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Some Challenging Problems Using 3-dimensional Geometric Probabilities

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Abstract: *In this article we use Henri Poincaré's definition of the function probability, on an infinite field of Lebesgue measurable events, for solving some geometric problems in* $\mathbb{R}^3$

In the following, we will prove a few generalisations of some classical probability problems, using 3-dimensional geometry. The generalized probability, definded by Henri Poincaré on an infinite field of Lebesgue measurable events, allows us to model and solve situations that are closer to those in real life.

1. Introduction

At first, we will remind the notions we need in this paper.

Definition 1.1. The set Ω of the elementary events associated to an experiment is named *sample space*, or *the set of all possible outcomes.*

Remark 1.2. We denote by $K = \mathcal{P}(\Omega)$ the set of all events associated to an experiment. K is also named *the power set* of the sample space, or *the event space.* The sets Ω and K may be finite, or infinite.

Definition 1.3. (*M. Kolmogorov*)
Let be (Ω, K) an infinite field of events.
A function $p : K \to \mathbb{R}$ is a (completely additive) *probability* on (Ω, K) if the following assertions are true:

(1) $p(A) \geq 0, \quad \forall A \in K.$
(2) $p(\Omega) = 1.$
(3) If $I \subseteq \mathbb{N}, \quad I \neq \emptyset$, and $(A_i)_{i\in I} \subset K$, such as for all $i, j \in I$ with $i \neq j$, we have $A_i \cap A_j = \emptyset$, then $p\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} p(A_i)$

The triplet (Ω, K, p) with the above properties is named an *infinite field of probabilities.*

Definition 1.4. (*H. Poincaré*)
$\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable set and (Ω, K) an infinite field of events, such that the subsets of K (the events) are Lebesgue measurable. For $A \in K$, we denote by $\mu(A)$ its Lebesgue measure. We say that the function

$$p : K \to \mathbb{R},\ p(A) = \frac{\mu(A)}{\mu(\Omega)}$$

is a *geometric probability* on the field (Ω, K).

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Remark 1.5. In this context, the set of all possible outcomes from the classical definition will be replaced by the *possible set*, the set of all favorable outcomes, by the *favorable set* and the cardinal of the set, by its Lebesgue measure.

Remark 1.6. The properties of the Lebesgue measure ensure the fact that the function p from the previous definition is a probability .

Remark 1.7. For $n = 1, 2$, or 3, the Lebesgue measure coincides with the standard measure of length, area, or volume.

2. Generalizations of some problems of classic geometrical probabilities

In the following, we will present some problems that are solved in $\mathbb{R}^2$ (whose solutions may be found in [1]) and we will prove their generalisations in 3-D space.

Problem 2.1. (*Triangles with constant perimeter*)
A segment $[AB]$ of length $a > 0$ is divided in three random parts. Prove that the probability of having a triangle with the side lengths equal to those of the three segments thus obtained is $p = \frac{1}{4}$.

Generalisation 2.1. (*D. Heuberger*)
A segment $[RS]$ of length $a > 0$ is divided in four random parts. Prove that the probability of having a quadrilateral with the side lengths equal to those of the four segments thus obtained is $p = \frac{1}{2}$.

Proof. We will use the following

Lemma Let $a, b, c, d \in (0, \infty)$. A quadrilateral with the side lengths equal to a, b, c, d exists, if and only if

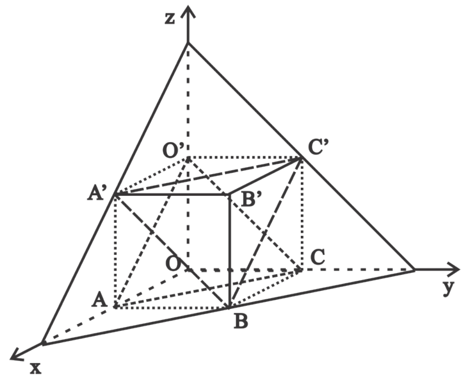
$$\begin{cases} a < b + c + d \\ b < c + d + a \\ c < a + b + d \\ d < a + b + c \end{cases}$$

Let M, N, P be three random interior points of $[RS]$, such that $RM = x$, $MN = y$, $NP = z$.

Then, $PS = t = a - x - y - z$ and we must have $x + y + z < a$.
The possible set is: $\Omega = \left\{ (x,\ y,\ z) \in \mathbb{R}^3 \;\middle|\; x,\ y,\ z \in (0,\ a),\ \ 0 < x + y + z < a \right\}$.
We denote by $\mathcal{D}$ the favorable set of the event $E_{\mathcal{D}}$ that there exists a quadrilateral with the sides length equal to x, y, z, t. Using the previous Lemma, we deduce:
$\mathcal{D} = \left\{ (x, y, z) \in \mathbb{R}^3 \cap \Omega \middle| \ x < y + z + t,\ y < x + z + t,\ z < x + y + t,\ t < x + y + z \right\}$
therefore

$$\mathcal{D} = \left\{ (x,\ y,\ z) \in \left(0,\ \frac{a}{2}\right)^3 \cap \Omega \;\middle|\; x + y + z > \frac{a}{2} \right\}$$

The plane $(O'AC)$ has the equation $x+y+z=\frac{a}{2}$, and $x+y+z>\frac{a}{2}$ is the equation of the half-plane bounded by $(O'AC)$ that doesn't contain O. Moreover, the coordinates of the points within the cube $[OABCO'A'B'C']$ verify the relation: $(x,\ y,\ z)\in\left(0,\ \frac{a}{2}\right)^3$.
We obtain:



$$\mathcal{D}=Int\left([OABCO'A'B'C']\setminus([OO'AC]\cup[B'BA'C'])\right)$$

The plane (UVW) has the equation $x+y+z=a$, and $x+y+z<a$ is the equation of the half-plane bounded by (UVW) that contains O, therefore Ω is the interiour of the tetrahedron $[OUVW]$. The requested probability is:

$$p\left(E_{\mathcal{D}}\right)=\frac{V\ \left([OABCO'A'B'C']\setminus[O'OAC]\right)}{V\ [OUVW]}=\frac{\left(\frac{a}{2}\right)^3-\frac{2}{6}\cdot\left(\frac{a}{2}\right)^3}{\frac{1}{6}\cdot a^3}=\frac{1}{2}.$$

□

Remark 2.3. The probability of having a *circumscribed* quadrilateral with the side lengths equal to those of the segments obtained by dividing $[RS]$ in four random parts is equal to 0, i.e. the favorable set is negligible relative to the possible set.

Indeed, the possible set is the same as in Generalisation 2.1 and to find the favorable set,we will use the following:

Theorem (***Pitot***)
The quadrilateral $ABCD$ is circumscribed if and only if $AB+CD=BC+AD$.

The favorable set M of the event E_M to whom we are seeking the probability, is: $M=M_1\cup M_2\cup M_3$, where

$$M_1=\{\,(x,y,z)\in\mathcal{D}|\ \ x+y=z+t\,\}=\{\,(x,y,z)\in\mathcal{D}|\ \ x+y=z+a-x-y-z\,\}$$

therefore

$$M_1=\left\{(x,y,z)\in\mathcal{D}\ |\ \ x+y=\frac{a}{2}\right\}=[ACC'A'],$$

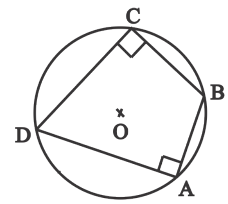
$$M_2=\left\{(x,y,z)\in\mathcal{D}|\ \ x+z=\frac{a}{2}\right\}=[ABC'O'],$$

$$M_3=\left\{(x,y,z)\in\mathcal{D}|\ \ y+z=\frac{a}{2}\right\}=[BCO'A'].$$

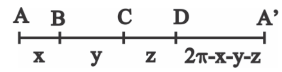
We obtain $p\left(E_M\right)=\frac{V\,(M_1\cup M_2\cup M_3)}{V\,[OUVW]}=\frac{0}{\frac{1}{6}\cdot a^3}=0.$

Remark 2.4. The probability that on choosing four points randomly on a circle of radius 1 we obtain a quadrilateral with a straight angle is also equal to 0.

Indeed, we may consider that A is fixed. The positions of the other three points will be given by the positive measure in radians of the arcs AB, AC and AD.
If, for example, the angle A is straight, then we cut the circle in A and we unfold it on the segment $[AA']$.



We must find the probability that choosing randomly the points B, C, D on $[AA']$, the sum of the lenghts of two adjacent segments equals π.



The possible set is:

$$\Omega = \left\{ (x,\ y,\ z) \in \mathbb{R}^3 \ \middle|\ x,\ y,\ z \in (0,\ 2\pi)\,,\quad x + y + z < 2\pi \right\}$$

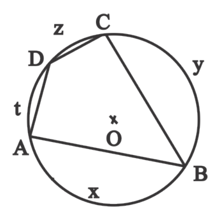
and the favorable set $\mathcal{M}$ of the event $E_{\mathcal{M}}$ to whom we are seeking the probability, is:

$$\mathcal{M} = \{\ (x,\ y,\ z) \in \Omega \mid \ x + y = \pi\ \} \cup \{\ (x,\ y,\ z) \in \Omega \mid \ y + z = \pi\ \}$$

With the same reasoning as in Remark 2.3, we deduce that the probability equals 0.

Remark 2.5. The probability that on choosing randomly four points A, B, C, D on a circle of radius 1 we obtain a quadrilateral with $O \in Int\,(ABCD)$ is $p = \frac{1}{2}$.

Indeed, denoting by x, y, z, t the positive measure in radians of the arcs AB, BC, CD and DA, the center of the circle must be in the interior of the quadrilateral, because otherwise the length (measure) of one of the arcs is greater than the sum of the lengths of the other three arcs.



By unfolding the cercle, as in Remark 2.4, on the segment $[AA']$, it follows that it suffices to find the probability that on choosing randomly the points $B, C, D \in [AA']$, if $AB = x$, $BC = y$, $CD = z$ and $DA' = a - x - y - z = t$, we obtain

$$\begin{cases} x < y + z + t \\ y < z + t + x \\ z < x + y + t \\ t < x + y + z \end{cases}.$$

So it's enough to have a quadrilateral with the sides of lengths x, y, z, t. Using Generalisation 2.1, the conclusion follows.

Problem 2.6. (*The problem of the meeting*)
Two persons can arrive at any moment of time of the interval $[0, T]$ in a certain place. Prove that the probability that the time interval between the arrivals of the two persons doesn't exceed t, where $t \in (0, T)$, is $p = 1 - \left(1 - \frac{t}{T}\right)^2$.

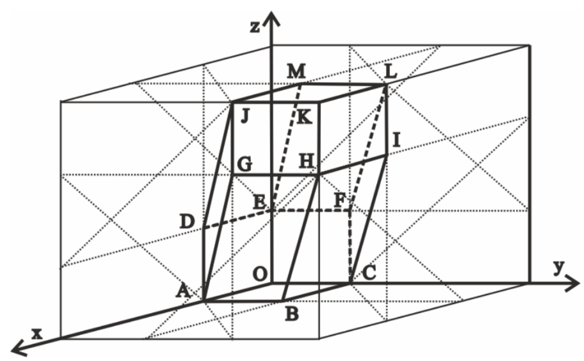
Generalisation 2.6. (*D. Heuberger*)
Three persons can arrive at any moment of time of the interval $[0, T]$ in a certain place. Then, the probability that the time interval between the arrivals of any two of the three persons doesn't exceed t, where $t \in \left(0, \frac{T}{2}\right)$ is $p = \left(\frac{t}{T}\right)^2 \left(3 - 2\frac{t}{T}\right)$.

Proof. We denote by x, y, z the moments of the arrivals of the three persons. We have $x, y, z \in [0, T]$. The possible set is $\Omega = [0, T]^3$.
We denote by Q the favorable set of the desired event E_Q. Then:

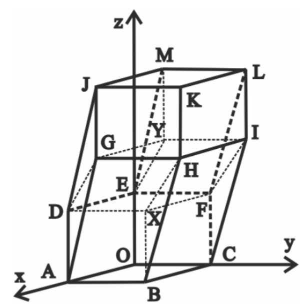
$$Q = \{\ (x,\ y,\ z) \in \Omega\ |\ \ |x-y| \le t,\ \ |y-z| \le t,\ \ |x-z| \le t\}$$

Q is the set of the points situated in the interior of the cube with the length of the sides T, between the planes α_1, α_2, α_3, α_4, α_5, α_6, where

$$\alpha_1 : x - y - t = 0$$
$$\alpha_2 : x - y + t = 0$$
$$\alpha_3 : y - z - t = 0$$
$$\alpha_4 : y - z + t = 0$$
$$\alpha_5 : x - z - t = 0$$
$$\alpha_6 : x - z + t = 0$$



We have $\alpha_1 \parallel \alpha_2 \parallel Oz$, $\alpha_3 \parallel \alpha_4 \parallel Ox$ and $\alpha_5 \parallel \alpha_6 \parallel Oy$. Q is the union of the cubes $[OABCEDXF]$, $[GHIYJKLM]$, of the quadrilateral prism $[DXFEGHIY]$ and of the triangular prisms $[ADGBXH]$, $[BXHCFI]$, $[DGJEYM]$, $[EYMFIL]$.
Then,
$V[OABCEDXF] = V[GHIYJKLM] = t^3$,



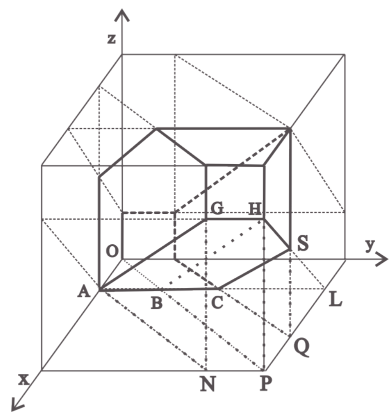
$V[DXFEGHIY] = t^2 \cdot d\,(G,\ (DEF)) = t^2 \cdot (T - 2t)$
and $V[ADGBXH] = V[BXHCFI] = V[DGJEYM] = V[EYMFIL]$.
Because $V\,[ADGBXH] = S\,[ADG] \cdot d\,(B,\ (ADG))$, $d\,(B,\ (ADG)) = \frac{\sqrt{2}}{2} \cdot t$ is the distance from B to α_1 and $S\,[ADJG] = AD \cdot d\,(A, GJ) = \sqrt{2} \cdot t \cdot (T - t)$, we obtain $V\,[Q] = 2 \cdot t^3 + t^2 \cdot (T - 2t) + 2 \cdot t^2 \cdot (T - t) = t^2 \cdot (3T - 2t)$.
Finally, $p\,(E_Q) = \frac{V[Q]}{V[\Omega]} = \left(\frac{t}{T}\right)^2 \left(3 - 2\frac{t}{T}\right)$. $\square$

Generalisation 2.6. (*D. Heuberger*)
Three persons can arrive at any moment of time of the interval $[0, 4t]$ at a hotel. Arrivals are independent and equally possible. Then, the probability that the time interval between the arrivals of at least two of the three persons doesn't exceed t is $p = \frac{7}{8}$.

Proof. We denote by x, y and z the moments of the arrivals of the three persons, therefore $x, y, z \in [0, 4t]$. The possible set is $\Omega = [0, 4t]^3$.
We denote by Q' the favorable set of the desired event E'_Q.
We have $Q' = Q_1 \cup Q_2 \cup Q_3$ where $Q_1 = \{\ (x,\ y,\ z) \in \Omega\ |\ \ |x - y| \le t\}$, $Q_2 = \{\ (x,\ y,\ z) \in \Omega\ |\ \ |y - z| \le t\}$, $Q_3 = \{\ (x,\ y,\ z) \in \Omega\ |\ \ |x - z| \le t\}$.
Q_1 is the set of the points situated in the interior of the cube with the length of the sides $4t$, between the planes α_1, α_2, where $\alpha_1 : x - y - t = 0$, $\alpha_2 : x - y + t = 0$.
Q_2 is the set of the points situated in the interior of the cube with the length of the sides $4t$, between the planes α_3, α_4, where $\alpha_3 : y - z - t = 0$, $\alpha_4 : y - z + t = 0$.

and Q_3 is the set of the points situated in the interior of the cube with the length of the sides $4t$, between the planes α_5, α_6, where $\alpha_5 : x-z-t=0$, $\alpha_6 : x-z+t=0$. We have $\alpha_1 \parallel \alpha_2 \parallel Oz$, $\alpha_3 \parallel \alpha_4 \parallel Ox$, $\alpha_5 \parallel \alpha_6 \parallel Oy$, and

$$V\left[Q_1\right] = (4t)^3 - 2 \cdot \frac{1}{2} \cdot (3t)^2 \cdot 4t = 28t^3.$$



It is obvious that the sets Q_1, Q_2 and Q_3 have the same volume.
The set $Q_1 \cap Q_2 \cap Q_3$ coincides with the set Q from Generalisation 2.6, for $T = 4t$, therefore $V[Q_1 \cap Q_2 \cap Q_3] = 10t^3$.
The sets $Q_1 \cap Q_2$, $Q_2 \cap Q_3$ and $Q_3 \cap Q_1$ have the same volume.
The set $Q_3 \cap Q_1$ is the polyhedron from the last image. Its volume is:
$V[Q_3 \cap Q_1] = V[Q_1] - 2 \cdot V[AGNBHP] - 2 \cdot V[BHPCSQ]$.
$V[AGNBHP] = S[AGN] \cdot d\,(A,\ OP) = \frac{AN \cdot GN}{2} \cdot \frac{1}{8} \cdot OP = \frac{9}{2}t^3$, and
$V[BHPCSQ] = V[LBHP] - V[LCSQ] = \frac{9}{2}t^3 - \frac{CQ \cdot SQ \cdot d(L,\ CQ)}{6} = \frac{9}{2}t^3 - \frac{4}{3}t^3 = \frac{19}{6}t^3$
We obtain $V[Q_3 \cap Q_1] = 28t^3 - 2 \cdot \left(\frac{9}{2}t^3 + \frac{19}{6}t^3\right) = \frac{38}{3}t^3$ and then

$$\begin{aligned} V\left[Q_1 \cup Q_2 \cup Q_3\right] =& V\left[Q_1\right] + V\left[Q_2\right] + V\left[Q_3\right] - V\left[Q_1 \cap Q_2\right] - V\left[Q_1 \cap Q_3\right] - \\ & - V\left[Q_2 \cap Q_3\right] + V\left[Q_1 \cap Q_2 \cap Q_3\right] = 56t^3. \end{aligned}$$

Therefore $p\left(E_Q\right) = \frac{V[Q]}{V[\Omega]} = \frac{56t^3}{64t^3} = \frac{7}{8}$ □

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