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1. In ΔABC

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{2r}{R}\geq 4.$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

We have $\sum\frac{r_a}{r_b}=\sum\frac{r_a^2}{r_ar_b}\overbrace{\geq}^{Bergstrom}\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}=\frac{(4R+r)^2}{p^2}\overbrace{\geq}^{(1)}4-\frac{2r}{R}$, *where (1) follows from Blundon - Gerretsen's inequality* $p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$ *(true from Gergonne's identity:* $H\Gamma^2=4R^2\left[1-\frac{2p^2(2R-r)}{r(4R+r)^2}\right]$, Γ *is Gergonne's point, namely the lines intersections* AA_1,BB_1,CC_1, *where* A_1,B_1,C_1 *are the tangent point of incircle in* ΔABC with the sides BC,CA,AB).

The equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be developed:

2. In ΔABC

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+n\cdot\frac{r}{R}\geq 3+\frac{n}{2},\ \textit{where } n\leq 2.$$

Proposed by Marin Chirciu - Romania

Proof.

We have $\sum\frac{r_a}{r_b}=\sum\frac{r_a^2}{r_ar_b}\overbrace{\geq}^{Bergstrom}\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}=\frac{(4R+r)^2}{p^2}\overbrace{\geq}^{(1)}3+\frac{n}{2}-\frac{nr}{R}$, *where (1) follows from Blundon-Gerretsen's inequality* $p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$ *(true from Gergonne's identity:* $H\Gamma^2=4R^2\left[1-\frac{2p^2(2R-r)}{r(4R+r)^2}\right]$, Γ *is Gergonne's point, namely the intersection lines* AA_1,BB_1,CC_1, *where* A_1,B_1,C_1 *are the tangent points of the incircle in* ΔABC *with the sides* BC,CA,AB*).*
It remains to prove that:

$$\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\overbrace{\geq}^{(1)}3+\frac{n}{2}-\frac{nr}{R}\Leftrightarrow\frac{4R-2r}{R}+\frac{nr}{R}\geq\frac{n+6}{2}\Leftrightarrow(2-n)(R-2r)\geq 0$$

obviously from Euler's inequality $R \geq 2r$ and the condition $2-n \geq 0$.
The equality holds if and only if the triangle is equilateral.

□

Remark.

*For $n=2$ we obtain inequality **1.***

For $n=0$ we obtain the well known inequality $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a} \geq 3$.

For $n=-2$ we obtain the known inequality $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a} \geq 2+\frac{2r}{R}$.

Let's notice that for $n \leq 0$ the obtained inequalities are very weak, the inequality is interesting for $n>0$, being the strongest for $n=2$.

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