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1. In ΔABC

$$\frac{r_a^2}{h_a}+\frac{r_b^2}{h_b}+\frac{r_c^2}{h_c}\geq 9r.$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

We prove the following lemma:

Lemma.
2. In ΔABC

$$\frac{r_a^2}{h_a}+\frac{h_b^2}{h_b}+\frac{r_c^2}{h_c}=\frac{2R(4R+r)-p^2}{r}.$$

Proof.

We have $\sum\frac{r_a^2}{h_a}=\sum\frac{(\frac{S}{p-a})^2}{\frac{2S}{a}}=\frac{S}{2}\sum\frac{a}{(p-a)^2}=\frac{rp}{2}\cdot\frac{4R(4R+r)-2p^2}{r^2p}=\frac{2R(4R+r)-p^2}{r}$

□

*Let's prove inequality **1.***

*Using the **Lemma**, inequality **1**, can be written:*

$\frac{2R(4R+r)-p^2}{r}\geq 9r\Leftrightarrow p^2\leq 2R(4R+r)-9r^2$, *which follows from Gerretsen's inequality:*

$$p^2\leq 4R^2+4Rr+3r^2$$

It remains to prove that:

$4R^2+4Rr+3r^2\leq 2R(4R+r)-9r^2\Leftrightarrow 2R^2-Rr-6r^2\geq 0\Leftrightarrow (R-2r)(2R+3r)\geq 0,$

obviously from Euler's inequality: $R\geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **1.** can be strengthened:*

3. In ΔABC

$$\frac{r_a^2}{h_a}+\frac{r_b^2}{h_b}+\frac{r_c^2}{h_c}\geq\frac{9R}{2}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma**, inequality **3** can be written:*

$\frac{2R(4R+r)-p^2}{r} \geq \frac{9R}{2} \Leftrightarrow 2p^2 \leq 2R(4R+r) - 9Rr$, *which follows from Gerretsen's inequality:*

$$p^2 \leq 4R^2 + 4Rr + 3r^2.$$

It remains to prove that:

$2(4R^2+4Rr+3r^2) \leq 2R(4R+r) - 9Rr \Leftrightarrow 8R^2 - 13Rr - 6r^2 \geq 0 \Leftrightarrow (R-2r)(8R+3r) \geq 0,$

obviously from Euler's inequality: $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality 3. is stronger than inequality **1.***

4. In ΔABC

$$\frac{r_a^2}{h_a} + \frac{r_b^2}{h_b} + \frac{r_c^2}{h_c} \geq \frac{9R}{2} \geq 9r.$$

Proof.

*See inequality **3.** and Euler's inequality.*

The equality holds if and only if the triangle is equilateral.

□

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