 100$$

Aknowledgment (by Alexander Bogomolny)
The problem (from the ***Romanian Mathematical Magazine***) has been posted by Dan Sitaru at the ***CutTheKnotMath page***, Dan later communicated his solution on a LaTex file by email.

Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru