RMM - Triangle Marathon 201 - 300

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201. In $\Delta\, ABC$:

$$\sum \sqrt{(b+c)^2 - a^2} < 2(a+b+c)$$

Proposed by Zdravco Starc-Bulgaria

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC \sum \sqrt{(b+c)^2 - a^2} < 2(a+b+c)$

Como a, b, c **son lados de un triángulo**

$b+c-a > 0, a+c-b > 0, a+b-c > 0$. **Además** $\rightarrow a > 0, b > 0, c > 0$

$\Leftrightarrow a+b+c > b+c-a, b+c+a > a+c-b, c+a+b > a+b-c$

Partimos de

$$\left(\sqrt{a+b+c} - \sqrt{b+c-a}\right)^2 + \left(\sqrt{b+c+a} - \sqrt{a+c-b}\right)^2 +$$

$$+\left(\sqrt{c+a+b} - \sqrt{a+b-c}\right)^2 > 0$$

$$2(b+c) + 2(c+a) + 2(a+b) > 2\sqrt{(a+b+c)(b+c-a)} + 2\sqrt{(a+b+c)(a+c-b)} +$$

$$+2\sqrt{(a+b+c)(a+b-c)} \Leftrightarrow \sum \sqrt{(b+c)^2 - a^2} < 2(a+b+c)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS = \sqrt{(b+c+a)(b+c-a)} + \sqrt{(c+a+b)(c+a-b)} +$$

$$+\sqrt{(a+b+c)(a+b-c)}$$

$$\underset{\text{equality not holding}}{\overset{C-B-S}{<}} \sqrt{3(a+b+c)}\sqrt{a+b+c} = \sqrt{3}(a+b+c) < 2(a+b+c)$$

Solution 3 by Ravi Prakash-New Delhi-India

$$(b+c)^2 - a^2 = (b^2 + c^2 - a^2) + 2bc = 2bc\cos A + 2bc$$

$$= 2bc(1+\cos A) = 4bc\cos^2\frac{A}{2} < 4bc \Rightarrow \sqrt{(b+c)^2 - a^2} < 2\sqrt{bc} \le b+c$$

$$\therefore \sum \sqrt{(b+c)^2 - a^2} < \sum (b+c) = 2(a+b+c)$$

Solution 4 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\begin{aligned} &b+c>a\\ &a+c>b \Rightarrow a+b+c=2S\\ &a+b>c \end{aligned}$$

$$\sum\sqrt{(b+c-a)(a+b+c)}<2(a+b+c)$$

(1) $\frac{(b+c-a)+(a+b+c)}{2}>\sqrt{(b+c-a)(a+b+c)}$

(2) $\frac{(a+c-b)+(a+b+c)}{2}>\sqrt{(a+c-b)(a+b+c)}$ **(AM – GM)**

(3) $\frac{(a+b-c)+(a+b+c)}{2}>\sqrt{(a+b-c)(a+b+c)}$

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(1)+**(2)**+**(3)**$=2(a+b+c)>\sum\sqrt{(b+c-a)(a+b+c)}$

because "$\geq$" **no** "$>$" $a,b,c\neq 0$

Solution 5 by Geanina Tudose-Romania

$$\sum\sqrt{(b+c-a)(b+c+a)}<2(a+b+c)|\cdot\sqrt{b+c+a}$$

$$\Leftrightarrow\sum\sqrt{b+c-a}<2\sqrt{a+b+c}$$

Denote $\left.\begin{aligned}\sqrt{b+c-a}=x\Leftrightarrow b+c-a=x^2\\ \sqrt{a+c-b}=y\Leftrightarrow a+c-b=y^2\\ \sqrt{b+a-c}=z\Leftrightarrow b+a-c=z^2\end{aligned}\right\}\overset{\oplus}{\Rightarrow} a+b+c=x^2+y^2+z^2$

we have $x+y+z<2\sqrt{x^2+y^2+z^2}|^2\Leftrightarrow$

$$\Leftrightarrow x^2+y^2+z^2+2xy+2xz+2yz<4(x^2+y^2+z^2)$$

$$\Leftrightarrow 2xy+2xz+2yz<3(x^2+y^2+z^2)\ \textbf{true}$$

since $2xy+2xz+2yz\leq 2(x^2+y^2+z^2)<3(x^2+y^2+z^2)$

Solution 6 by Marian Dincă – Romania

$$\sum_{ciclic}\sqrt{(b+c)^2-a^2}=\sqrt{b+c+a}\left(\sqrt{b+c-a}+\sqrt{c-b+a}+\sqrt{a+b-c}\right)\le$$

$$\le\sqrt{b+c+a}\left(3\sqrt{\frac{\sum_{ciclic}b+c-a}{3}}\right)=\sqrt{b+c+a}\cdot\sqrt{3}\cdot\sqrt{a+b+c}=\sqrt{3}(a+b+c)$$

202. In $\Delta\, ABC$:

$$\sum\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\le\frac{a^2+b^2+c^2}{2r^2\sqrt{3}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\sum\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\le\frac{a^2+b^2+c^2}{2r^2\sqrt{3}}$

Recordar las siguientes identidades y desigualdades en un triángulo

$ABC;\ \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s}{r},\ S=sr,$

$a^2+b^2+c^2\ge 4\sqrt{3}S$. ***La desigualdad es equivalente***

$$\sum\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}=\sum\frac{\sin\left(\frac{B+C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}=\sum\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\le\frac{a^2+b^2+c^2}{2r^2\sqrt{3}}$$

En otras palabras $\frac{2s}{r}=\frac{a^2+b^2+c^2}{2r^2\sqrt{3}}\Leftrightarrow a^2+b^2+c^2\ge 4\sqrt{3}s=4\sqrt{3}S$

(Inequality Ionescu-Weitzenbock)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\sum\frac{2\sin A}{4\prod\sin\frac{A}{2}}$$

$$=\sum\frac{2\sin A}{\frac{r}{R}}=\sum\frac{a}{r}=\frac{\sum a}{r}=\frac{2s}{r}$$

$$RHS=\frac{\sum a^2}{2r^2\sqrt{3}}\overset{Ionescu-Weitzenbock}{\geq}\frac{4\sqrt{3}rs}{2r^2\sqrt{3}}=\frac{2s}{r}=LHS\ \textbf{(Proved)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\cos\frac{A}{2}=\sqrt{\frac{p\cdot(p-a)}{bc}}$$

$$\sin\frac{B}{2}=\sqrt{\frac{(p-a)\cdot(p-c)}{ac}};\sin\frac{C}{2}=\sqrt{\frac{(p-a)\cdot(p-b)}{ab}}$$

$$\sum\sqrt{\frac{\frac{p\cdot(p-a)}{bc}}{\frac{(p-a)\cdot(p-c)}{ac}\cdot\frac{(p-a)(p-b)}{ab}}}=\sum\sqrt{\frac{a^2\cdot p^2}{p(p-a)(p-b)(p-c)}}=$$

$$=\sum\frac{ap}{s}=\sum\frac{ap}{rp}=\frac{a+b+c}{2}=$$

$$=\frac{2p}{r}=\frac{4\sqrt{3}pr}{2r^2\sqrt{3}}=\frac{4\sqrt{3}S}{2r^2\sqrt{3}}\overset{IONESCU-WEITZENBOCK}{\leq}\frac{a^2+b^2+c^2}{2r^2\sqrt{3}}$$

203. **In acute $\Delta\,ABC$ with orthocenter H:**

$$HA\cdot HB+HB\cdot HC+HC\cdot HA\leq 3R^2.$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Daniel Sitaru – Romania

$$\sum HA\cdot HB=4R^2\sum\cos A\cos B=$$

$$=4R^2\cdot\frac{s^2+r^2-4R^2}{4R^2}=s^2+r^2-4R^2\leq\frac{27}{4}R^2+\frac{R^2}{4}-4R^2=3R^2$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo acutángulo ABC con Ortocentro H:

$HAHB + HBHC + HCHA \leq 3R^2$***. Dado que es un triángulo acutángulo:***

$$HA = 2R\cos A > 0, HB = 2R\cos B > 0, HC = 2R\cos C > 0$$

Teniendo en cuenta las siguientes identidades y desigualdades en un triángulo ABC: $\cos A + \cos B + \cos C = 1 + \frac{r}{R} \leq \frac{3}{2}$,

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C \geq \frac{3}{4}$$

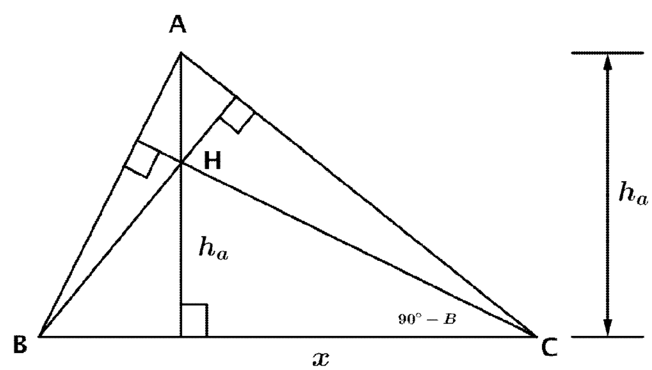
La desigualdad es equivalente:

$$2\cos A\cos B + 2\cos B\cos C + 2\cos C\cos A \leq \frac{3}{2}$$

Lo cual es cierto ya que $2\cos A\cos B + 2\cos B\cos C + 2\cos C\cos A =$

$$= (\cos A + \cos B + \cos C)^2 - (\cos^2 A + \cos^2 B + \cos C^2) \leq \frac{9}{4} - \frac{3}{4} = \frac{3}{2}$$

Solution 3 by Soumava Chakraborty-Kolkata-India



$$x = b\cos C\,;\ \frac{h'_a}{x} = \tan(90^\circ - B) = \cot B$$

$$\Rightarrow h'_a = b\cos C\cot B = 2R\sin B\cos C\frac{\cos B}{\sin B} \Rightarrow h'_a = 2R\cos B\cos C$$

$$\therefore AH = h_a - h'_a = b\sin C - 2R\cos B\cos C$$

$$= 2R\sin B\sin C - 2R\cos B\cos C = -2R\cos(B+C) = 2R\cos A$$

$\therefore AH = 2R\cos A$ ***Similarly***, $BH = 2R\cos B$ ***and*** $CH = 2R\cos C$

$$\therefore LHS = 4R^2\sum\cos A\cos B = 2R^2\left(2\sum\cos A\cos B\right)$$

$$= 2R^2\left\{\left(\sum\cos A\right)^2 - \sum\cos^2 A\right\} = 2R^2\left\{\left(1+\frac{r}{R}\right)^2 - \sum(1-\sin^2 A)\right\}$$

$$= 2R^2\left\{\frac{(R+r)^2}{R^2} - 3 + \frac{\sum a^2}{4R^2}\right\}$$

$$= 2(R+r)^2 - 6R^2 + (s^2 - 4Rr - r^2) = s^2 - 4R^2 + r^2$$

$\therefore$ ***given inequality*** $\Leftrightarrow s^2 - 4R^2 + r^2 \le 3R^2 - \frac{r^2(R-2r)}{R-r}$

$$\Leftrightarrow s^2 \le 7R^2 - \frac{r^2(2R-3r)}{R-r}$$

$\Leftrightarrow s^2(R-r) \le 7R^3 - 7R^2r - 2Rr^2 + 3r^3$ ***(1)***

Gerretsen $\Rightarrow s^2(R-r) \underset{(2)}{\le} (4R^2+4Rr+3r^2)(R-r) = 4R^3 - Rr^2 - 3r^3$

(1), (2) $\Rightarrow$ ***it suffices to prove:***

$$4R^3 - Rr^2 - 3r^3 \le 7R^3 - 7R^2r - 2Rr^2 + 3r^3$$

$$\Leftrightarrow 3R^3 - 7R^2r - Rr^2 + 6r^3 \ge 0 \Leftrightarrow (t-2)(3t^2-t-3) \ge 0;\ \left(t = \frac{R}{r}\right)$$

$\Leftrightarrow (t-2)\{3t(t-2) + 5(t-2) + 7\} \ge 0$ ***which is true*** $\because t = \frac{R}{r} \ge 2$

204. In $\Delta\,ABC$:

$$\frac{1}{\frac{1}{\sin^2\frac{A}{2}} + \frac{1}{\sin^2\frac{B}{2}} + \frac{1}{\sin^2\frac{C}{2}}} \le \frac{r^2}{4}\left(\frac{1}{w_a^2} + \frac{1}{w_b^2} + \frac{1}{w_c^2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

In any $\Delta\ ABC, \frac{1}{\sum \csc^2\frac{A}{2}} \le \frac{r^2}{4}\left(\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\right) \because w_a \le \sqrt{s(s-a)}$ ***etc,***

$$RHS \overset{(1)}{\ge} \frac{r^2}{4s}\sum\frac{1}{s-a} = \frac{r^2}{4s}\cdot\frac{\sum(s-b)(s-c)}{\prod(s-a)}$$

$$LHS = \frac{1}{\frac{bc}{(s-b)(s-c)}+\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}}$$

$$\overset{(2)}{=} \frac{\prod(s-a)}{bc(s-a)+ca(s-b)+ab(s-c)} = \frac{\prod(s-a)}{s\sum ab-12Rrs}$$

(1), (2) $\Rightarrow$ ***it suffices to prove:*** $\frac{r^2}{4s}\cdot\frac{\sum(s-b)(s-c)}{\prod(s-a)} \ge \frac{\prod(s-a)}{s\sum ab-12Rrs}$

$$\Leftrightarrow r^2s^2\left\{\sum(s-b)(s-c)\right\}(s^2-8Rr+r^2) \ge 4s^2\left(\prod(s-a)\right)^2$$

$$\Leftrightarrow s^2r^2\left\{\sum(s^2-s(b+c)+bc)\right\}(s^2-8Rr+r^2) \ge 4r^4s^4$$

$$\Leftrightarrow \left\{3s^2-s(4s)+\sum ab\right\}(s^2-8Rr+r^2) \ge 4r^2s^2$$

$$\Leftrightarrow (4Rr+r^2)(s^2-8Rr+r^2) \ge 4r^2s^2 \quad \textbf{(3)}$$

$\because s^2 \ge 16Rr-5r^2$ ***(Gerretsen), (*)***

$$\therefore s^2-8Rr+r^2 \ge 8Rr-4r^2 > 0$$

$$\Rightarrow (4Rr+r^2)(s^2-8Rr+r^2) \ge (4R+r^2)(8Rr-4r^2) \quad \textbf{(4)}$$

$(\because s^2-8Rr+r^2 \ge 8Rr-4r^2 > 0)$; ***(3), (4)*** $\Rightarrow$ ***it suffices to prove:***

$$(4R+r)(2R-r) \ge s^2 \Leftrightarrow s^2 \le 8R^2-2Rr-r^2 \quad \textbf{(5)}$$

$s^2 \le 4R^2+4Rr+3r^2$ ***(Gerretsen) (*)***

$\therefore$ ***it suffices to prove:*** $4R^2+4Rr+3r^2 \le 8R^2-2Rr-r^2$ ***(from (5))***

$\Leftrightarrow 4R^2-6Rr-4r^2 \ge 0 \Leftrightarrow (R-2r)(2R+r) \ge 0 \rightarrow$ ***true,***

$\because R \ge 2r$ ***(Euler)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\textit{In any } \Delta\, ABC,\ \frac{1}{\sum \csc^2\frac{A}{2}} \le \frac{r^2}{4}\left(\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\right)$$

$$RHS \underset{(1)}{\overset{Bergstrom}{\ge}} \frac{9r^2}{4\sum w_a^2} \ge \frac{9r^2}{4\sum s(s-a)}\left(\because w_a \le \sqrt{s(s-a)}, etc\right)$$

$$= \frac{9r^2}{4s\sum(s-a)} = \frac{9r^2}{4s(3s-2s)} = \frac{9r^2}{4s^2}$$

$$\therefore \textit{ it suffices to prove: } \frac{9r^2}{4s^2} \ge \frac{1}{\sum \csc^2\frac{A}{2}} \textit{ (from (1))} \Leftrightarrow \sum \csc^2\frac{A}{2} \ge \frac{4s^2}{9r^2}$$

$$\Leftrightarrow \frac{bc}{(s-b)(s-c)} + \frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \ge \frac{4s^2}{9r^2}$$

$$\Leftrightarrow \frac{bc(s-a)+ca(s-b)+ab(s-c)}{\prod(s-a)} \ge \frac{4s^2}{9r^2}$$

$$\Leftrightarrow \frac{s^2\sum ab - 12Rrs^2}{r^2s^2} \ge \frac{4s^2}{9r^2}$$

$$\Leftrightarrow 9(s^2-8Rr+r^2) \ge 4s^2 \Leftrightarrow 5s^2 \ge 72Rr - 9r^2 \quad (2)$$

$$\textit{But Gerretsen} \Rightarrow 5s^2 \ge 80Rr - 25r^2$$

$$\therefore \textit{ it suffices to prove: } 80Rr - 25r^2 \ge 72Rr - 9r^2 \quad \textit{(from (2))}$$

$$\Leftrightarrow 8Rr \ge 16r^2 \Leftrightarrow R \ge 2r \to \textit{true (Euler)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\textit{In any } \Delta\, ABC,\ \frac{1}{\sum \csc^2\frac{A}{2}} \le \frac{r^2}{4}\left(\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\right)$$

$$RHS \ge \frac{r^2}{4}\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}+\frac{1}{m_c^2}\right) (\because w_a \le m_a, etc)$$

$$\overset{Bergstrom}{\ge} \frac{9r^2}{4\sum m_a^2} = \frac{9r^2}{4\cdot\frac{3}{4}\sum a^2} = \frac{3r^2}{\sum a^2} \ge \frac{1}{\sum \csc^2\frac{A}{2}} \Leftrightarrow \sum \csc^2\frac{A}{2} \ge \frac{\sum a^2}{3r^2}$$

$$\Leftrightarrow \frac{bc}{(s-b)(s-c)}+\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\ge\frac{\sum a^2}{3r^2}$$

$$\Leftrightarrow \frac{bc(s-a)+ca(s-b)+ab(s-c)}{\prod(s-a)}\ge\frac{\sum a^2}{3r^2}$$

$$\Leftrightarrow \frac{s^2\sum ab-12Rrs^2}{r^2s^2}\ge\frac{\sum a^2}{3r^2}\Leftrightarrow 3(s^2-8Rr+r^2)\ge 2(s^2-4Rr-r^2)$$

$$\Leftrightarrow s^2\ge 16Rr-5r^2\rightarrow \textit{true, by Gerretsen (Proved)}$$

205. **In $\Delta\, ABC$ the following relationship holds:**

$$a+b+c\le\frac{a(2s-a)}{4(s-a)}+\frac{b(2s-b)}{4(s-b)}+\frac{c(2s-c)}{4(s-c)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Stefan-Romania

$$\textit{WLOG } a\le b\le c$$

$$\Rightarrow\begin{cases}a\le b\le c\\ \frac{2s-a}{s-a}\le\frac{2s-b}{s-b}\le\frac{2s-c}{s-c}\end{cases}\overset{Cebyshev}{\Rightarrow}\frac{\sum a}{3}\cdot\frac{\sum\frac{2s-a}{s-a}}{3}\le\frac{\sum\frac{a(2s-a)}{s-a}}{3}\Leftrightarrow$$

$$\sum\frac{a(2s-a)}{s-a}\ge\frac{1}{3}\left(\left(\sum a\right)\cdot\left(3+s\sum\frac{1}{s-a}\right)\right)\overset{Bergstrom}{\ge}\frac{1}{3}\left(\left(\sum a\right)\cdot(3+9)\right)=$$

$$=4\sum a\Leftrightarrow LHS\le RHS$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$RHS\overset{A-G}{\ge}\frac{3}{4}\sqrt[3]{\frac{abc(b+c)(c+a)(a+b)}{\prod(s-a)}}\overset{A-G}{\ge}\frac{3}{4}\sqrt[3]{\frac{8a^2b^2c^2}{\prod(s-a)}}$$

$$=\frac{6}{4}\sqrt[3]{\frac{a^2b^2c^2s}{r^2s^2}}=\frac{3}{2}\sqrt[3]{\frac{16R^2r^2s^3}{r^2s^2}}=3\sqrt[3]{2R^2s}$$

$$\geq 2s(RHS) \Leftrightarrow 27\ \times\ 2R^2s \geq 8s^3$$

$$\Leftrightarrow \frac{27}{4}R^2 \geq s^2 \Leftrightarrow s \leq \frac{3\sqrt{3}R}{2} \rightarrow$$ ***true by Mitrinovic (Proved)***

Solution 3 by Mehmet Șahin-Ankara-Turkey

We define $f(x) = \frac{x(2s-x)}{4\cdot(s-x)}$ ***then*** $f''(x) > 0;\ f$ ***is convex. Using the Jensen***

Inequality $f\left(\frac{a+b+c}{3}\right) \leq \frac{1}{3}[f(a) + f(b) + f(c)]$

$$f\left(\frac{a+b+c}{3}\right) = \frac{a+b+c}{3\cdot 4}\cdot\left(\frac{a+b+c-\dfrac{a+b+c}{3}}{\dfrac{a+b+c}{2}-\dfrac{a+b+c}{3}}\right)$$

$$= \frac{a+b+c}{12}\cdot\left(\frac{2(a+b+c)}{3\cdot\dfrac{(a+b+c)}{6}}\right)$$

$$= \frac{a+b+c}{3} \leq \frac{1}{3}\left[\frac{a\cdot(2s-a)}{4(s-a)} + \frac{b(2s-b)}{4\cdot(s-b)} + \frac{c(2s-c)}{4(s-c)}\right]$$

$$\Rightarrow a+b+c \leq \frac{a(2s-a)}{4\cdot(s-a)} + \frac{b\cdot(2s-b)}{4\cdot(s-b)} + \frac{c\cdot(2s-c)}{4\cdot(s-c)}$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

In $\Delta\ ABC,\ \sum_{cyc}\frac{a(2p-a)}{4(p-a)} \geq a+b+c$

Applying RAVI TRANSFORMATION, $a = x+y, b = y+z$ ***and*** $c = z+x$

$\therefore p = x+y+z, p-a = z, p-b = x$ ***and*** $p-c = y$

$$\therefore \sum_{cyc}\frac{a(2p-a)}{4(p-a)} = \sum_{cyc}\frac{(x+y)(x+y+2z)}{4z} = \sum_{cyc}\frac{(x+y)^2}{4z} + x+y+z$$

$$\overset{BERGSTROM}{\geq} \frac{(x+y+y+z+z+x)^2}{4(x+y+z)} + x+y+z = 2(x+y+z) = \sum_{cyc}(x+y)$$

$= a+b+c$ ***(proved)***

206. In $\Delta\, ABC$:

$$m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2\geq 9S^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Francisco Javier Garcia Capitan

$$m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2=9S^2+\frac{9}{32}\sum_{cyclic}(a^2-b^2)^2\geq 9S^2$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_a\geq\sqrt{s(s-a)},m_b\geq\sqrt{s(s-b)},m_c\geq\sqrt{s(s-c)}$$

$$\therefore\sum m_a^2m_b^2\geq\sum s(s-a)\cdot s(s-b)=s^2\sum\{(s^2-s(a+b)+ab)\}$$

$$=s^2(3s^2-4s^2+s^2+4Rr+r^2)=s^2(4Rr+r^2)\overset{?}{\geq}9S^2$$

$$\Leftrightarrow 4Rr+r^2\overset{?}{\geq}9r^2\Leftrightarrow 4Rr\overset{?}{\geq}8r^2\Leftrightarrow R\overset{?}{\geq}2r\to \textit{true (Euler) (Proved)}$$

Solution 3 by Adil Abdullayev-Baku-Azerbaidian

LEMMA 1. $m_a+m_b+m_c\geq 9r$.

LEMMA 2. $m_a\geq\sqrt{r_br_c}$.

$$m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2\geq m_am_bm_c(m_a+m_b+m_c)$$

$$\geq r_ar_br_c\cdot 9r=rp^2\cdot 9r=9S^2.$$

207. In $\Delta\, ABC, I$ – incentre. Prove that:

$$6r\leq AI+BI+CI\leq\sqrt{36r^2+12(R-2r)(R+r)}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{In any }\Delta\, ABC, 6r\overset{(1)}{\leq}\sum AI\overset{(2)}{\leq}\sqrt{36r^2+12(R-2r)(R+r)}$$

$$\sum AI = r\sum \csc\frac{A}{2} \overset{Jensen}{\geq} 3r\csc\left(\frac{A+B+C}{6}\right)$$

($\because f(x) = \csc\frac{x}{2}\ \forall x \in (0,\pi)$ ***is convex)*** $\Rightarrow \sum AI \geq 6r \Rightarrow$ ***(1) is true***

Again, $\sum AI \overset{C-B-S}{\leq} \sqrt{3}\sqrt{\sum AI^2} = \sqrt{3}r\sqrt{\sum \csc^2\frac{A}{2}}$

$$= \sqrt{3}r\sqrt{\frac{bc}{(s-b)(s-c)} + \frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-b)(s-b)}}$$

$$= \sqrt{3}r\sqrt{\frac{bc(s-a)+ca(s-b)+ab(s-c)}{\prod(s-a)}} = \sqrt{3}r\sqrt{\frac{s^2(\sum ab) - s(3abc)}{r^2s^2}}$$

$$= \sqrt{3}\sqrt{\sum ab - 12Rr} = \sqrt{3}\sqrt{s^2 - 8Rr + r^2}$$

$$= \sqrt{3s^2 - 24Rr + 3r^2} \overset{?}{\leq} \sqrt{36r^2 + 12(R-2r)(R+r)}$$

$$\Leftrightarrow s^2 - 8Rr + r^2 \overset{?}{\leq} 12r^2 + 4(R^2 - Rr - 2r^2)$$

$\Leftrightarrow s^2 \overset{?}{\leq} 4R^2 + 4Rr + 3r^2 \rightarrow$ ***true, by Gerretsen*** $\Rightarrow$ ***(2) is true (Proved)***

208. In ΔABC the centroid belongs to the incircle. Prove that:

$$a^2 + b^2 + c^2 + 6r(4R + r) = 3s^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

LEMMA. (LEYBINIS) $GI^2 = \frac{s^2+5r^2-16Rr}{9}$.

$$GI^2 = r^2 \leftrightarrow s^2 = 4r^2 + 16Rr \leftrightarrow s^2 = 4r(r + 4R)$$

$$LHS = 2(s^2 - r^2 - 4Rr) + 6r(r + 4R) =$$

$$= 2s^2 + 4r(r + 4R) = 3s^2 = RHS.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum a^2 + 6r(4R+r) \overset{(1)}{=} 3s^2;$$

$$(1) \Leftrightarrow 2(s^2 - 4Rr - r^2) + 24Rr + 6r^2 = 3s^2$$

$$\Leftrightarrow s^2 = 16Rr + r^2 \text{ (2)}$$

By problem, $GI = r \Rightarrow GI^2 = r^2 \Rightarrow -\frac{\sum a^3 - 2(\sum a^2b + \sum ab^2)}{18s} = r^2$

$$\Rightarrow \sum a^3 - 2\left(\sum a^2b + \sum ab^2\right) + 9abc = -18r^2s$$

$$\Rightarrow 3abc + 2s\left(\sum a^2 - \sum ab\right) - 2\left\{\sum\left(ab(2s-c)\right)\right\} + 9abc = -18r^2S$$

$$\Rightarrow 3abc + 2s\left(\sum a^2\right) - 2s\left(\sum ab\right) - 2\left\{2s\left(\sum ab\right) - 3abc\right\} + 9abc = -18r^2s$$

$$\Rightarrow 3abc + 2s\left(\sum a^2\right) - 2s\left(\sum ab\right) - 4s\left(\sum ab\right) + 6abc + 9abc = -18r^2s$$

$$\Rightarrow 72Rrs + 4s(s^2 - 4Rr - r^2) - 6s(s^2 + 4Rr + r^2) = -18r^2s$$

$$\Rightarrow 36Rr + 2(s^2 - 4Rr - r^2) - 3(s^2 + 4Rr + r^2) = -9r^2$$

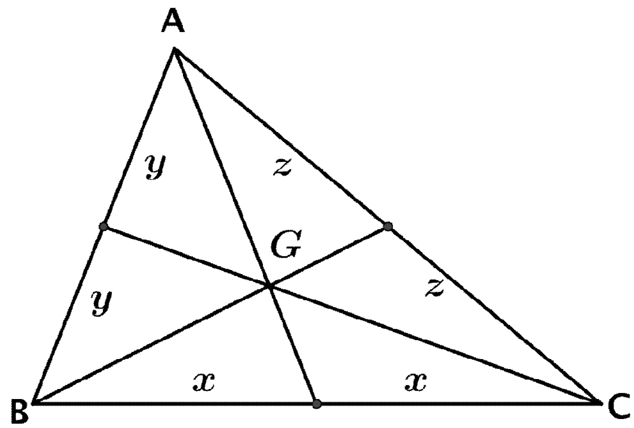
$$\Rightarrow s^2 = 16Rr + 4r^2 \Rightarrow \text{(2) is true (Proved)}$$

209. **Let $\Omega_a, \Omega_b, \Omega_c$ be the circumradii of $\Delta BGC, \Delta CGA$ respectively ΔAGB, G - the centroid of ΔABC. Prove that:**

$$27\Omega_a\Omega_b\Omega_c \geq 4Rs^2$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty – Kolkata – India



$$2y + x = 2z + x \Rightarrow y = z; 2z + y = 2x + y \Rightarrow x = z \Rightarrow x = y = z$$

$$\therefore area\ (\Delta BCG) = area\ (\Delta CGA) = area\ (\Delta\ AGB) = \frac{2}{6} area\ (\Delta\ ABC)$$

$$= \frac{area\ (\Delta\ ABC)}{3} = \frac{S}{3}.\text{ Also, } AG = \frac{2}{3}m_a, BG = \frac{2}{3}m_b, CG = \frac{2}{3}m_c$$

$$\Omega_a = \frac{\frac{2}{3}m_b \cdot \frac{2}{3}m_c \cdot a}{\frac{4S}{3}} = \frac{1}{3} \cdot \frac{am_bm_c}{S} = \frac{am_bm_c}{3S}$$

$$\text{Similarly, } \Omega_b = \frac{bm_cm_a}{3S} \text{ and } \Omega_c = \frac{cm_am_b}{3S}$$

$$\therefore 27\Omega_a\Omega_b\Omega_c = \frac{abc(m_am_bm_c)^2}{S^3} = \frac{4RS(m_am_bm_c)^2}{S^3} \geq 4Rs^2 \quad (RHS)$$

$$\Leftrightarrow \frac{(\prod m_a)^2}{S^2} \geq s^2 \Rightarrow \prod m_a \geq Ss = \quad \textbf{(a)}$$

$$= s \cdot rs = rs^2.\text{ Now, } m_a \geq \sqrt{s(s-a)} \quad \textbf{(1)}\ m_b \geq \sqrt{s(s-b)} \quad \textbf{(2)}$$

$$m_c \geq \sqrt{s(s-c)} \quad \textbf{(3)}$$

Proof.

$$m_a = \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2} \overset{Chebyshev}{\geq} \frac{\sqrt{(b+c)^2 - a^2}}{2} = \frac{\sqrt{2s}\sqrt{2(s-a)}}{2} =$$

$$= \sqrt{s(s-a)}.\ \textit{Similarly, } m_b \geq \sqrt{s(s-b)} \textit{ and } m_c = \sqrt{s(s-c)}$$

$$\therefore 1 \times 2 \times 3 \Rightarrow \prod m_a \geq \sqrt{s^2(s-a)(s-b)(s-c)}$$

$$= s\sqrt{s(s-a)(s-b)(s-c)} = sS = rs^2 \Rightarrow a \textit{ is true (Proved)}$$

210. If in ΔABC, O - circumcentre, N - Nagel's point then:

$$ON \leq R\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} - 3\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakrabory – Kolkata - India

$$ON = \frac{4S \cdot (OI)^2}{abc} = \frac{4S \cdot (R(R-2r))}{4RS} = R - 2r.\ \textit{Given inequality} \Leftrightarrow \frac{R-2r}{R} \leq \sum \frac{a^2}{bc} - 3$$

$$\Leftrightarrow 1 - \frac{2r}{R} \leq \sum \frac{a^2}{bc} - 3 \Leftrightarrow \sum \frac{a^2}{bc} \geq 4 - \frac{2r}{R}$$

$$\Leftrightarrow \frac{\sum a^3}{abc} \geq 4 - \frac{2r}{R} \Leftrightarrow 3abc + 2s\left(\sum a^2 - \sum abc\right) \geq 16Rrs - 8r^2 s$$

$$\Leftrightarrow 12Rrs + 2s\{2(s^2 - 4Rr - r^2) - (s^2 + 4Rr + r^2)\} \geq 16Rrs - 8r^2 s$$

$$\Leftrightarrow 2s(s^2 - 12Rr - 3r^2) \geq 4Rrs - 8r^2 s$$

$$\Leftrightarrow s^2 - 12Rr - 3r^2 \geq 2Rr - 4r^2$$

$$\Leftrightarrow s^2 \geq 14Rr - r^2 \quad \textit{Gerretsen} \Rightarrow s^2 \geq 16Rr - 5r^2$$

$$\therefore \textit{it suffices to prove: } 16Rr - 5r^2 \geq 14Rr - r^2$$

$$\Leftrightarrow 2Rr \geq 4r^2 \Leftrightarrow R \geq 2r \rightarrow \textit{true (Euler) (Proved)}$$

Solution 2 by Saptak Bhattacharya-Kolkata-India

$$ON = \frac{4\Delta\, OI^2}{abc} = \frac{4\Delta R(R-2r)}{abc}\ \textit{(Euler)} = R - 2r.\ \textit{To show,}$$

$$\frac{R-2r}{R} \leq \sum \frac{a^2}{bc} - 3 \Rightarrow 1 - \frac{2r}{R} \leq \sum \frac{a^2}{bc} - 3$$

Now, we know, $\sum \cos A = 1 + \frac{r}{R}$***;Thus, to show,***

$$1 - \left(2\sum \cos A - 2\right) \le \sum \frac{a^2}{bc} - 3 \Rightarrow 6 \le \sum \frac{a^2}{bc} + \sum 2\cos A;$$

Now, by cosine rule; $2\cos A = \frac{b^2+c^2-a^2}{bc}$***. So, to show,***

$\sum\left(\frac{a^2}{bc} + \frac{b^2+c^2-a^2}{bc}\right) \ge 6 \Rightarrow \sum\left(\frac{b}{c} + \frac{c}{b}\right) \ge 6$;***which is true by AM*** $\ge$ ***GM***

211. In acute-angled ΔABC**:**

$$B(1 + \sin A)^{\sin A} + C(1 + \sin B)^{\sin B} + A(1 + \sin C)^{\sin C} < 2\pi$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Shahlar Maharramov-Jebrail-Azerbaidian

Since A, B, C ***acute-angled*** $0 < \sin A < 1$***, same*** $\sin B$, $\sin C$***. Then***

$(1 + \sin A)^{\sin A}$ ***increasing function, as*** $(1 + \sin B)^{\sin B}$ ***and***

$$(1 + \sin C)^{\sin C} \Rightarrow \sum A(\sin A + 1)^{\sin A} \le \sum A(1 + 1)^1 =$$

$$= 2A + 2B + 2C = 2(A + B + C) = 2 \cdot \pi = 2\pi$$

Solution 2 by Soumava-Chakraborty-Kolkata-India

Bernoulli's inequality states that:

$(1 + x)^r \le 1 + rx, \forall\, 0 \le r \le 1$ ***and*** $x \ge -1$

Putting $x = \sin A$ ***and*** $r = \sin A$***, we get,***

$(1 + \sin A)^{\sin A} \le 1 + \sin^2 A \Rightarrow B(1 + \sin A)^{\sin A} \le B(1 + \sin^2 A)$ ***(1)***

Similarly, $C(1 + \sin B)^{\sin B} \le C(1 + \sin^2 B)$ ***(2) and***

$A(1 + \sin C)^{\sin C} \le A(1 + \sin^2 C)$ ***(3)***

(1) + ***(2)*** + ***(3)*** $\Rightarrow LHS < B(1 + \sin^2 A) + C(1 + \sin^2 B) + A(1 + \sin^2 C)$

($\because$ ***all of*** $\sin A$, $\sin B$, $\sin C$ ***can't be*** $= 1$ ***simultaneoulsy)***

$$= (A + B + C) + B\sin^2 A + C\sin^2 B + A\sin^2 C$$

$$< \pi + B(1) + C(1) + A(1) = \pi + \pi = 2\pi$$

***(** $\because \sin^2 A \le 1$ **, etc and of all** $\sin A$, $\sin B$, $\sin C$ **can't be** $= 1$

simultaneously) (Proved)

Solution 3 by Ravi Prakash-New Delhi-India

***For** $0 < x < 1, 1 < 1 + x < 2 \Rightarrow (1 + x)^x < 2^x < 2$

***and for** $x = 1, (1 + x)^x = 2$

As, at most one of** $\sin A$, $\sin B$, $\sin C$ **is** 1 **and rest are less than 1

$$B(1 + \sin A)^{\sin A} + C(1 + \sin B)^{\sin B} + A(1 + \sin C)^{\sin C}$$

$$< 2(B + C + A) = 2\pi$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

Have,** $0 < \sin A, \sin B, \sin C < 1$ **since,** ΔABC **is acute triangle

$$\sum_{cyc} B(1 + \sin A)^{\sin A} \overset{CAUCHY-SCHWARZ}{\le} \sqrt{(A + B + C)\left(\sum_{cyc}(1 + \sin A)^{\sin A}\right)}$$

$$< \sqrt{\prod (2 + 2 + 2)} = \sqrt{6\pi} < 2\pi$$

[$\because 4\pi^2 > 6\pi \Leftrightarrow 4\pi\left(\pi - \frac{3}{2}\right) > 0$***, which is true]***

$$\therefore \sum_{cyc} B(1 + \sin A)^{\sin A} < 2\pi$$

Solution 5 by Eliezer Okeke-Nigeria

$$\sum B\,(1 + \sin A)^{\sin A} < 2\pi$$

***We know** $(1 + x)^x < 1 + x^2$**, for** $0 < x < 1 \Rightarrow (1 + \sin A)^{\sin A} < 1 + \sin^2 A$

$$\textit{We know } \sin^2 A < \sin A = \left(0,\frac{\pi}{2}\right) \Rightarrow 1 + \sin^2 A < 1 + \sin A$$

$$\textit{We know } \sin A < 1 \ \textit{in} \left(0,\frac{\pi}{2}\right) \Rightarrow 1 + \sin A < 1 + 1 = 2$$

$$\sum B(1 + \sin A)^{\sin A} < \sum B\,(2) = 2\sum B = 2\pi \ \textit{(Proved)}$$

212. In $\Delta ABC: I -$ incentre

$$AI \cdot BI \cdot CI \ge 8rS\sqrt{\left(1-\frac{2r}{h_a}\right)\left(1-\frac{2r}{h_b}\right)\left(1-\frac{2r}{h_c}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Francisco Javier Garcia Capitan-Spain

$$\textit{From } AI^2 = \frac{bc(s-a)}{s}, AI^2 = \frac{ca(s-b)}{s}, CI^2 = \frac{ab(s-c)}{s} \textit{ we get}$$

$$AI^2 \cdot BI^2 \cdot CI^2 = 16r^4R^2.$$

On the other hand, from $S = sr = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$*, we can write*

$$\left(1-\frac{2r}{h_a}\right)\left(1-\frac{2r}{h_b}\right)\left(1-\frac{2r}{h_b}\right) = \left(1-\frac{a}{s}\right)\left(1-\frac{b}{s}\right)\left(1-\frac{c}{s}\right)$$

$$= \frac{(s-a)(s-b)(s-c)}{s^3} = \frac{r^2}{s^2}. \textit{ Therefore we have}$$

$$16r^4R^2 - 64r^2S^2 \cdot \frac{r^2}{s^2} = 16r^4R^2 - 64r^6 = 16r^4(R^2-4r^2)16r^4(R+2r)(R-2r) \ge 0$$

Solution 2 by Nirapada Pal-India

In any triangle, $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ *(1) and* $r = \frac{\Delta}{s}$*. Now,* $\frac{1}{2} \times h_a \times a = \Delta$

$$\Rightarrow \frac{2}{h_a} = \frac{a}{\Delta} \therefore \frac{2r}{h_a} = \frac{ar}{\Delta} = \frac{a}{s} \textit{ etc. So } 8r\Delta\sqrt{\left(1-\frac{r}{h_a}\right)\left(1-\frac{r}{h_b}\right)\left(1-\frac{r}{h_c}\right)} =$$

$$= 8r\Delta\sqrt{\left(1-\frac{a}{s}\right)\left(1-\frac{b}{s}\right)\left(1-\frac{c}{s}\right)} = 8r\Delta\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s^2}$$

$$=\frac{8r\Delta^2}{s^2}=8r^2\le 4r^2\times R \text{ as } R\ge 2r=\frac{r^3}{\frac{r}{4R}}=\frac{r^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \text{ using (1)}$$

$$=\frac{r}{\sin\frac{A}{2}}\cdot\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}=AI\cdot BI\cdot CI$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\text{In any } \Delta\, ABC, AI\cdot BI\cdot CI \underset{(1)}{\ge} 8rS\sqrt{\left(1-\frac{2r}{h_a}\right)\left(1-\frac{2r}{h_b}\right)\left(1-\frac{2r}{h_c}\right)}$$

$$1-\frac{2r}{h_a}=1-\frac{2r}{\frac{2\Delta}{a}}=1-\frac{2ra}{2rs}=1-\frac{a}{s}=\frac{s-a}{s}$$

$$\text{Similarly, } 1-\frac{2r}{h_b}=\frac{s-b}{s} \text{ and } 1-\frac{2r}{h_c}=\frac{s-c}{s}$$

$$\therefore RHS=8r^2s\sqrt{\frac{(s-a)(s-b)(s-c)\cdot s}{s^3\cdot s}}=8r^2s\sqrt{\frac{r^2s^2}{s^4}}$$

$$=8r^2s\cdot\frac{rs}{s^2}=\frac{8r^3s^2}{s^2}\underset{(2)}{=}8r^3$$

$$LHS\underset{(3)}{=}\frac{r^3}{\prod\sin\frac{A}{2}} \therefore (1)\Leftrightarrow\frac{r^3}{\prod\sin\frac{A}{2}}\ge 8r^3 \text{ (from (2), (3))}$$

$$\Leftrightarrow\prod\sin\frac{A}{2}\le\frac{1}{8}\to \text{true (Proved)}$$

213. In $\Delta\, ABC$:

$$r^5\sum\sin^5\frac{A}{2}\left(\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)^5>\sum\left(a\sin\frac{A}{2}\right)^5$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

We shall first prove:

$$r^5\sin^5\frac{A}{2}\left(\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)^5>\left(a\sin\frac{A}{2}\right)^5\quad \textit{(1)}$$

$$\Leftrightarrow r\sin\frac{A}{2}\left(\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)>a\sin\frac{A}{2}$$

$$\Leftrightarrow rs\left(\frac{\sqrt{ca}}{\sqrt{(s-c)(s-a)}}+\frac{\sqrt{ab}}{\sqrt{(s-b)(s-c)}}\right)>as$$

$$\Leftrightarrow \sqrt{s(s-a)(s-b)(s-c)}\left(\frac{\sqrt{ca}\sqrt{s-b}+\sqrt{ab}\sqrt{s-c}}{\sqrt{(s-a)(s-b)(s-c)}}\right)>as$$

$$\Leftrightarrow \sqrt{c(s-b)}+\sqrt{b(s-c)}>\sqrt{as}$$

$$\Leftrightarrow c(s-b)+b(s-c)+2\sqrt{bc(s-b)(s-c)}>as\ \textit{(squaring)}$$

$$\Leftrightarrow s(b+c-a)+\sqrt{bc(a+b-c)(c+a-b)}>2bc$$

$$\Leftrightarrow (b+c+a)(b+c-a)+2\sqrt{bc(a+b-c)(c+a-b)}>4bc$$

$$\Leftrightarrow (b+c)^2-4bc-a^2+2\sqrt{bc(a+b-c)(c+a-b)}>0$$

$$\Leftrightarrow (b-c)^2-a^2+2\sqrt{bc(a+b-c)(c+a-b)}>0$$

$$\Leftrightarrow 2\sqrt{bc(a+b-c)(c+a-b)}>(a+b-c)(c+a-b)$$

$$\Leftrightarrow 2\sqrt{bc}>\sqrt{(a+b-c)(c+a-b)}$$

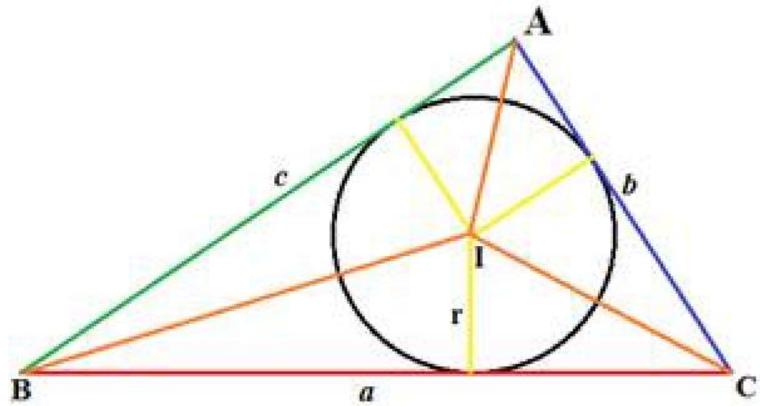
$$\Leftrightarrow 4bc>a^2-(b-c)^2\Leftrightarrow (b+c)^2>a^2\Leftrightarrow b+c>a\rightarrow \textit{true}$$

$\therefore$ ***(1) is true***

$$\Rightarrow \sum r^5\sin^5\frac{A}{2}\left(\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)^5>\sum\left(a\sin\frac{A}{2}\right)^5$$

(Proved)

Solution 2 by Nirapada Pal-India



$$r^5\sum\sin^5\frac{A}{2}\left(\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)^5=\sum\sin^5\frac{A}{2}\left(\frac{r}{\sin\frac{B}{2}}+\frac{r}{\sin\frac{C}{2}}\right)^5$$

$$=\sum\sin^5\frac{A}{2}(BI+CI)^5\ \ [\text{in } \Delta\, BIC, BI+CI>AB=a]$$

$$>\sum\sin^5\frac{A}{2}a^5=\sum\left(a\sin\frac{A}{2}\right)^5$$

214. **Prove that in any triangle:**

$$\frac{R}{r}\geq\frac{r_a}{r_b+r_c}+\frac{r_b}{r_c+r_a}+\frac{r_c}{r_a+r_b}+\frac{1}{2}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian, Marian Ursărescu-Romania

Solution 1 by Daniel Sitaru – Romania

$$\sum\frac{r_a}{r_b+r_c}=\sum\frac{\frac{S}{s-a}}{\frac{S}{s-b}+\frac{S}{s-c}}=\sum\frac{(s-b)(s-c)}{(2s-b-c)(s-a)}=$$

$$=\sum\frac{(s-b)(s-c)}{a(s-a)}\overset{AM-GM}{\leq}\sum\frac{\left(\frac{s-a+s-a}{2}\right)^2}{a(s-a)}=\sum\frac{a^2}{4a(s-a)}=$$

$$=\frac{1}{4}\sum\frac{a}{s-a}=\frac{1}{4}\cdot\frac{2(2R-r)}{r}=\frac{R}{r}-\frac{1}{2}$$

Solution 2 by Nirapada Pal-Jhargram-India

In any triangle $r_a+r_b+r_c-r=4R$

We have $r_a=\frac{\Delta}{s-a},r_b=\frac{\Delta}{s-b},r_c=\frac{\Delta}{s-c},r=\frac{\Delta}{s}$

So, $r_a+r_b+r_c-r=\Delta\left[\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right]=\Delta\left[\frac{s-a+s-b}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}\right]$

$$=\Delta\left[\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right]=\Delta c\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]=$$

$$=\frac{\Delta c}{\Delta^2}[s^2-sc+s^2-sa-sb+ab]$$

$$=\frac{c}{\Delta}[2s^2-s(a+b+c)+ab]=\frac{c}{\Delta}[2s^2-s\cdot 2s+ab]=\frac{abc}{\Delta}=4R$$

$\therefore r_a+r_b+r_c=4R+r$ ***... (1)***

Now, $\sum\frac{r_a}{r_b+r_c}=\sum\frac{(s-b)(s-c)}{a(s-a)}\overset{AGM}{\leq}\sum\frac{\left(\frac{s-b+s-c}{2}\right)^2}{a(s-a)}=\sum\frac{\left(\frac{a}{2}\right)^2}{a(s-a)}=\frac{1}{4}\sum\frac{a}{s-a}=\frac{1}{4\Delta}\sum ar_a$

$$=\frac{1}{4\Delta}\sum(s-(s-a))r_a=\frac{2}{4\Delta}(r_a+r_b+r_c)-\frac{1}{4\Delta}\sum(s-a)r_a=$$

$=\frac{s}{4\Delta}(4R+r)-\frac{3}{4}$ ***[Since*** $(s-a)r_a=\Delta, rs=\Delta$***]***

$=\frac{1}{4r}(4R+r)-\frac{3}{4}$ ***[Using (1)]***$=\frac{R}{r}-\frac{1}{2}$

$$\therefore\sum\frac{r_a}{r_b+r_c}+\frac{1}{2}\leq\frac{R}{r}$$

Solution 3 by Rahim Shahbazov-Baku-Azerbaidian

$a=x+y,b=y+z,c=z+x$ ***inequality becomes***

$\frac{(x+y)(y+z)(x+z)}{4xyz}\geq\frac{xy}{z(x+y)}+\frac{yz}{x(y+z)}+\frac{xz}{y(x+z)}+\frac{1}{2}$ ***or***

$$(x+y)(y+z)(x+z)\geq\sum\frac{4x^2y^2}{x+y}+2xyz \quad \textit{or} \sum xy(x+y)\geq\sum\frac{4x^2y^2}{x+y}\Rightarrow$$

$$\Rightarrow\sum xy\left(x+y-\frac{4xy}{x+y}\right)\geq 0\Rightarrow\sum\frac{xy(x-y)^2}{x+y}\geq 0 \quad \textit{true}$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$R=\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}};r=\frac{\sqrt{xyz\cdot(x+y+z)}}{x+y+z};\frac{R}{2}=\frac{(x+y)(y+z)(z+x)}{4xyz}:LHS$$

$$\sum\frac{r_a}{r_b+r_c}+\frac{1}{2}=\sum\frac{yz}{(z+y)\cdot x}+\frac{1}{2}=\sum\frac{xy}{(x+y)\cdot z}+\frac{1}{2}=$$

$$=\frac{\sum 2(xy)^2\cdot(z+x)(y+z)+xyz(x+y)(y+z)(z+x)}{2xyz\cdot(x+y)\cdot(y+z)\cdot(z+x)}$$

$$\Downarrow$$

$$RHS$$

$$LHS\geq RHS \ \textbf{(ASSURE)}$$

$$\frac{(x+y)(y+z)(z+x)}{4xyz}\geq\frac{\sum 2(xy)^2\cdot(z+x)(y+z)+xyz(x+y)(y+z)(z+x)}{2xyz(x+y)(y+z)(z+x)}\Rightarrow$$

$$\Rightarrow(x+y)^2(y+z)^2(z+x)^2\geq\sum 4(xy)^2(z+x)(y+z)+2xyz(x+y)(y+z)(z+x)$$

$$(x+y)(y+z)(z+x)\big((x+y)(y+z)(z+x)-2xyz\big)\geq\sum 4(xy)^2(z+x)(y+z)$$

$$(x+y)(y+z)(z+x)\cdot\left(\sum xy(x+y)\right)=$$

$$=\sum xy(x+y)^2(y+z)(z+x)\underset{Cauchy}{\geq}\sum 4(xy)^2(y+z)(z+x)$$

215. Let R_a, R_b, R_c be the circumradius of $\Delta\ BOC, \Delta\ COA, \Delta\ AOB$ respectively, where O is the circumcenter of an acute $\Delta\ ABC$. Prove that

$$13\left(\frac{2r}{R}\right)^2-12\leq\frac{R_a}{2R_a+R}+\frac{R_b}{2R_b+R}+\frac{R_c}{2R_c+R}\leq\left(\frac{R}{2r}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

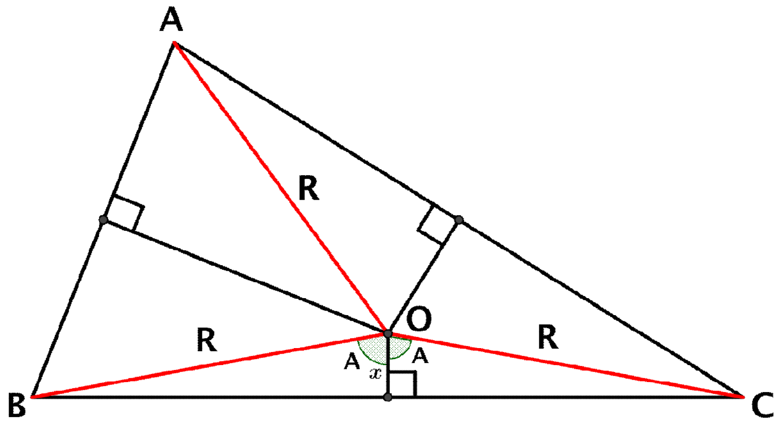
R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Solution by Soumava Chakraborty-Kolkata-India

$$R_a, R_b, R_c \rightarrow \textit{circumradius of } \Delta\, BOC, \Delta\, COA, \Delta\, AOB$$

$$\textit{Then, } 13\left(\frac{2r}{R}\right)^2 - 12 \underbrace{\leq}_{(a)} \frac{R_a}{2R_a+R} + \frac{R_b}{2R_b+R} + \frac{R_c}{2R_c+R} \underbrace{\leq}_{(b)} \left(\frac{R}{2r}\right)^2$$



$$R_a = \frac{R^2 a}{4\cdot\frac{1}{2}ax} = \frac{R^2}{2x};\ \frac{x}{R} = \cos A \Rightarrow x = R\cos A \therefore R_a = \frac{R^2}{2R\cos A} = \frac{R}{2\cos A}$$

$$\textit{Similarly, } R_b = \frac{R}{2\cos B} \textit{ and } R_c = \frac{R}{2\cos C}$$

$$\therefore \sum \frac{R_a}{2R_a + R} = \sum \frac{\frac{R}{2\cos A}}{\frac{R}{\cos A} + R} \overset{(1)}{=} \sum \frac{1}{2 + 2\cos A}$$

$$\textit{Now, } \sum \frac{1}{1+\cos A} \underbrace{\overset{Bergstrom}{\geq}}_{(2)} \frac{9}{3+1+\frac{r}{R}} = \frac{9R}{4R+r}$$

$$\therefore \sum \frac{R_a}{2R_a+R} = \frac{1}{2}\sum \frac{1}{1+\cos A} \textit{ (by (1))} \geq \frac{9R}{8R+2r} \textit{ (by (2))} \overset{?}{\geq} 13\left(\frac{2r}{R}\right)^2 - 12$$

$$\Leftrightarrow \frac{105R+24R}{8R+2r} \overset{?}{\geq} \frac{52r^2}{R^2} \Leftrightarrow 105t^3 + 24t^2 - 416t - 104 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(105t^2 + 234t + 52) \overset{?}{\geq} 0 \rightarrow \textit{true}, \because t \geq 2 \Rightarrow \textit{(a) is true}$$

Again, $\sum\frac{R_a}{2R_a+R}=\frac{1}{2}\sum\frac{1}{1+\cos A}$ *(by (1))*

$$=\frac{1}{4}\sum\sec^2\frac{A}{2}\overset{?}{\leq}\frac{R^2}{4r^2}\Leftrightarrow\sum\sec^2\frac{A}{2}\overset{?}{\leq}\frac{R^2}{r^2}\quad (3)$$

Now, $\sum\sec^2\frac{A}{2}=\frac{1}{s}\left(\frac{bc}{s-a}+\frac{ca}{s-b}+\frac{ab}{s-c}\right)$ *(i)*

WLOG, we may assume $a\geq b\geq c$.

Then $bc\leq ca\leq ab$ *and* $\frac{1}{s-a}\geq\frac{1}{s-b}\geq\frac{1}{s-c}$

$$\therefore\sum\sec^2\frac{A}{2}\underset{(4)}{\overset{Chebyshev}{\leq}}\frac{1}{3s}\left(\sum ab\right)\left(\sum\frac{1}{s-a}\right)\ \textit{(using (i))}$$

$$=\frac{1}{3s}(s^2+4Rr+r^2)\cdot\frac{\sum(s-b)(s-c)}{\prod(s-a)}$$

$$=\frac{1}{3s}(s^2+4Rr+r^2)\cdot\frac{s(3s^2-4s^2+\sum ab)}{r^2s^2}=\frac{(s^2+4Rr+r^2)(4Rr+r^2)}{3r^2s^2}$$

(3), (4) $\Rightarrow$ *it suffices to prove:* $\frac{(s^2+4Rr+r^2)(4Rr+r^2)}{3r^2s^2}\leq\frac{R^2}{r^2}$

$$\Leftrightarrow(s^2+4Rr+r^2)(4Rr+r^2)\leq 3R^2s^2\quad (4)$$

Now, $(s^2+4Rr+r^2)(4Rr+r^2)\underset{(ii)}{\overset{Gerretsen}{\leq}}(4R^2+8Rr+4r^2)(4Rr+r^2)$

Again $3R^2s^2\underset{(iii)}{\overset{Gerretsen}{\geq}}3R^2(16Rr-5r^2)$

(ii), (ii), (4) $\Rightarrow$ *it suffices to prove:*

$$3R^2(16Rr-5r^2)\geq(4R^2+8Rr+4r^2)(4Rr+r^2)$$

$$\Leftrightarrow 32R^3-47R^2r-32Rr^2-4r^3\geq 0$$

$\Leftrightarrow(t-2)(32t^2+17t+2)\geq 0\rightarrow$ *true* $\left(t=\frac{R}{r}\right)\because t\geq 2$ *(Euler)*

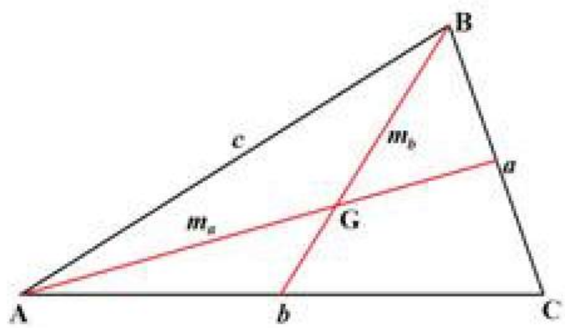
$\Rightarrow$ *(b) is true (Proved)*

216. **In $\Delta\ ABC$ the following relationship holds:**

$$16\sum\left(\frac{m_a}{m_c}+\frac{m_b}{m_c}\right)^4>81\left(\left(\frac{a}{m_a}\right)^4+\left(\frac{b}{m_b}\right)^4+\left(\frac{c}{m_c}\right)^4\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Nirapada Pal-Jhargram-India



***Let** G **be the centroid. Then** $AG=\frac{2}{3}m_a$ **and** $BG=\frac{2}{3}m_b$*

***In** $\Delta\ ABG$, $AG+BG>AB$. **Or,** $\frac{2}{3}(m_a+m_b)>c$. **Or,** $2(m_a+m_b)>3c$*

Similarly,** $2(m_b+m_c)>3a$, $2(m_c+m_a)>3b$ **... (1)

Now,** $16\sum\left(\frac{m_a}{m_c}+\frac{m_b}{m_c}\right)^4=\sum\left(\frac{2(m_a+m_b)}{m_c}\right)^4>\sum\left(\frac{3c}{m_c}\right)^4$ **[using (1)]

$$=81\sum\left(\frac{c}{m_c}\right)^4=81\left(\left(\frac{a}{m_a}\right)^4+\left(\frac{b}{m_b}\right)^4+\left(\frac{c}{m_c}\right)^4\right)$$

217. **ADIL ABDULLAYEV'S GENERALIZED INEQUALITIES**

In $\Delta\ ABC$:

$$\frac{(xh_a+yh_b+zh_c)(xm_a+ym_b+zm_c)}{xm_ah_a+ym_bh_b+zm_ch_c}\le x+y+z;\ x,y,z\in\mathbb{N}^*$$

$$\frac{(xr_a+yr_b+zr_c)(xm_a+ym_b+zm_c)}{xr_ah_a+yr_bh_b+zr_ch_c}\ge x+y+z;\ x,y,z\in\mathbb{N}^*$$

Proposed by Daniel Sitaru – Romania

Solution by Adil Abdullayev-Baku-Azerbaidian

$$\textbf{1) Let } a \le b \le c \to \left.\begin{matrix} h_a \ge h_b \ge h_c \\ m_a \ge m_b \ge m_c \end{matrix}\right\}.$$

$$\textbf{Chebyshev's} \to (xh_a + yh_b + zh_c)(xm_a + ym_b + zm_c)$$

$$\le (xh_am_a + yh_bm_b + zh_cm_c)(x + y + z) \to$$

$$LHS \le RHS.$$

$$\textbf{2) Let} \le b \le c \to \left.\begin{matrix} m_a \ge m_b \ge m_c \\ r_a \le r_b \le r_c \end{matrix}\right\}.$$

$$\textbf{Chebyshev's} \to (xr_a + yr_b + zr_c)(xm_a + ym_b + zm_c)$$

$$\ge (xr_am_a + yr_bm_b + zr_cm_c)(x + y + z) \to$$

$$LHS \ge RHS.$$

218. In $\Delta\ ABC$ the following relationship holds:

$$\frac{\sin^2\frac{A}{3}\sin^2\frac{B}{3}\sin^2\frac{C}{3}}{\sin A \sin B \sin C} < \frac{\sqrt{3}}{24}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$p \ge 3\sqrt{3}r \Rightarrow p > \sqrt{3}r \Rightarrow \frac{r}{p} < \frac{\sqrt{3}}{3} \ \textbf{(1)}$$

$$\frac{r}{p} = \frac{r \cdot p}{p^2} = \frac{S}{p^2} = \sqrt{\frac{(p-a)(p-b)(p-c)}{p^2}} =$$

$$= \frac{\sqrt{\frac{(p-a)(p-b)}{ab} \cdot \frac{(p-b)(p-c)}{bc} \cdot \frac{(p-c)(p-a)}{ca}}}{\sqrt{\frac{p \cdot (p-a)}{bc} \cdot \frac{p(p-b)}{ca} \cdot \frac{p(p-c)}{ab}}} =$$

$$=\frac{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}==\frac{8\cdot\sin^2\frac{A}{2}\cdot\sin^2\frac{B}{2}\cdot\sin^2\frac{C}{2}}{\sin A\cdot\sin B\cdot\sin C}$$

$$\frac{r}{p}=\frac{8\cdot\sin^2\frac{A}{2}\cdot\sin^2\frac{B}{2}\cdot\sin^2\frac{C}{2}}{\sin A\cdot\sin B\cdot\sin C}\quad (2)$$

$$\left(0,\frac{\pi}{2}\right)\Rightarrow\sin\frac{A}{3}<\sin\frac{A}{2}\binom{SIMILARLY}{B;C}\quad (3)$$

(1); (2); (3) $\Rightarrow\frac{\sqrt{3}}{3}>\frac{8\cdot\sin^2\frac{A}{2}\cdot\sin^2\frac{B}{2}\cdot\sin^2\frac{C}{2}}{\sin A\cdot\sin B\cdot\sin C}>\frac{8\cdot\sin^3\frac{A}{3}\cdot\sin^3\frac{B}{2}\cdot\sin^3\frac{C}{3}}{\sin A\cdot\sin B\cdot\sin C},\ \frac{\sqrt{3}}{24}>\frac{\prod\sin^3\frac{A}{3}}{\prod\sin A}$

Solution 2 by Soumava Chakraborty - Kolkata-India

In any $\Delta\ ABC,\frac{\sin^2\frac{A}{3}\sin^2\frac{B}{3}\sin^2\frac{C}{3}}{\sin A\sin B\sin C}<\frac{\sqrt{3}}{24}$

$\because 0<\frac{A}{3}<\frac{A}{2}<\frac{\pi}{2}$, **and** $\therefore\sin x$ **is increasing on** $\left(0,\frac{\pi}{2}\right),\therefore\sin\frac{A}{3}<\sin\frac{A}{2}$

with similar argument, $\sin\frac{B}{3}<\sin\frac{B}{2}$ **and** $\sin\frac{C}{3}<\sin\frac{C}{2}$

$\therefore LHS\underset{(1)}{<}\frac{\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{8\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}=\frac{1}{8}\left(\prod\tan\frac{A}{2}\right)$. **Now,** $\tan\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)=\frac{\sum\tan\frac{A}{2}-\prod\tan\frac{A}{2}}{1-\sum\tan\frac{A}{2}\tan\frac{B}{2}}$

and $\tan 90^\circ$ **is undefined,** $\because\sum\tan\frac{A}{2}\tan\frac{B}{2}=1\quad$ **(2)**

By AM – GM, $\sum\tan\frac{A}{2}\tan\frac{B}{2}\geq 3\sqrt[3]{\left(\prod\tan\frac{A}{2}\right)^2}\Rightarrow 1\geq 27\left(\prod\tan\frac{A}{2}\right)^2$ **(using (1))**

$\Rightarrow\frac{1}{3\sqrt{3}}\geq\prod\tan\frac{A}{2}\Rightarrow\prod\tan\frac{A}{2}\leq\frac{\sqrt{3}}{9}$ **(3); (1), (3)** $\Rightarrow LHS<\frac{1}{8}\cdot\frac{\sqrt{3}}{9}=\frac{\sqrt{3}}{72}<\frac{\sqrt{3}}{24}$

219. Prove that in any triangle

$$\frac{(r_a+r_b+3r_c)(m_a+m_b+3m_c)}{r_am_a+r_bm_b+3r_cm_c}\geq 5$$

Proposed by Adil Abdullayev – Baku – Azerbaidian

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$a \geq b \geq c$$
$$m_c \geq m_b \geq m_a \quad \textit{Chebyshev}$$
$$r_a \geq r_b \geq r_c$$

$$(r_a + r_b + r_c + r_c + r_c) \cdot (m_a + m_b + m_c + m_c + m_c) \overset{Chebyshev}{\geq}$$
$$\geq 5 \cdot (r_a \cdot m_a + r_b \cdot m_b + 3 \cdot r_c \cdot m_c)$$
$$\frac{(r_a + r_b + 3r_c) \cdot (m_a + m_b + m_c)}{r_a \cdot m_a + r_b \cdot m_b + 3r_c \cdot m_c} \geq \frac{5 \cdot (r_a \cdot m_a + r_b \cdot m_b + 3r_c \cdot m_c)}{r_a \cdot m_a + r_b \cdot m_b + 3r_c \cdot m_c} = 5$$

220. Prove that in any triangle ABC,

$$a^2 AI^2 \cot A + b^2 BI^2 \cot B + c^2 CI^2 \cot C \geq 8SRr$$

where I is the incenter.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triàngulo ABC

$a^2 IA^2 \cot A + b^2 IB^2 \cot B + c^2 IC^2 \cot C \geq 8SRr$ ***donde I es Incentro..***

Teniendo en cuentas las siguientes notaciones en un $\Delta\ ABC$

$$IA = \sqrt{\frac{bc(p-a)}{p}}, \quad IB = \sqrt{\frac{ca(p-b)}{p}}, \quad IC = \sqrt{\frac{ab(p-c)}{p}}$$

$$\cot A = \frac{2bc\cos A}{2bc\sin A} = \frac{2bc\cos A}{4S}, \cot B = \frac{2ab\cos B}{4S}, \cot C = \frac{2ca\cos C}{4S}$$

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}, \frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, s = \frac{abc}{4R}$$

$$\frac{a}{p}\cos A + \frac{b}{p}\cos B + \frac{c}{p}\cos C = \frac{R(\sin 2A + \sin 2B + \sin 2C)}{4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} =$$

$$=\frac{\sin A\sin B\sin C}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{2r}{R}.$$ *La desigualdad es equivalente*

$$a^2\cdot\frac{bc(p-a)}{p}\cdot\frac{2bc\cos A}{4S}+b^2\cdot\frac{ca(p-b)}{p}\cdot\frac{2ab\cos B}{4S}+$$

$$+c^2\cdot\frac{(p-c)}{p}\cdot\frac{2ca\cos C}{4S}\ge 8SRr$$

$$2\cdot\frac{abc}{4S}\cdot abc\left(\cos A\left(1-\frac{a}{p}\right)+\cos B\left(1-\frac{b}{p}\right)+\cos C\left(1-\frac{c}{p}\right)\right)=$$

$$=8R^2S\left(\sum\cos A-\frac{a}{p}\sum\cos A\right)=8R^2S\left(1+\frac{r}{R}-\frac{2r}{R}\right)$$

$$a^2IA^2\cot A+b^2IB^2\cot B+c^2IC^2\cot C=8R^2S\left(\frac{R-r}{R}\right)\ge 8SRr\Leftrightarrow R\ge 2r$$

221. **Let ABC be a triangle and $A'B'C'$ is the Morley triangle of ABC. Prove that**

$$\frac{R_m}{R}\le 2\sqrt{3}\cdot\sin 20^\circ-1$$

where R and R_m are the circumradii of triangles ABC and $A'B'C'$ respectively.

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Daniel Sitaru – Romania

$$\frac{R_M}{R}=\frac{8R\sqrt{3}\prod\sin\frac{A}{3}}{3R}=\frac{8\sqrt{3}\prod\sin\frac{A}{3}}{3}\overset{GM-AM}{\le}$$

$$\le\frac{8\sqrt{3}}{3}\left(\frac{\sum\sin\frac{A}{3}}{3}\right)^3\overset{JENSEN}{\le}\frac{8\sqrt{3}}{3}\left(\sin\frac{A+B+C}{3\cdot 3}\right)^3=$$

$$=\frac{8\sqrt{3}}{3}\left(\sin\frac{\pi}{9}\right)^3-2\sqrt{3}\sin\frac{\pi}{9}+2\sqrt{3}\sin\frac{\pi}{9}=\frac{2\sqrt{3}}{3}\left(4\sin^3\frac{\pi}{9}-3\sin\frac{\pi}{9}\right)+2\sqrt{3}\sin\frac{\pi}{9}=$$

$$=-\frac{2\sqrt{3}}{3}\sin\frac{\pi}{3}+2\sqrt{3}\sin\frac{\pi}{9}=2\sqrt{3}\sin 20^\circ-1$$

222. Prove that in any triangle:

$$r_a^3+r_b^3+r_c^3\geq p^2(4R-5r)$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Daniel Sitaru – Romania

$$\sum r_a^3=(4R+r)^3-12Rs^2\geq s^2(4R-5r)\ \textit{(to prove)}\leftrightarrow$$

$$\leftrightarrow (4R+r)^3\geq s^2(16R-5r)$$

$$s^2(16R-5r)\overset{GERRETSEN}{\lesssim}(4R^2+4Rr+3r^2)(16R-5r)$$

$$(4R^2+4Rr+3r^2)(16R-5r)\leq(4R+r)^3$$

$$64R^3-20R^2r+64R^2r-20Rr^2+48Rr^2-15r^3\leq$$

$$\leq 64R^3+48R^2r+12Rr^2+r^3$$

$$4R^2r-16Rr^2+16r^3\geq 0\leftrightarrow 4r(R-2r)^2\geq 0$$

223. In $\Delta\, ABC$ the following relationship holds:

$$\sum\frac{b^2(s-b)}{a}+\sum\frac{c^2(s-c)}{a}\geq r(4R+r)+\sum h_a^2$$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

In $\Delta\, ABC$ the following relationship holds

$$\sum_{\Delta}\frac{b^2(s-b)}{a}+\sum_{\Delta}\frac{c^2(s-c)}{a}\geq r\cdot(4R+r)+\sum_{\Delta}h_a^2$$

$$\left.\begin{matrix}x=p-a\\ y=p-b\\ z=p-c\end{matrix}\right\}\Rightarrow \textit{1) } h_a^2=\frac{4xyz\cdot(x+y+z)}{(y+z)^2}\ \textit{similarly } h_b^2;h_c^2$$

2) $r(4R+r)+p^2-p^2=ab+bc+ca-\frac{(a+b+c)^2}{4}=$

$$=\frac{2\cdot(ab+bc+ca)-(a^2+b^2+c^2)}{4}=$$

$$=\frac{2\cdot\left((x+y)\cdot(y+z)+(y+z)\cdot(z+x)+(z+x)\cdot(x+y)\right)}{4}\Rightarrow$$

$$\Rightarrow -\sum(x+y)^2=xy+yz+zx$$

3) $\sum\frac{b^2\cdot(s-b)-c^2\cdot(s-c)}{a}=\sum\frac{(x+z)^2\cdot y+(x+y)^2\cdot z}{y+z}$

1), 2) 3) $\Rightarrow$

$$\sum_{\Delta}\frac{(x+z)^2\cdot y+(x+y)^2\cdot z}{y+z}\geq\sum_{\Delta}yz+\sum_{\Delta}\frac{4xyz\cdot(x+y+z)}{(y+z)^2}$$

(ASSURE)

$$\sum_{\Delta}\frac{(y+z)\cdot((x+z)^2\cdot y+(x+y)^2\cdot z)-4xyz\cdot(x+y+z)}{(y+z)^2}\geq\sum_{\Delta}yz$$

2) $\sum_{\Delta}\frac{(y+z)\cdot\left((x^2y+2xyz+z^2y)+(x^2z+2xyz+y^2z)\right)-4xyz(x+y+z)}{(y+z)^2}\geq\sum_{\Delta}yz$

$$\sum_{\Delta}\frac{(y+z)\cdot\left((y+z)x^2+(y+z)yz+4xyz\right)-4xyz(x+y+z)}{(y+z)^2}=$$

$$=\sum_{\Delta}(x^2+y^2)+\sum_{\Delta}\frac{4xyz\cdot\left(y+z-(x+y+z)\right)}{(y+z)^2}=$$

$$=\sum_{\Delta}yz+\sum_{\Delta}\left(x^2-\frac{4x^2yz}{(y+z)^2}\right)=\sum_{\Delta}yz+\sum_{\Delta}\frac{(xy+xz)^2-4x^2yz}{(y+z)^2}\overset{Cauchy}{\geq}$$

$$\geq\sum_{\Delta}yz+\sum_{\Delta}\frac{4x^2yz-4x^2yz}{(y+z)^2}\geq\sum_{\Delta}yz$$

224. **In acute $\Delta\, ABC$:**

$$\sum \tan A \tan B + 45 \le 2\tan^2 A \tan^2 B \tan^2 C$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$2\left(\prod \tan A\right)^2 = 2\left(\sum \tan A\right)^2 \ge 6\sum \tan A \tan B$$

$$\left(\because (x+y+z)^2 \ge 3(xy+yz+zx)\right) \overset{?}{\ge} \sum \tan A \tan B + 45$$

$$\Leftrightarrow \sum \tan A \tan B \ge 9 \quad \textbf{(1)}$$

$$\textit{Now, } \sum \tan A \tan B \overset{A-G}{\ge} 3\sqrt[3]{\tan^2 A \tan^2 B \tan^2 C}$$

$$= 3\sqrt[3]{\left(\sum \tan A\right)^2} \ge 3\sqrt[3]{\left(3\sqrt{3}\right)^2} = 9 \Rightarrow \textit{(1) is true}$$

$$\left(\because \sum \tan A \overset{Jensen}{\ge} 3\tan\left(\frac{A+B+C}{3}\right), \textit{as } f(x) = \tan x \textit{ is convex } \forall x \in \left(0, \frac{\pi}{2}\right)\right) \textit{(Proved)}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\textit{We know, } \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$\textit{Now, } 2(\tan A \tan B \tan C)^2 = 2\left(\sum_{cyc} \tan A\right)^2 \ge 6\left(\sum_{cyc} \tan A \tan B\right)$$

$$\textit{Now, } 5\left(\sum_{cyc} \tan A \tan B\right) = 5\left(\prod_{cyc} \tan A\right)\left(\sum_{cyc} \frac{1}{\tan A}\right)$$

$$= 5\left(\sum_{cyc} \tan A\right)\left(\sum_{cyc} \frac{1}{\tan A}\right) \ge 45$$

$$6\left(\sum_{cyc} \tan A \tan B\right) \ge 45 + \sum_{cyc} \tan A \tan B$$

$$2\left(\prod_{cyc} \tan A\right)^2 \ge 45 + \sum_{cyc} \tan A \tan B$$

(Proved)

225. In $\Delta ABC, I$ - the incentre, O - the circumcentre, G - the centroid

Prove that:

$$3(OI + IG + GO)^2 + 52Rr \leq s^2 + 5r^2 + 18R^2$$

Proposed by Daniel Sitaru – Romania

Solution by Adil Abdullayev-Baku-Azerbaidian

$$3(OI + IG + GO)^2 + 52Rr \leq s^2 + 5r^2 + 18R^2$$

$$3(OI + IG + GO)^2 + 52Rr \leq 3(1^2 + 1^2 + 1^2)(OI^2 + IG^2 + GO^2) + 52Rr =$$

$$= 9\left(R^2 - 2Rr + \frac{s^2 + 5r^2 - 16Rr}{9} + R^2 - \frac{2(s^2 - r^2 - 4Rr)}{9}\right) + 52Rr =$$

$$= 18R^2 + 26Rr + 7r^2 - s^2 \leq s^2 + 5r^2 + 18R^2 \Leftrightarrow s^2 \geq 13Rr + r^2$$

$$\text{Gerretsen} \Rightarrow s^2 \geq 16Rr - 5r^2 \geq 13Rr + r^2 \Leftrightarrow R \geq 2r \text{ (Euler)}$$

226. Prove that in any triangle:

$$\frac{r_a^2}{\cos^2\frac{\alpha}{2}} + \frac{r_b^2}{\cos^2\frac{\beta}{2}} + \frac{r_c^2}{\cos^2\frac{\gamma}{2}} \geq 2R(4R + r)$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Mehmet Sahin-Ankara-Turkey

$$\frac{r_a^2}{\cos^2\frac{\alpha}{2}} + \frac{r_b^2}{\cos^2\frac{\beta}{2}} + \frac{r_c^2}{\cos^2\frac{\gamma}{2}} \geq \frac{(r_a + r_b + r_c)^2}{\cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} + \cos^2\frac{\gamma}{2}}$$

$$\geq \frac{(r + 4R)^2}{\frac{s(s-a)}{bc} + \frac{s(s-b)}{ca} + \frac{s(s-c)}{ab}}$$

$$\geq \frac{(r + 4R)^2}{as(s-a) + bs(s-b) + cs(s-c)} \cdot abc$$

$$\left(a^2+b^2+c^2=2\cdot(s^2-r^2-4Rr)\right)\ge\frac{(r+4R)^2}{2rs(r+4R)}\cdot 4R\cdot r\cdot s$$

$$\ge 2R(r+4R)$$

Solution 2 by Daniel Sitaru – Romania

$$\sum\frac{r_a^2}{\cos^2\frac{A}{2}}\overset{BERGSTROM}{\ge}\frac{(\sum r_a)^2}{\sum\cos^2\frac{A}{2}}=\frac{(4R+r)^2}{2+\frac{r}{2R}}=$$

$$=\frac{2R(4R+r)^2}{4R+r}=2R(4R+r)$$

Solution 3 by Soumava Chakraborty-Kolkata-India

WLOG, we may assume $a\ge b\ge c$

Then $r_a^2\ge r_b^2\ge r_c^2$ ***and*** $\frac{1}{\cos^2\frac{\alpha}{2}}\ge\frac{1}{\cos^2\frac{\beta}{2}}\ge\frac{1}{\cos^2\frac{\gamma}{2}}$

$\therefore$ ***applying Chebyshev,*** $LHS\ge\frac{1}{3}\left(\sum r_a^2\right)\left(\sum\sec^2\frac{\alpha}{2}\right)$

$$\overset{Chebyshev}{\ge}\frac{1}{9}\left(\sum r_a^2\right)\left(\sum\sec\frac{\alpha}{2}\right)^2\overset{Jensen}{\ge}\frac{1}{9}\left(\sum r_a^2\right)\left(3\sec\left(\sum\frac{\alpha}{6}\right)\right)^2$$

$$=\frac{1}{9}\{(4R+r)^2-2s^2\}(4\cdot 3)=\frac{4\{(4R+r)^2-2s^2\}}{3}$$

$\therefore$ ***it suffices to prove that:*** $2(4R+r)^2-4s^2\ge 3R(4R+r)$

$$\Leftrightarrow 4s^2\le 20R^2+13Rr+2r^2$$

But, Gerretsen $\Rightarrow 4s^2\le 16R^2+16Rr+12r^2$

$\therefore$ ***it suffices to prove that:*** $20R^2+13Rr+2r^2\ge 16R^2+16Rr+12r^2$

$\Leftrightarrow 4R^2-3Rr-10r^2\ge 0\Leftrightarrow (R-2r)(4R+5r)\ge 0$ ***true***

$\because R\ge 2r$ ***(Euler)***

(Proved)

227. For $\Delta\ ABC; A \geq B \geq C > 0$

Prove: $3\sin A + 4\sin B + 5\sin C \leq 6\sqrt{3}$

Proposed by Eliezer Okeke-Nigeria

Solution by Daniel Sitaru – Romania

$$\frac{\pi}{2} \geq A \geq B \geq C > 0, 3 < 4 < 5$$

$$3\sin A + 4\sin B + 5\sin C \overset{Chebyshev}{\leq} \frac{1}{3}(3+4+5)\sum \sin A =$$

$$\leq \frac{1}{3}\cdot 12\cdot \frac{3\sqrt{3}}{2} \leq 6\sqrt{3}$$

228. In $\Delta\ ABC$:

$$a^2 + b^2 + c^2 \geq \sqrt{3}\max(am_a, bm_b, cm_c) + \frac{2s^2}{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

$$(a\sqrt{3})\cdot(2m_a) \leq \frac{3a^2 + 4m_a^2}{2} = a^2 + b^2 + c^2$$

$$\sqrt{3}\cdot am_a \leq \frac{a^2+b^2+c^2}{2} \Rightarrow \sqrt{3}\cdot am_a + \frac{2S^2}{3} \leq \frac{a^2+b^2+c^2}{2} + \frac{2s^2}{3} \quad \dots (1)$$

$$\frac{a^2+b^2+c^2}{2} + \frac{2s^2}{3} \overset{?}{\leq} a^2 + b^2 + c^2 \Rightarrow (a+b+c)^2 \leq 3(a^2+b^2+c^2)$$

$$\left.\begin{aligned} (1) \Rightarrow \sqrt{3}\cdot am_a + \frac{2s^2}{3} &\leq a^2 + b^2 + c^2 \\ \sqrt{3}\cdot bm_b + \frac{2s^2}{3} &\leq a^2 + b^2 + c^2 \\ \sqrt{3}\cdot cm_c + \frac{2s^2}{3} &\leq a^2 + b^2 + c^2 \end{aligned}\right\} \Rightarrow LHS \geq RHS$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$a^2+b^2+c^2 \geq \sqrt{3}\max(am_a+bm_b+cm_c)+\frac{2s^2}{3}$$

Es suficiente demostrar lo siguiente $a^2+b^2+c^2 \geq \sqrt{3}am_a+\frac{2s^2}{3}$

$$a^2+b^2+c^2-\frac{2s^2}{3}=a^2+b^2+c^2-\frac{(a+b+c)^2}{6} \geq$$

$\geq a^2+b^2+c^2-\frac{a^2+b^2+c^2}{2}=\frac{a^2+b^2+c^2}{2}$***. Por último demostraremos***

$$\frac{a^2+b^2+c^2}{2} \geq \sqrt{3}am_a \to a^2+b^2+c^2 \geq 2\sqrt{3}am_a$$

$$2(a^2+b^2+c^2)=(2b^2+2c^2-a^2)+3a^2=4m_a^2+3a^2 \geq 4\sqrt{3}am_a$$

$$\Leftrightarrow a^2+b^2+c^2 \geq 2\sqrt{3}am_a$$

Análogamente para los siguientes términos

$$a^2+b^2+c^2 \geq \sqrt{3}bm_a+\frac{2s^2}{3} \wedge a^2+b^2+c^2 \geq \sqrt{3}cm_c+\frac{2s^2}{3}$$

Por lo tanto $\to a^2+b^2+c^2 \geq \sqrt{3}\max(am_a+bm_a+cm_c)+\frac{2s^2}{3}$

229. **In $\Delta\ ABC$:**

$$\frac{a^2\tan x}{\sin y+\sin z}+\frac{b^2\tan y}{\sin z+\sin x}+\frac{c^2\tan z}{\sin x+\sin y}>2\sqrt{3}S, x,y,z \in \left(0,\frac{\pi}{2}\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC, probar que

$$E=\frac{a^2\tan x}{\sin y+\sin z}+\frac{b^2\tan y}{\sin z+\sin x}+\frac{c^2\tan z}{\sin x+\sin y}>2\sqrt{3}S, \quad x,y,z \in <0,\frac{\pi}{2}>$$

se cumple lo siguiente

$$\tan x = \sin x \sec x > \sin x, \tan y > \sin y, \tan z > \sin z$$

Lo cuals tiene lo siguiente

$$E > \frac{a^2 \sin x}{\sin y + \sin z} + \frac{b^2 \sin y}{\sin z + \sin x} + \frac{c^2}{\sin z + \sin x} > 2\sqrt{3}S$$

Desigualdad de Weizenbock (Refinamiento de Pohoata)

Siendo $m, n, p \geq 0$ y a, b, c los lados de un triángulo ABC se cumple lo siguiente $a^2m + b^2n + c^2p \leq 4S\sqrt{mn + np + pm}$ (A), sea

$$m = \frac{\sin x}{\sin y + \sin z}, n = \frac{\sin y}{\sin z + \sin x}, p = \frac{\sin z}{\sin x + \sin y}$$

Lo cual se verifica que $\rightarrow mn + np + pm = 1 - 2mnp \geq 1 - \frac{1}{4} = \frac{3}{4}$

Aplicando en (A) $\frac{a^2 \sin x}{\sin y + \sin z} + \frac{b^2 \sin y}{\sin z + \sin x} + \frac{c^2}{\sin z + \sin x} \geq 4S\sqrt{\frac{3}{4}} = 2\sqrt{3}S$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$\forall\, x, y, z \in \left(0, \frac{\pi}{2}\right), \tan x > x, \tan y > y, \tan z > z$, ***and***

$$\sin x < x, \sin y < y, \sin z < z \therefore LHS \overset{(1)}{>} \frac{a^2 x}{y+z} + \frac{b^2 y}{z+x} + \frac{c^2 z}{x+y}$$

$$\geq 4S\sqrt{\frac{xy}{(y+z)(z+x)} + \frac{yz}{(z+x)(x+y)} + \frac{zx}{(x+y)(y+z)}}$$

$$\left(\because a^2m + b^2n + c^2p \geq 4s\sqrt{mn + np + pm} \quad \forall\, m, n, p \geq 0\right)$$

$\therefore$ ***it suffices to prove:*** $2\sqrt{\frac{xy}{\sum xy + z^2} + \frac{yz}{\sum xy + x^2} + \frac{zx}{\sum xy + y^2}} \geq \sqrt{3}$ ***(using (1))***

$$\Leftrightarrow \frac{xy}{\sum xy + z^2} + \frac{yz}{\sum xy + x^2} + \frac{zx}{\sum xy + y^2} \geq \frac{3}{4} \quad (2)$$

LHS of (2) $= \frac{x^2y^2}{xy\sum xy + z^2xy} + \frac{y^2z^2}{yz\sum xy + x^2yz} + \frac{z^2x^2}{zx\sum xy + y^2zx}$

$$\overset{Bergstrom}{\underset{(3)}{\geq}} \frac{(\sum xy)^2}{(\sum xy)^2 + xyz(\sum x)}$$

$\therefore$ ***it suffices to prove:*** $\frac{(\sum xy)^2}{(\sum xy)^2+xyz(\sum x)} \geq \frac{3}{4}$ ***from (3)***

$$\Leftrightarrow \left(\sum xy\right)^2 \geq 3xyz\left(\sum x\right) \Leftrightarrow \sum x^2y^2 \geq xyz\left(\sum x\right)$$

$\rightarrow$ ***true*** $\because \sum u^2 \geq \sum uv$, ***where*** $u = xy, v = yz, w = zx$ ***(Proved)***

230. Prove that in any triangle:

$$\left(\frac{\cos\frac{\alpha}{2}}{l_a} + \frac{\cos\frac{\beta}{2}}{l_b} + \frac{\cos\frac{\gamma}{2}}{l_c}\right)(a+b+c) \geq 9$$

$$\left(\frac{\cos\frac{\alpha}{2}}{l_a} + \frac{\cos\frac{\beta}{2}}{l_b} + \frac{\cos\frac{\gamma}{2}}{l_c}\right)(a+b+c) \geq 10 - \frac{2r}{R}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Daniel Sitaru – Romania

$$\left(\sum a\right)\left(\sum \frac{\cos\frac{A}{2}}{w_a}\right) = \left(\sum a\right)\left(\sum \frac{\sqrt{\frac{s(s-a)}{bc}}}{\frac{2\sqrt{bcs(s-a)}}{b+c}}\right) =$$

$$= \frac{1}{2}\left(\sum a\right)\sum \frac{b+c}{bc} = \frac{1}{2}\cdot 2\left(\sum a\right)\cdot\left(\sum \frac{1}{a}\right) \overset{AM-GM}{\geq} 9$$

$$\sum \frac{2x}{y+z} \overset{NESBBIT}{\geq} 2\cdot\frac{3}{2} = 3, \frac{(y+z)(z+x)(x+y)}{8xyz} \overset{CESARO}{\geq} 1$$

$$\sum \frac{2x}{y+z} + \frac{(y+z)(z+x)(x+y)}{8xyz} \geq 4, a = y+z, b = x+z, c = x+y$$

$$\sum \frac{x+y+x+z}{y+z} + \frac{(y+z)(z+x)(x+y)}{8xyz} \geq 7$$

$$3 + \sum \frac{b+c}{a} + \frac{2r}{R} \geq 10 \to 3 + \sum \frac{b+c}{a} \geq 10 - \frac{2r}{R}$$

$$\left(\sum a\right)\left(\sum \frac{\cos\frac{A}{2}}{w_a}\right) = \left(\sum a\right)\left(\sum \frac{1}{a}\right) = 3 + \sum \frac{b+c}{a} \geq 10 - \frac{2r}{R}$$

231. In $\Delta\, ABC$:

$$\frac{a^2 \cdot l_a^4 + b^2 \cdot l_b^4 + c^2 \cdot l_c^4}{a^4 + b^4 + c^4} \leq \frac{27}{16} R^2$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC $\frac{a^2 \cdot l_a^4 + b^2 \cdot l_b^4 + c^2 \cdot l_c^4}{a^4+b^4+c^4} \leq \frac{27}{16} R^2$

Recordar lo siguiente en un triángulo ABC

$$l_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2bc}{b+c}\sqrt{\frac{p(p-a)}{bc}} = \frac{2\sqrt{bc}}{b+c}\sqrt{p(p-a)} \leq \sqrt{p(p-a)}$$

Análogamente $l_b \leq \sqrt{p(p-b)}, l_c \leq \sqrt{p(p-c)}, 2p = a+b+c \leq 3\sqrt{3}R$

Reemplazando en el númerador tenemos

$$\sum a^2 l_a^4 \leq \frac{p^2}{4} \sum a^2 (b+c-a)^2 = \frac{p^2}{4} \sum (a^2b^2 + a^2c^2 + a^4 + 2a^2bc - 2a^3b - 2a^3c)$$

$$\sum a^2 l_a^4 \leq \frac{p^2}{4}\left(\sum a^4 + 2\sum a^2b^2 + 2abc(a+b+c) - 2\sum ab\left(a^2+b^2\right)\right) \leq \frac{p^2}{4}\left(a^4+b^4+c^4\right) \quad \textbf{(A)}$$

Se utilizó lo siguiente en (A)

$$2abc(a+b+c) \leq 2\sum (ab)(ac) \leq 2\sum a^2b^2 \wedge 2\sum ab(a^2+b^2) \geq 4\sum a^2b^2$$

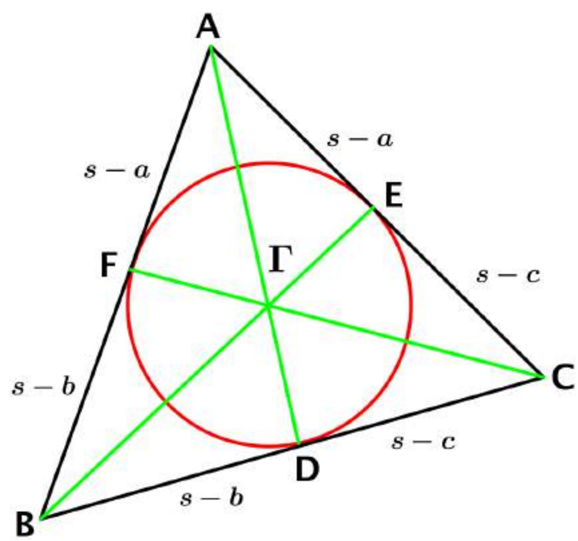
Por la tanto $\frac{a^2 l_a^4 + b^2 l_b^4 + c^2 l_c^4}{a^4+b^4+c^4} \leq \frac{p^2}{4} \leq \frac{27R^2}{16}$

232. In ΔABC , AD, BE, CF are Gergonne's cevians. Prove that:

$$AD^2 + BE^2 + CF^2 + \frac{1}{4}\sum a^2 \ge \sum \frac{s(b^2 + c^2) - (b^3 + c^3)}{a}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India



By Stewart's theorem, $c^2(s-c) + b^2(s-b) = a\left(AD^2 + (s-b)(s-c)\right)$

$$\Rightarrow AD^2 = \frac{s(b^2 + c^2) - (b^3 + c^3)}{a} - (s-b)(s-c)$$

$$\Rightarrow AD^2 + (s-b)(s-c) = \frac{s(b^2+c^2)-(b^3+c^3)}{a} \quad (1)$$

Similarly, $BE^2 + (s-c)(s-a) = \frac{s(c^2+a^2)-(c^3+a^3)}{b}$ (2)

$$CF^2 + (s-a)(s-b) = \frac{s(a^2+b^2)-(a^3+b^3)}{c} \quad (3)$$

(1) + (2) + (3) $\Rightarrow AD^2 + BE^2 + CF^2 + 3s^2 - s(2\sum a) + \sum ab =$

$$= \sum \frac{s(b^2 + c^2) - (b^3 + c^3)}{a} \Rightarrow AD^2 + BE^2 + CF^2 + \frac{1}{4}\sum a^2$$

$$= \sum \frac{s(b^2 + c^2) - (b^3 + c^3)}{a} + \left(\frac{1}{4}\sum a^2 + s^2 - \sum ab\right)$$

∴ *it suffices to prove that:*

$$\frac{1}{4}\sum a^2+s^2-\sum ab\ge 0\Leftrightarrow\sum a^2+4s^2\ge 4\sum ab$$

$$\Leftrightarrow\sum a^2+\left(\sum a\right)^2\ge 4\sum ab\Leftrightarrow\sum a^2+\sum a^2+2\sum ab\ge 4\sum ab$$

$$\Leftrightarrow 2\sum a^2\ge 2\sum ab\Leftrightarrow\sum a^2\ge\sum ab\rightarrow \textit{true (Proved)}$$

233. From the book: "Math Phenomenon"

In $\Delta\, ABC$:

If $b\cos B+c\cos C=2\,a\sin B\sin C$ then:

$$\frac{b^6}{c^4}>\frac{8(a-c)^3}{a}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$b\cos B+c\cos C=2R\sin B\cos B+2R\sin C\cos C$$

$$=R(\sin 2B+2\sin 2C)=2R\sin(B+C)\cos(B-C)=2R\sin A\cos(B-C)$$

$$=a\cos(B-C)$$

$$\therefore a\cos(B-C)=a(\cos(B-C)-\cos(B+C))$$

$$\Rightarrow\cos(B+C)=0\Rightarrow B+C=A=90^\circ$$

$$b^2=a^2-c^2\ \therefore b^6=(a^2-c^2)^3$$

$$\therefore\frac{b^6}{c^4}-\frac{8(a-c)^3}{a}=\frac{(a+c)^3(a-c)^3}{c^4}-\frac{8(a-c)^3}{a}$$

$$=(a-c)^3\cdot\frac{(a+c)^3}{c^4}-\frac{8}{a}=(a-c)^3\left(\frac{(a+c)^3}{c^3}\left(\frac{1}{c}\right)-\left(\frac{1}{c}\right)\left(\frac{8c}{a}\right)\right)$$

$$=\frac{(a-c)^3}{c}\left(\left(\frac{a+c}{c}\right)^3-8\left(\frac{c}{a}\right)\right)=\frac{(a-c)^3}{c}\left(\left(\frac{a}{c}+1\right)^3-8\left(\frac{c}{a}\right)\right)\quad (A)$$

Now AM ≥ GM $\Rightarrow \frac{a}{c}+1>2\sqrt{\frac{a}{c}} \Rightarrow \left(\frac{a}{c}+1\right)^3>8\left(\frac{a}{c}\right)\sqrt{\frac{a}{c}}$ ***(1)***

Now, $a>c \therefore a^5>c^5 \Rightarrow \frac{a^3}{c^3}>\frac{c^2}{a^2} \Rightarrow \left(\frac{a}{c}\right)\sqrt{\frac{a}{c}}>\frac{c}{a} \Rightarrow 8\left(\frac{a}{c}\right)\sqrt{\frac{a}{c}}>8\left(\frac{c}{a}\right)$ ***(2)***

(1) and (2) $\Rightarrow \left(\frac{a}{c}+1\right)^3>8\left(\frac{c}{a}\right) \Rightarrow \left(\frac{a}{c}+1\right)^3-\frac{8c}{a}>0$

$\therefore$ ***from (A),*** $\frac{b^6}{a^4}-\frac{8(a-c)^3}{a}=\frac{(a-c)^3}{c}\left(\left(\frac{a}{c}+1\right)^3-\frac{8c}{a}\right)>0 \quad (\because a>c, \therefore (a-c)^3>0)$

$\Rightarrow \frac{b^6}{a^4}>\frac{8(a-c)^3}{a}$ ***(Proved)***

234. **In** $\Delta\ ABC$**:**

$$\frac{r_a^2}{\sin^2\frac{A}{2}}+\frac{r_b^2}{\sin^2\frac{B}{2}}+\frac{r_c^2}{\sin^2\frac{C}{2}}\le\frac{27}{2}\cdot\frac{R^3}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{r_a^2}{\sin^2\frac{A}{2}}+\frac{r_b^2}{\sin^2\frac{B}{2}}+\frac{r_c^2}{\sin^2\frac{C}{2}}\le\frac{27R^3}{2r}$

Tener en cuenta las siguientes notaciones y desigualdades en un $\Delta\ ABC$

$$r_a=p\tan\frac{A}{2}, r_b=p\tan\frac{B}{2}, r_c=p\tan\frac{C}{2}, r_a+r_b+r_c=4R+r\le\frac{9R}{2}$$

Si $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\to\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1, 2p\le3\sqrt{3}R$

$$\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}=3+\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=$$

$$=\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2+1=\frac{(4R+r)^2}{p^2}+1$$

La desigualdad es equivalente

$$p^2\left(\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\right)=(4R+r)^2+p^2\le\frac{81R^2}{4}+\frac{27R^2}{4}=$$

$$=27R^2\le\frac{27R^3}{2r}$$

235. In $\Delta\ ABC$:

$$\prod\left(\frac{a+b-c+2h_a+2h_b-2h_c}{c+2h_c}\right)\le 1$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\prod\left(\frac{a+b-c+2h_a+2h_b-2h_c}{c+2h_c}\right)\le 1$

Realizamos los siguientes cambios de variables

$x=a+2h_a, y=b+2h_b, z=c+2h_c$. **La desigualdad pedida es equivalente** $xyz\ge(x+y-z)(y+z-x)(z+x-y)$

$$xyz\ge(y^2-(x-z)^2)(z+x-y)$$

$$xyz\ge(y^2-x^2-z^2+2xz)(z+x-y)$$

$$xyz\ge y^2z+y^2x-y^3-x^2z-x^3+x^2y-z^3-z^2x+z^2y+2xz^2+2x^2z-2xyz$$

$$\Leftrightarrow x^3+y^3+z^3+3xyz\ge xy(x+y)+yz(y+z)+zx(z+x)$$

$$\Leftrightarrow x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y)\ge 0$$

$$\Leftrightarrow a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)\ge 0$$

(Válido por desigualdad Schur)

236. In $\Delta\ ABC$:

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{ab}{rb+4Rc}+\frac{bc}{rc+4Ra}+\frac{ca}{ra+4Rb}\right)\ge\frac{9}{r_a+r_b+r_c}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} r_a = 4R + r \text{ now}$$

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{ab}{rb+4Rc}+\frac{bc}{rc+4R_a}+\frac{ca}{ra+4Rb}\right)=\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{1}{c(rb+4Rc)}\right)$$

$$\overset{AM\geq GM}{\geq}\left(\sum_{cyc} ab\right)\frac{3}{\sqrt[3]{abc(rb+4Rc)(rc+4Ra)(ra+4Rb)}}$$

$$=\left(\sum_{cyc} ab\right)\frac{3}{\sqrt[3]{(rab+4Rac)(rcb+4Rab)(rac+4Rcb)}}$$

$$\overset{REVERSE\ AM\geq GM}{\geq}\left(\sum_{cyc} ab\right)\frac{9}{(\sum ab)+(\sum 4Rab)}=\frac{9}{4R+r}=\frac{9}{r_a+r_b+r_c}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{1}{a}\sum\frac{ab}{rb+c4R}\geq\frac{9}{\sum r_a};\ \sum\frac{1}{a}=\frac{ab+bc+ca}{abc}$$

$$\sum\frac{ab}{rb+c\cdot 4R}=\sum\frac{a^2b}{abr+ac\cdot 4R}\overset{Bergstrom}{\geq}\frac{\left(\sum a\sqrt{b}\right)^2}{(ab+bc+ca)\cdot(4R+r)}\geq$$

$$\overset{AM\geq GM}{\geq}\frac{9abc}{(ab+bc+ca)(4R+r)}$$

$$\sum\frac{1}{a}\cdot\sum\frac{a^2b}{abr+ac\cdot 4R}\geq\frac{ab+bc+ca}{abc}\cdot\frac{9abc}{(ab+bc+ca)(4R+r)}=$$

$$=\frac{9}{4R+r}=\frac{9}{r_a+r_b+r_c}$$

237. In $\Delta\ ABC$ the following relationship holds:

$$\frac{\sin\frac{A}{2}}{\cos\left(\frac{B-C}{2}\right)}+\frac{\sin\frac{B}{2}}{\cos\left(\frac{C-A}{2}\right)}+\frac{\sin\frac{C}{2}}{\cos\left(\frac{A-B}{2}\right)}\geq\frac{3r}{s}\left(\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}\right)$$

Proposed by Marian Ursărescu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}=\sum\frac{\sin A}{\sin B+\sin C}=\sum\frac{a}{b+c}=$$

$$=\frac{\sum a(c+a)(a+b)}{2abc+\sum ab\,(2s-c)}=\frac{\sum a^3+(\sum ab)(2s)}{2s(s^2+4Rr+r^2)-4Rrs}=$$

$$=\frac{3abc+2s(\sum a^2-\sum ab)+2s(\sum ab)}{2s(s^2+2Rr+r^2)}=\frac{2s\cdot 2(s^2-4Rr-r^2)+12Rrs}{2s(s^2+2Rr+r^2)}$$

$$=\frac{4s(s^2-Rr-r^2)}{2s(s^2+2Rr+r^2)}\overset{(1)}{=}\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}$$

$$RHS=\frac{3r}{s}\sum\frac{s\tan\frac{A}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{3r}{4R}\sum\sec^2\frac{A}{2}=\frac{3r}{4Rs}\left(\sum\frac{bc}{s-a}\right)=$$

$$=\frac{3r}{4Rs\cdot r^2s}\left\{\sum bc(s-b)(s-c)\right\}=\frac{3}{4Rrs^3}\sum\{ab(s^2-s(a+b)+ab)\}$$

$$=\frac{3}{4Rrs^3}\left\{s^2\left(\sum ab\right)-s\sum ab\,(2s-c)+\left(\sum ab\right)^2-2abc(2s)\right\}$$

$$=\frac{3}{4Rrs^3}\left\{-s^2\left(\sum ab\right)+\left(\sum ab\right)^2-4Rrs^2\right\}=$$

$$=\frac{3}{4Rrs^2}\{(s^2+4Rr+r^2)(4Rr+r^2)-4Rrs^2\}=\frac{3}{4Rrs^3}\{s^2r^2+r^2(4R+r)^2\}$$

$\overset{(2)}{=}\frac{3r}{4Rs^2}\{s^2+(4R+r)^2\}$. ***(1), (2) ⇒ it suffices to prove:***

$$8Rs^2(s^2-Rr-r^2)\ge 3r(s^2+2Rr+r^2)\left(s^2+\left(s^2+(4R+r)\right)^2\right)$$

$$\Leftrightarrow (8R-3r)s^4\overset{(3)}{\ge} rs^2(56R^2+38Rr+6r^2)+3r^2(2R+r)(4R+r)^2$$

LHS of (3) $\overset{(3)}{\ge} s^2(8R-3r)(16Rr-5r^2)\overset{?}{\ge}$ ***RHS of (3)***

$$\Leftrightarrow s^2(24R^2-42Rr+3r^2)\underset{(4)}{\overset{?}{\ge}} r(2R+r)(4R+r)^2$$

Now, LHS of (4) $\overset{Gerretsen}{\ge}(16Rr-5r^2)(24R^2-42Rr+3r^2)\overset{?}{\ge} r(2R+r)(4R+r)^2$

$$\Leftrightarrow 44t^3 - 103t^2 + 31t - 2 \overset{?}{\geq} 0 \quad \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\{(t-2)(44t+73)+147\} \overset{?}{\geq} 0 \;\; \textit{true} \because t \geq 2 \;\textit{(Euler) (proved)}$$

238. In $\Delta\, ABC, I$ - incentre:

$$\left(\sum \frac{IA^2}{abc}\right)\left(\sum \frac{IA^2}{bc}\right)\left(\sum \frac{aIA^2}{bc}\right) \geq 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo I - Incentro, probar en un triángulo ABC

$$\left(\sum \frac{IA^2}{abc}\right)\left(\sum \frac{IA^2}{bc}\right)\left(\sum \frac{aIA^2}{bc}\right) \geq 1$$

Teniendo en cuentas las siguientes notaciones en un $\Delta\, ABC$

$$IA = \sqrt{\frac{bc(p-a)}{p}}, IB = \sqrt{\frac{ca(p-b)}{p}}, IC = \sqrt{\frac{ab(p-c)}{p}}$$

$$IA^2 = bc - \frac{abc}{p} = bc - 4Rr, IB^2 = ca - 4Rr, IC = ab - 4Rr$$

Además: $\sum \frac{IA^2}{bc} = \sum \left(\frac{bc-4Rr}{bc}\right) = 3 - 4Rr\left(\sum \frac{1}{bc}\right) = 3 - \frac{4Rr}{2Rr} = 1$

Por la desigualdad de Cauchy $\left(\sum \frac{IA^2}{abc}\right)\left(\sum \frac{aIA^2}{bc}\right)(1) \geq \left(\sum \frac{IA^2}{bc}\right)^2 = 1$ ***(LQQD)***

Solution 2 by Nirapada Pal-Jhargram-India

$$\left(\sum \frac{1}{\sin^2\frac{A}{2}}\right)\left(\sum \frac{a^2}{\sin^2\frac{A}{2}}\right)\left(\sum \frac{a}{\sin^2\frac{A}{2}}\right) \overset{CBS}{\geq} \left(\sum \frac{a}{\sin^2\frac{A}{2}}\right)^2\left(\sum \frac{a}{\sin^2\frac{A}{2}}\right) = \left(\sum \frac{a}{\sin^2\frac{A}{2}}\right)^3$$

$$= \left(\sum \frac{abc}{(s-b)(s-c)}\right)^3, \left[\textit{since}\, \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}\right]$$

$$=\left(\frac{abc}{\Delta^2}\sum s(s-a)\right)^3,\left[since\ \Delta=\sqrt{s(s-a)(s-b)(s-c)}\right]$$

$$=\left(\frac{abc}{\Delta^2}[3s^2-s\sum a]\right)^3=\left(\frac{abc}{\Delta^2}s^2\right)^2,[since\ 2s=a+b+c]=\left(\frac{abc}{r^2}\right)^3\ \textbf{since}\ r=\frac{\Delta}{s}$$

$$\therefore\left(\sum\frac{r^2}{abc\sin^2\frac{A}{2}}\right)\left(\sum\frac{a^2r^2}{abc\sin^2\frac{A}{2}}\right)\left(\sum\frac{ar^2}{abc\sin^2\frac{A}{2}}\right)\geq 1$$

Or, $\left(\sum\frac{IA^2}{abc}\right)\left(\sum\frac{aIA^2}{bc}\right)\left(\sum\frac{IA^2}{bc}\right)\geq 1$. **Or,** $\left(\sum\frac{IA^2}{abc}\right)\left(\sum\frac{IA^2}{bc}\right)\left(\sum\frac{aIA^2}{bc}\right)\geq 1$

Solution 3 by Nirapada Pal-Jhargram-India

$$\left(\sum\frac{1}{\sin^2\frac{A}{2}}\right)\left(\sum\frac{a^2}{\sin^2\frac{A}{2}}\right)\left(\sum\frac{a}{\sin^2\frac{A}{2}}\right)\overset{CBS}{\geq}\left(\sum\frac{a}{\sin^2\frac{A}{2}}\right)^2\left(\sum\frac{a}{\sin^2\frac{A}{2}}\right)=\left(\sum\frac{a}{\sin^2\frac{A}{2}}\right)^3$$

$$=\left(2R\sum\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin^2\frac{A}{2}}\right)^3,\ \textbf{[since}\ a=2R\sin\theta\ \textbf{and}\ \sin 2\theta=2\sin\theta\cos\theta\textbf{]}=\left(4R\sum\cot\frac{A}{2}\right)^3$$

$$=\left(4R\prod\cot\frac{A}{2}\right)^3,\ \textbf{[since}\sum\cot\frac{A}{2}=\prod\cot\frac{A}{2}\textbf{]}=\left(4R\prod\frac{s(s-a)^3}{\Delta}\right),\ \textbf{[since}\ \cot\frac{A}{2}=\frac{s(s-a)}{\Delta}\ \textbf{etc.]}$$

$$=\left(4R\frac{s^2}{\Delta}\right)^3=\left(abc\frac{s^2}{\Delta^2}\right),\ \textbf{[since}\ 4R=\frac{abc}{\Delta}\textbf{]}=\left(\frac{abc}{r^2}\right)^3\ \textbf{since}\ r=\frac{\Delta}{s}$$

$$\therefore\left(\sum\frac{r^2}{abc\sin^2\frac{A}{2}}\right)\left(\sum\frac{a^2r^2}{abc\sin^2\frac{A}{2}}\right)\left(\sum\frac{ar^2}{abc\sin^2\frac{A}{2}}\right)\geq 1$$

Or, $\left(\sum\frac{IA^2}{abc}\right)\left(\sum\frac{aIA^2}{bc}\right)\left(\sum\frac{IA^2}{bc}\right)\geq 1$. **Or,** $\left(\sum\frac{IA^2}{abc}\right)\left(\sum\frac{IA^2}{bc}\right)\left(\sum\frac{aIA^2}{bc}\right)\geq 1$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{AI^2}{abc}=\frac{r^2}{4Rrs}\sum\frac{bc}{(s-b)(s-c)}$$

$$=\frac{r^2}{4Rrs}\left\{\frac{\sum bc(s-a)}{\prod(s-a)}\right\}=\frac{r^2}{4Rrs}\cdot\frac{s(\sum ab)-3abc}{\prod(s-a)}$$

$$=\frac{r^2}{4Rrs}\cdot\frac{s^2\sum ab-12Rrs^2}{r^2s^2}\overset{(1)}{=}\frac{s^2-8Rr+r^2}{4Rrs}$$

$$\sum\frac{AI^2}{bc}=r^2\sum\frac{bc}{bc(s-b)(s-c)}=r^2\cdot\frac{\sum(s-a)}{\prod(s-a)}$$

$$=\frac{r^2s}{\prod(s-a)}=\frac{r^2s^2}{r^2s^2}\overset{(2)}{\cong}\ \textbf{(1)}$$

$$\sum\frac{aAI^2}{bc}=r^2\sum\frac{a}{bc}\cdot\frac{bc}{(s-b)(s-c)}=r^2\sum\frac{a}{(s-b)(s-c)}$$

$$=r^2\cdot\frac{\sum a(s-a)}{\prod(s-a)}=sr^2\cdot\frac{s(2s)-\sum a^2}{r^2s^2}=\frac{2s^2-\sum a^2}{s}$$

$$=\frac{2s^2-(2s^2-8Rr-2r^2)}{s}\overset{(3)}{\cong}\frac{8Rr+2r^2}{s}$$

(1), (2), (3) $\Rightarrow$ ***LHS*** $\geq\frac{s^2-8Rr+r^2}{4Rrs}\cdot\frac{2r(4R+r)}{s}\overset{?}{\geq}1$

$$\Leftrightarrow(s^2-8Rr+r^2)(4R+r)\overset{?}{\geq}2Rs^2$$

$$\Leftrightarrow(4R+r)s^2-(8Rr-r^2)(4R+r)\overset{?}{\geq}2Rs^2$$

$$\Leftrightarrow(2R+r)s^2-r(8R-r)(4R+r)\overset{?}{\geq}0\quad\textit{(i)}$$

By Gerretsen, LHS of (i) $\geq(2R+r)r(16R-5r)-r(8R-r)(4R+r)$

$$=r(32R^2+6Rr-5r^2-32R^2-4Rr+r^2)$$

$=r(2Rr-4r^2)=2r^2(R-2r)\geq 0,\because R\geq 2r$ ***(Euler)*** $\Rightarrow$ ***(i) is true***

Solution 5 by Soumitra Mandal-Chandar Nagore-India

In $\Delta\ ABC, I-$ ***Incentre***

$$\left(\sum_{cyc}\frac{IA^2}{abc}\right)\left(\sum_{cyc}\frac{IA^2}{bc}\right)\left(\sum_{cyc}\frac{aIA^2}{bc}\right)\geq 1$$

Mathematical Tools: $\csc\frac{A}{2}=\sqrt{\frac{bc}{(p-b)(p-c)}},\csc\frac{B}{2}=\sqrt{\frac{ac}{(p-a)(p-c)}}$

$$\csc\frac{C}{2}=\sqrt{\frac{ab}{(p-b)(p-a)}}, IA=r\csc\frac{A}{2}, IB=r\csc\frac{B}{2}, IC=r\csc\frac{C}{2} \text{ and } \Delta=pr$$

$$\therefore\left(\sum_{cyc}\frac{IA^2}{abc}\right)\left(\sum_{cyc}\frac{IA^2}{bc}\right)\left(\sum_{cyc}\frac{aIA^2}{bc}\right)=\frac{1}{(abc)^3}\left(\sum_{cyc}IA^2\right)\left(\sum_{cyc}aIA^2\right)\left(\sum_{cyc}a^2IA^2\right)$$

$$\overset{HOLDER'S\ INEQUALITY}{\geq}\frac{1}{(abc)^3}\left(\sum_{cyc}aIA^2\right)^3=\frac{1}{(abc)^3}\left(r^2\sum_{cyc}\frac{abc}{(p-b)(p-c)}\right)^3$$

$$=\frac{1}{(abc)^3}\left(r^2abc\frac{p-a+p-b+p-c}{(p-a)((p-b)(p-c)}\right)=\left(\frac{r^2p^2}{p(p-a)(p-b)(p-c)}\right)^2=1$$

$$\left[\begin{array}{c}\because \Delta=rp\\ \text{and } \Delta=\sqrt{p(p-a)(p-b)(p-c)}\end{array}\right]$$. Hence, $\therefore\left(\sum\frac{IA^2}{abc}\right)\left(\sum\frac{IA^2}{bc}\right)\left(\sum\frac{aIA^2}{bc}\right)\geq 1$ (proved)

239. Prove that in any triangle ABC

$$48\sqrt{3}R^2r\leq I_AI_B\cdot I_BI_C\cdot I_CI_A\leq 24\sqrt{3}R^3$$

where I_A, I_B, I_C are the excenters correspond to A, B, C respectively.

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC $48\sqrt{3}R^2r\leq I_AI_B\cdot I_BI_C\cdot I_CI_A\leq 24\sqrt{3}R^3$

donde I_A, I_B, I_C *son los excenntros correspondientes a* A, B, C

respectivamente. Teniendo en cuenta las siguientes identidades

$I_AI_B=4R\cos\frac{C}{2},\quad I_BI_C=4R\cos\frac{A}{2},\quad I_CI_A=4R\cos\frac{B}{2}$. *Por lo tanto*

$$I_AI_B\cdot I_BI_C\cdot I_CI_A=64R^3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\leq 64R^3\cdot\frac{3\sqrt{3}}{8}=24\sqrt{3}R^3$$

$$I_AI_B\cdot I_BI_C\cdot I_CI_A=64R^3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=$$

$$= 64R^3 \cdot \frac{p}{4R} = 16R^2 \cdot p \geq 16R^2 \cdot 3\sqrt{3}r = 48\sqrt{3}R^2r$$

240. **From the book "Math Phenomenon"**

In $\Delta\, ABC$:

$$\frac{16}{(a+3)(b+5)(c+7)} \leq \frac{1}{4RS} + \frac{1}{105}$$

R - circumradius, S - area

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$\frac{abc}{4R} = S \Rightarrow \frac{1}{4RS} = \frac{1}{abc}$. ***To prove:*** $\frac{16}{(a+3)(b+5)(c+7)} \leq \frac{105+abc}{105abc}$

$$\Leftrightarrow (105 + abc)(a+3)(b+5)(c+7) \geq 16 \times 105\, abc$$

$$\Leftrightarrow 105abc + 105 \times 5ca + 105 \times 3bc + 105 \times 15c + 105 \times 7ab$$

$$+105 \times 35a + 105 \times 21\, b + 105^2 + a^2b^2c^2 + 5abc \cdot ca + 3bc \cdot abc$$

$$+15c \cdot abc + 7ab \cdot abc + 35a \cdot abc + 21b \cdot abc + 105abc$$

$$\geq 16 \times 105\, abc$$

$$\Leftrightarrow 105(7ab + 3bc + 5ca) + 105(35a + 21b + 15c) + 105^2 + a^2b^2c^2$$

$$+abc(7ab + 3bc + 5ca) + abc(35a + 21b + 15c) \geq 14 \times 105abc$$

$$\begin{matrix} ab & a & b^2c^2a & c^2ab \\ bc & b & c^2a^2b & a^2bc \\ ca & c & a^2b^2c & b^2ca \end{matrix} \Leftrightarrow \begin{matrix} (105 \cdot 7ab + 105 \cdot 35a + 3b^2c^2a + 15c^2ab) \\ +(105 \cdot 3bc + 105 \cdot 21b + 5c^2a^2b + 35a^2bc) \\ +(105 \cdot 5ca + 105 \cdot 15c + 7a^2b^2c + 21b^2ca) \\ +(105^2 + a^2b^2c^2) \geq 14 \times 105\, abc \to (A) \end{matrix}$$

Applying AM ≥ GM,

$$105 \cdot 7ab + 105 \cdot 35a + 3b^2c^2a + 15c^2ab \overset{(1)}{\geq} \sqrt[4]{105^4a^4b^4c^4} \times 4 = 4(105)abc$$

$$105 \cdot 3bc + 105 \cdot 21b + 5c^2a^2b + 35a^2bc \overset{(2)}{\geq} 4 \cdot \sqrt[4]{105^4a^4b^4c^4} = 4(105)abc$$

$$105 \cdot 5ca + 105 \cdot 15c + 7a^2b^2c + 21b^2ca \overset{(3)}{\geq} 4 \cdot \sqrt[4]{105^4a^4b^4c^4} = 4(105)abc$$

$$105^2 + a^2b^2c^2 \overset{(4)}{\geq} 2 \cdot 105 \cdot abc$$

(1) + (2) + (3) + (4) ⇒ (A) is proved.

241. In acute angled $\Delta\ ABC$, I - incentre:

$$AI^2 + BI^2 + CI^2 \leq 2R(w_a + m_b + s_c - 6r)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC, donde $I \to$ Incentro

$$IA^2 + IB^2 + IC^2 \leq 2R(w_a + m_b + S_c - 6r)$$

Teniendo en cuenta las siguientes identidades y desigualdades en un

$$\Delta\ ABC;\ IA^2 = bc - 4Rr,\ IB^2 = ca - 4Rr,\ IC^2 = ab - 4Rr$$

$w_a \geq h_a, m_a \geq h_b, S_c \geq h_c$. ***La desigualdad es equivalente***

$$\Leftrightarrow 2R(w_a + m_b + S_c) - 12Rr \geq 2R(h_a + h_b + h_c) - 12Rr =$$

$$= ab + bc + ca - 12Rr = IA^2 + IB^2 + IC^2$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$s_c \geq h_c \Leftrightarrow \frac{2ab}{a^2 + b^2} \cdot m_c \geq \frac{ab}{2R} \Leftrightarrow m_c \geq \frac{a^2 + b^2}{4R}$$

which is true by Tereshin $\therefore s_c \geq h_c$

$$\therefore RHS \overset{(1)}{\geq} 2R(h_a + h_b + h_c - 6r) \quad (\because w_a \geq h_a \text{ and } m_b \geq h_b)$$

$$= 2R\left(\frac{\sum ab}{2R} - 6r\right) = \sum ab - 12Rr$$

Now, $LHS = r^2\left(\csc^2\frac{A}{2} + \csc^2\frac{B}{2} + \csc^2\frac{C}{2}\right)$

$$= r^2\left[\frac{bc}{(s-b)(s-c)} + \frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right]$$

$$= r^2\left[\frac{bc(s-a)+ca(s-b)+ab(s-c)}{(s-a)(s-b)(s-c)}\right]$$

$$= r^2\cdot\left[\frac{s^2(\sum ab)-3abcS}{s(s-a)(s-b)(s-c)}\right] = \frac{r^2S^2(\sum ab-12Rr)}{r^2S^2} \overset{(2)}{=} \sum ab-12Rr$$

(1), (2) $\Rightarrow RHS \ge LHS$ **(Proved)**

242. Prove that in any triangle:

$$3\left(r_a\sqrt{r_b}+r_b\sqrt{r_c}+r_c\sqrt{r_a}\right)^2 \le (4R+r)^3$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$r_a+r_b+r_c = 4R+r;\ r_a = x^2, r_b = y^2, r_c = z^2$$

$$3(x^2y+y^2z+z^2x)^2 \le (x^2+y^2+z^2)^3$$

$$(x^2+y^2+z^2)(x+y+z) \ge 3(x^2y+y^2z+z^2x)$$

$$(x^2+y^2+z^2)^2(x+y+z)^2 \ge 9(x^2y+y^2z+z^2x)^2$$

$$(x^2+y^2+z^2)^2(x+y+z)^2 \le 3(x^2+y^2+z)^3$$

$$(x+y+z)^2 \le 3(x^2+y^2+z^2)$$

Solution 2 by Daniel Sitaru – Romania

$$3\left(\sum r_a\sqrt{r_b}\right)^2 \overset{CBS}{\le} 3\sum r_a\sum r_ar_b = 3(4R+r)s^2 \le (4R+r)^3 \text{ (to prove)} \leftrightarrow$$

$$3s^2 \le (4R+r)^2 \leftrightarrow 3\sum r_ar_b \le \left(\sum r_a\right)^2 \leftrightarrow \sum (r_a-r_b)^2 \ge 0$$

Solution 3 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$3\left(r_a\sqrt{r_b}+r_b\sqrt{r_c}+r_c\sqrt{r_a}\right)^2 \le (4R+r)^3 = (r_a+r_b+r_c)^3$$

Siendo $m, n, p \quad \mathbb{R}$ ***se cumple la siguiente desigualdad***

$$(m+n+p)^2 \geq 3(mn+np+pm) \quad \textbf{(A)}$$

$$m = x^2 + y(z-x),\ n = y^2 + z(x-y),\ p = z^2 + x(y-z)$$

Lo cual se verifica que

$$m+n+p = x^2+y^2+z^2 \wedge mn+np+pm = x^3y+y^3z+z^3x$$

(demostrado anteriormente). Lo equivalente en (A) es

$$(x^2+y^2+z^2)^2 \geq 3(x^3y+y^3z+z^3x) \rightarrow \textit{(Vasile Cîrtoaje) (B)}$$

$$x^2+y^2+z^2 \geq xy+yz+zx \quad \textbf{(C)}$$

Luego, multiplicamos (B) · (C) y aplicamos desigualdad de Cauchy

$$(x^2+y^2+z^2)^3 \geq 3(x^3y+y^3z+z^3x)(xy+yz+zx) \geq 3(x^2y+y^2z+z^2x)^2$$

Sustituyemos $\rightarrow x^2 = r_a, y^2 = r_b, z^2 = r_c$***, obtenemos la desigualdad pedida.***

243. In acute – angled $\Delta\, ABC$:

$$2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\cdot\tan B\cdot\tan C \geq 9\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo acutángulo ABC***:***

$$2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\cdot\tan B\cdot\tan C \geq 9\sqrt{3}.\ \textit{Si: } A+B+C=\pi$$

$$\rightarrow \tan(A+B) = \tan(\pi - C) \rightarrow \frac{\tan A + \tan B}{1-\tan A\tan B} = -\tan C$$

$$\Rightarrow \tan A + \tan B + \tan C = \tan A\tan B\tan C$$

Desde que es un triángulo acutángulo: $\tan A, \tan B, \tan C > 0$.

Por: MA $\geq$ ***MG*** $\tan A + \tan B + \tan C \geq 3\sqrt[3]{\tan A\tan B\tan C} \rightarrow$

$$\rightarrow (\tan A\tan B\tan C)^3 \geq 27\tan A\tan B\tan C$$

$$\Rightarrow \tan A \tan B \tan C \geq 3\sqrt{3} \quad \textit{(A)}$$

$$\textit{Si: } \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2} \to \cot\left(\frac{A}{2}+\frac{B}{2}\right) = \cot\left(\frac{\pi}{2}-\frac{C}{2}\right) \to$$

$$\to \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}. \textit{ Por: } MA \geq MG$$

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} \geq 3\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}} \to$$

$$\to \left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^3 \geq 27\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

$$2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geq 6\sqrt{3} \quad \textit{(B)} \to \textit{Sumando: (A) + (B)} \to$$

$$\to 2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right) + \tan A \tan B \tan C \geq 9\sqrt{3}$$

Solution 2 by Adil Abdullayev-Baku-Azerbaidian

Lemma 1.

$$\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2} = \frac{p}{r_a}+\frac{p}{r_b}+\frac{p}{r_c} = \frac{p}{r}.$$

Lemma 2.

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma \geq \frac{p}{r}.$$

Lemma 3.

$$p \geq 3\sqrt{3}\cdot r.$$

$$LHS \geq \frac{2p}{r}+\frac{p}{r} = \frac{3p}{r} \geq \frac{9\sqrt{3}\cdot r}{r} = 9\sqrt{3}.$$

Solution 3 by Ravi Prakash-New Delhi-India

For $0 < x < \frac{\pi}{2}$***, let*** $f(x) = 2\cot\left(\frac{x}{2}\right) + \tan x$***;*** $f'(x) = -\frac{2}{2}\csc^2\left(\frac{x}{2}\right) + \sec^2 x$

$$=\frac{1}{\cos^2 x\sin^2\left(\frac{x}{2}\right)}\left[\sin\left(\frac{x}{2}\right)-\cos x\right]\left[\sin\left(\frac{x}{2}\right)+\cos x\right]$$

$$=\frac{\left(\sin\frac{x}{2}+\cos x\right)}{\cos^2 x\sin^2\left(\frac{x}{2}\right)}\left(2\sin^2\frac{x}{2}+\sin\frac{x}{2}-1\right)$$

$$=\frac{\left(\sin\frac{x}{2}+\cos x\right)\left(\sin\left(\frac{x}{2}\right)+1\right)\left(2\sin\left(\frac{x}{2}\right)-1\right)}{\cos^2 x\sin^2\left(\frac{x}{2}\right)}$$

$$f'(x)<0 \textit{ for } 0<x<\frac{\pi}{3}=0 \textit{ for } x=\frac{\pi}{3}>0 \textit{ for } \frac{\pi}{3}<x<\frac{\pi}{2}$$

$$\therefore f(x) \textit{ is minimum for } x=\frac{\pi}{3}$$

$$\Rightarrow f(x)\ge f\left(\frac{\pi}{3}\right) \textit{ for } 0<x<\frac{\pi}{2}\Rightarrow f(x)\ge 3\sqrt{3} \textit{ for } 0<x<\frac{\pi}{2}$$

$$\textit{Thus, } f(A)+f(B)+f(C)\ge 9\sqrt{3}\Rightarrow \sum 2\cot\left(\frac{A}{2}\right)+\sum\tan A\ge 9\sqrt{3}$$

$$\textit{But in a triangle } \sum\tan A=\tan A\tan B\tan C$$

$$\therefore 2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\tan B\tan C\ge a\sqrt{3}$$

$$\textit{Equality at } A=B=C=\frac{\pi}{3}$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\cot\frac{A}{2}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=\frac{2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{1+\cos A}{\sin A}=\frac{1}{\sin A}+\frac{\cos A}{\sin A}$$

$$=\frac{1}{\sin A}+\frac{b^2+c^2-a^2}{2bc}\cdot\frac{2R}{a}=\frac{1}{\sin A}+\frac{R}{abc}(b^2+c^2-a^2)\quad \textbf{\textit{(1)}}$$

$$\textit{Similarly, } \cot\frac{B}{2}\overset{(2)}{=}\frac{1}{\sin B}+\frac{R}{abc}(c^2+a^2-b^2)\textit{, and,}$$

$$\cot\frac{C}{2}\overset{(3)}{=}\frac{1}{\sin C}+\frac{R}{abc}(a^2+b^2-c^2)$$

$$\therefore \sum \cot\frac{A}{2} = \sum \frac{1}{\sin A} + \frac{R}{abc}\left(\sum a^2\right) \text{ (by (1) + (2) + (3))}$$

$$\overset{Bergstrom}{\geq} \frac{9}{\sum \sin A} + \frac{R}{4RS}\left(\sum a^2\right)$$

$$= \frac{9 \cdot 2R}{\sum a} + \frac{1}{4S}\left(\sum a^2\right) \overset{Ionescu-Weitzenbock}{\geq} \frac{18R}{2S} + \frac{1}{4S} \cdot 4\sqrt{3}S$$

$$\overset{Mitrinovic}{\geq} \frac{18R}{3\sqrt{3}R} + \sqrt{3} = 2\sqrt{3} + \sqrt{3} = 3\sqrt{3}$$

$$\therefore 2\sum \cot\frac{A}{2} \geq 6\sqrt{3} \quad (4)$$

Now $\prod \tan A = \sum \tan A$***; Let*** $f(x) = \tan x \quad \forall x \in \left(0, \frac{\pi}{2}\right)$

Then $f''(x) = 2\sec^2 x \tan x > 0 \Rightarrow f(x)$ ***is convex***

$$\therefore \sum \tan A \underset{(5)}{\overset{Jensen}{\geq}} 3\tan\left(\frac{A+B+C}{3}\right) = 3\tan\frac{\pi}{3} = 3\sqrt{3}$$

(4) + (5) $\Rightarrow$ ***LHS*** $\geq 6\sqrt{3} + 3\sqrt{3} = 9\sqrt{3}$ = ***RHS (Proved)***

Solution 5 by Rozeta Atanasova-Skopje

$$\tan A \tan B \tan C = \tan A + \tan B + \tan C \quad (1)$$

$\tan x$ ***and*** $\cot\frac{x}{2}$ ***are convex over*** $\left(0, \frac{\pi}{2}\right)$ ***(2)***

From (1) and (2) $\Rightarrow$

$$LHS = 2\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) + \tan A + \tan B + \tan C \geq \text{(Jensen)}$$

$$2 \cdot 3\cot\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3} + 3\tan\frac{A+B+C}{3} = 6\cot\frac{\pi}{6} + 3\tan\frac{\pi}{3} = 9\tan\frac{\pi}{3} = 9\sqrt{3} = RHS.$$

244. In $\Delta\, ABC$:

$$a^a \cdot b^b \cdot c^c \cdot (s-a)^{s-a} \cdot (s-b)^{s-b} \cdot (s-c)^{s-c} \geq \left(\frac{4s^3}{27}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$a^a \cdot b^b \cdot c^c \cdot (s-a)^{s-a} \cdot (s-b)^{s-b} \cdot (s-c)^{s-c} \geq \left(\frac{4s^3}{27}\right)^2$$

Siendo a, b, c números R^+ se cumple la siguiente desigualdad

$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}$**. Supongamos** sin **pérdida de generalidad**

$a \leq b \leq c \Leftrightarrow \log a \leq \log b \leq \log c$**. Por la desigualdad de Chebysev**

$$\left(\frac{a+b+c}{3}\right)\left(\frac{\log a + \log b + \log c}{3}\right) \leq \frac{a\log a + b\log b + c\log c}{3}$$

$$\left(\frac{a+b+c}{3}\right)(\log abc) \leq \log\left(a^a b^b c^c\right)$$

$$\log(abc)^{\frac{a+b+c}{3}} \leq \log\left(a^a b^b c^c\right) \Leftrightarrow a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c} \quad \textit{(LQQD)}$$

Luego $\rightarrow a^a b^b c^c (s-a)^{s-a}(s-b)^{s-b}(s-c)^{s-c} \geq \left(\frac{2s}{3}\right)^{2s}\left(\frac{s}{3}\right)^{s} = \left(\frac{4s^3}{27}\right)^{s}$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

By GM $\geq$ HM: $a^a b^b c^c \geq \left(\frac{a+b+c}{\frac{a}{a}+\frac{b}{b}+\frac{c}{c}}\right)^{a+b+c}$ **and**

$$\prod_{cyc}(p-a)^{p-a} \geq \left(\frac{p-a+p-b+p-c}{\frac{p-a}{p-a}+\frac{p-b}{p-b}+\frac{p-a}{p-c}}\right)^{p-a+p-b+p-c} \Rightarrow \prod_{cyc}(p-a)^{(p-a)} \geq \left(\frac{p}{3}\right)^p$$

so, $a^a b^b c^c (p-a)^{p-a}(p-b)^{p-b}(p-c)^{p-c} \geq \left(\frac{2p}{3}\right)^{2p}\left(\frac{p}{3}\right)^{p} = \left(\frac{4p^3}{27}\right)^{p}$ ***(Proved)***

245. In acute – angled $\Delta\ ABC$:

$$3 + \sum \frac{a\cos A}{b\cos B + c\cos C - a\cos A} \geq \sum \frac{b\cos B + c\cos C}{a\cos A}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC

$$3+\sum\frac{a\cos A}{b\cos B+c\cos C-a\cos A}\geq\sum\frac{b\cos B+\cos C}{a\cos A}$$

$$\Leftrightarrow\sum\frac{a\cos A}{b\cos B+c\cos C-a\cos A}\geq\sum\frac{b\cos B+c\cos C-a\cos A}{a\cos A}$$

$$\Rightarrow\sum\frac{\sin 2A}{\sin 2B+\sin 2C-\sin 2A}\geq\sum\frac{\sin 2B+\sin 2C-\sin 2A}{\sin 2A}\quad (A)$$

Luego por transformacions trigonométricas

$$\sin 2B+\sin 2C-\sin 2A=2\sin(B+C)\cos(B-C)-2\sin A\cos A$$

$$\sin 2B+\sin 2C-\sin 2A=2\sin A\cos(B-C)+2\sin A\cos(B+C)$$

$$\sin 2B+\sin 2C-\sin 2A=2\sin A\left(\cos(B-C)+\cos(B+C)\right)=$$

$= 4\sin A\cos B\cos C$***. La desigualdad es equivalente en (A)***

$$\sum\frac{\cos A}{2\cos B\cos C}\geq\sum\frac{2\cos B\cos C}{\cos A}\Leftrightarrow\sum\frac{\cos A}{\cos B\cos C}\geq\sum\frac{4\cos B\cos C}{\cos A}$$

Ahora bien $\frac{\cos A}{\cos B\cos C}=\frac{\frac{\sin A}{\cos B\cos C}}{\frac{\sin A}{\cos A}}=\frac{\tan B+\tan C}{\tan A}$***. La desigualdad se puede reescribir como*** $\sum\frac{\tan B+\tan C}{\tan A}\geq 4\sum\frac{\tan A}{\tan B+\tan C}$

$$\frac{\tan B+\tan C}{\tan A}+\frac{\tan C+\tan A}{\tan B}+\frac{\tan A+\tan B}{\tan C}\geq$$

$$\geq\frac{4\tan A}{\tan B+\tan C}+\frac{4\tan B}{\tan C+\tan A}+\frac{4\tan C}{\tan A+\tan B}$$

Como es un Δ acutángulo $\tan A,\tan B,\tan C>0$***. Aplicando la desigualdad de Cauchy*** $\frac{\tan A}{\tan B}+\frac{\tan A}{\tan C}\geq\frac{4\tan A}{\tan B+\tan C}$,

$$\frac{\tan B}{\tan C}+\frac{\tan B}{\tan A}\geq\frac{4\tan B}{\tan C+\tan A},\ \frac{\tan C}{\tan A}+\frac{\tan C}{\tan B}\geq\frac{4\tan C}{\tan A+\tan B}$$

Sumando dichas desigualdades se obtiene lo siguiente

$$\frac{\tan B+\tan C}{\tan A}+\frac{\tan C+\tan A}{\tan B}+\frac{\tan A+\tan B}{\tan C}\ge$$

$$\ge\frac{4\tan A}{\tan B+\tan C}+\frac{4\tan B}{\tan C+\tan A}+\frac{4\tan C}{\tan A+\tan B}\quad \textbf{(LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow$

$$\sum\frac{a\cos A}{b\cos B+c\cos C-a\cos A}\ge\sum\frac{b\cos B+c\cos C-a\cos A}{a\cos A}\quad \textbf{(1)}$$

Now, $b\cos B+c\cos C-a\cos A$

$$=2R\sin B\cos B+2R\sin C\cos C-2R\sin A\cos A$$

$$=R(\sin 2B+\sin 2C-\sin 2A)=R\{2\sin A\cos(B-C)-2\sin A\cos A\}$$

$$=2R\sin\{\cos(B-C)+\cos(B+C)\}$$

$$=2\,R\sin A\,(2\cos B\cos C)=4R\sin A\cos B\cos C$$

> 0 ***($\because$ $\Delta\, ABC$ is acute – angled)***

Similarly, $c\cos C+a\cos A-b\cos B=4R\sin B\cos C\cos A>a$

And $a\cos A+b\cos B-c\cos C=4R\sin C\cos A\cos B>0$

$\therefore\ x=a\cos A\,,\ y=b\cos B\,,\ z=c\cos C$ ***from sides of a triangle. Let the semi-perimeter, inradius and circumradius of this triangle be*** s_0, r_0, R_0 ***respectively. (1) becomes*** $\frac{x}{y+z-x}+\frac{y}{z+x-y}+\frac{z}{x+y-z}$

$$\ge\frac{y+z-x}{x}+\frac{z+x-y}{y}+\frac{x+y-z}{z}$$

$$\Leftrightarrow\frac{1}{2}\left(\frac{x}{s_0-x}+\frac{y}{s_0-y}+\frac{z}{s_0-z}\right)\ge 2\left(\frac{s_0-x}{x}+\frac{s_0-y}{y}+\frac{s_0-y}{z}\right)\quad \textbf{(2)}$$

LHS of (2) $=\frac{\sum x(s_0-y)(s_0-z)}{2\prod(s_0-x)}=\frac{\sum[x\{s_0^2-s_0(y+z)+yz\}]s_0}{2r_0^2s_0^2}=\frac{\{s_0^2(\sum x)-s_0\sum(xy+zx)+3xyz\}s_0}{2r_0^2s_0^2}$

$$=\frac{\{2s_0^3-2s_0(\sum xy)+12R_0r_0s_0\}s_0}{2r_0^2s_0^2}$$

$$=\frac{2s_0^2\left(s_0^2-\sum xy+6R_0r_0\right)}{2r_0^2s_0^2}=\frac{s_0^2-s_0^2-4R_0r_0-r_0^2+6R_0r_0}{r_0^2}$$

$=\frac{2R_0r_0-r_0^2}{r_0^2}\overset{(3)}{=}\frac{2R_0-r_0}{r_0}$. ***RHS of (2)*** $=\frac{2\{yz(s_0-x)+zx(s_0-y)+xy(s_0-z)\}}{xyz}$

$$=\frac{2\{s_0(\sum xy)-3xyz\}}{4R_0r_0s_0}=\frac{2s_0\left(s_0^2+4R_0r_0+r_0^2-12R_0R_0\right)}{4R_0r_0s_0}$$

$\overset{(4)}{=}\frac{s_0^2-8R_0r_0+r_0^2}{2R_0r_0}$. ***(3), (4)*** $\Rightarrow$ ***it suffices to prove:*** $2R_0-r_0\geq\frac{s_0^2-8R_0r_0+r_0^2}{2R_0}$

$\Leftrightarrow s_0^2-8R_0r_0+r_0^2\leq 4R_0^2-2R_0r_0\Leftrightarrow s_0^2\leq 4R_0^2+6R_0r_0-r_0^2$ ***(5)***

Gerretsen $\Rightarrow s_0^2\leq 4R_0^2+4R_0r_0+3r_0^2$ ***(6). (5), (6)*** $\Rightarrow$ ***it suffices to prove:***

$4R_0r_0+3r_0^2\leq 6R_0r_0-r_0^2\Leftrightarrow 2R_0r_0\geq 4r_0^2\Leftrightarrow R_0\geq 2r_0\rightarrow$ ***true (Euler)***

246. In $\Delta\, ABC$:

$$\frac{a^3}{b\sin^2x+c\cos^2x}+\frac{b^3}{c\sin^2x+a\cos^2x}+\frac{c^3}{a\sin^2x+b\cos^2x}\geq 4\sqrt{3}S,$$
$$x\in(0,\pi)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Nirapada Pal-Jhargram-India

$$\sum\frac{a^3}{b\sin^2x+c\cos^2x}=\sum\frac{\left(a^{\frac{3}{2}}\right)^2}{b\sin^2x+c\cos^2x}$$

$$\overset{Bergstrom}{\geq}\frac{\left(\sum a^{\frac{3}{2}}\right)^2}{\sum b\sin^2x+c\cos^2x}\overset{AGM}{\geq}\frac{\left(3\left(a^{\frac{3}{2}}b^{\frac{3}{2}}c^{\frac{3}{2}}\right)\right)^2}{a+b+c}=\frac{9abc}{a+b+c}\geq 4\sqrt{3}S$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\frac{a^3}{b\sin^2 x+c\cos^2 x}+\frac{b^3}{c\sin^2 x+a\cos^2 x}+\frac{c^3}{a\sin^2 x+b\cos^2 x}\ge 4\sqrt{3}S,$$

$$x\in<0,\pi>$$

Tener en cuenta lo siguiente

$$\sin^2 x+\cos^2 x=1,\ a^2+b^2+c^2\ge ab+bc+ca,\ a^2+b^2+c^2\ge 4S\sqrt{3}$$

(Ineq. Weizenbock). Por la desigualdad de Cauchy

$$\frac{a^4}{ab\sin^2 x+ca\cos^2 x}+\frac{b^4}{bc\sin^2 x+ab\cos^2 x}+\frac{c^4}{ca\sin^2 x+bc\cos^2 x}\ge$$

$$\ge\frac{(a^2+b^2+c^2)^2}{(\sin^2 x+\cos^2 x)(ab+bc+ca)}$$

$$\Rightarrow\frac{a^4}{ab\sin^2 x+ca\cos^2 x}+\frac{b^4}{bc\sin^2 x+ab\cos^2 x}+\frac{c^4}{ca\sin^2 x+bc\cos^2 x}\ge$$

$$\ge a^2+b^2+c^2\ge 4S\sqrt{3}\quad \textit{(LQQD)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\because b\sin^2 x\ge 0 \text{ and } c\cos^2 x\ge 0,$$

$$\therefore b\sin^2 x+c\cos^2 x=0\Rightarrow \sin x=0 \text{ and } \cos x=0$$

But, $\because \sin^2 x+\cos^2 x=1,\therefore \sin x$ ***and*** $\cos x$ ***cannot be simultaneously***

$$=0\Rightarrow b\sin^2 x+c\cos^2 x\ne 0\Rightarrow b\sin^2 x+c\cos^2 x>0$$

Similarly, $c\sin^2 x+a\cos^2 x>0$ ***and,*** $a\sin^2 x+b\cos^2 x>0$

$$LHS\underset{(1)}{\overset{A-G}{\ge}}3abc\sqrt[3]{\frac{1}{(b\sin^2 x+c\cos^2 x)(c\sin^2 x+a\cos^2 x)(a\sin^2 x+b\cos^2 x)}}$$

Now, GM ≤ AM ⇒

$$\sqrt[3]{(b\sin^2 x+c\cos^2 x)(c\sin^2 x+a\cos^2 x)(a\sin^2 x+b\cos^2 x)}$$

$$\le\frac{b(\sin^2 x+\cos^2 x)+c(\sin^2 x+\cos^2 x)+a(\sin^2 x+\cos^2 x)}{3}$$

$$=\frac{2s}{3}\underset{(2)}{\leq}\frac{3\sqrt{3}R}{3}\ \textit{(Mitrinovic)}=\sqrt{3}R;\ (1),(2)\Rightarrow LHS\geq\frac{3\cdot(4RS)}{\sqrt{3}R}=4\sqrt{3}S$$

Solution 4 by Uche Eliezer Okeke-Anambra-Nigerie

In $\Delta\ ABC, x\in(0,\pi)$. Show: $\sum\frac{a^3}{b\sin^2 x+c\cos^2 x}\geq 4\sqrt{3}S$

Lemma: $\left.\begin{array}{c}\prod a=4Rrs=4RS\\ \sum a\leq 3\sqrt{3}R\ (M)\end{array}\right\}$

$$LHS\overset{A-G}{\geq}\frac{3abc}{\sqrt[3]{\prod(b\sin^2 x+c\cos^2 x)}}\overset{A-G}{\geq}\frac{9abc}{(\sin^2 x+\cos^2 x)\sum a}\overset{M}{\geq}\frac{9\cdot 4RS}{3\sqrt{3}R}=4\sqrt{3}S\ \textit{(RHS)}$$

Solution 5 by Marin Chirciu-Romania

We use Hölder's inequality $\frac{A^3}{X}+\frac{B^3}{Y}+\frac{C^3}{Z}\geq\frac{(A+B+C)^3}{3(X+Y+Z)}$, **where** $A,B,C,X,Y,Z>0$. **Putting** $A=a, B=b, C=c, X=b\sin^2 x+c\cos^2 x$, $Y=c\sin^2 x+a\cos^2 x, Z=a\sin^2 x+b\cos^2 x$, **we obtain**

$$\sum\frac{a^3}{b\sin^2 x+c\cos^2 x}\geq\frac{(a+b+c)^3}{3\sum(b\sin^2 x+c\cos^2 x)}=\frac{8p^3}{6p}=\frac{4p^2}{3}\geq 4\sqrt{3}S,$$

wherefrom the last inequality is equivalent with $p\geq 3\sqrt{3}r$ (Mitrinovic's Inequality). The equality holds if and only if the triangle is equilateral.

247. Let ABC be a triangle with perimeter 3. Prove that:

$$\frac{1}{h_a^a}+\frac{1}{h_b^b}+\frac{1}{h_c^c}\geq\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sea ABC un triángulo con perimetro 3. Probar que:

$$\frac{1}{h_a^a}+\frac{1}{h_b^b}+\frac{1}{h_c^c}\geq\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}.$$

Aplicando MA ≥ MG $\frac{1}{h_a^a}+\frac{1}{h_b^b}+\frac{1}{h_c^c}\geq 3\sqrt[3]{\frac{1}{h_a^a}\cdot\frac{1}{h_b^b}\cdot\frac{1}{h_c^c}}=3\sqrt[a+b+c]{\frac{1}{h_a^a}\cdot\frac{1}{h_b^b}\cdot\frac{1}{h_c^c}}$

Luego aplicando desigualdad ponderada MG ≥ MH

$$3\sqrt[a+b+c]{\frac{1}{h_a^a}\cdot\frac{1}{h_b^b}\cdot\frac{1}{h_c^c}}\geq\frac{3(a+b+c)}{\frac{a}{\frac{1}{h_a}}+\frac{b}{\frac{1}{h_b}}+\frac{c}{\frac{1}{h_c}}}=\frac{3(a+b+c)}{ah_a+bh_b+ch_c}=\frac{3(a+b+c)}{6S}=$$

$$=\frac{a}{2S}+\frac{b}{2S}+\frac{c}{2S}=\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}.$$ ***Por transitividad***

$$\Rightarrow\frac{1}{h_a^a}+\frac{1}{h_b^b}+\frac{1}{h_c^c}\geq\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}$$ ***(LQQD)***

248. Prove that in triangle ABC:

$$\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}\geq\sqrt{6(1+\cos A\cos B\cos C)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}\geq\sqrt{6(1+\cos A\cos B\cos C)}$

Teniendo en cuenta las siguientes identidades y desigualdades

$$\sin^2 A+\sin^2 B+\sin^2 C=2(1+\cos A\cos B\cos C)$$

$$h_a=\frac{bc}{2R},h_b=\frac{ca}{2R},h_c=\frac{ab}{2R}$$

$x+y+z\geq\sqrt{3(xy+yz+zx)}$, ***donde*** $x=\frac{bc}{a}>0,y=\frac{ca}{b}>0,z=\frac{ab}{c}>0$

$\Leftrightarrow\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\geq\sqrt{3(a^2+b^2+c^2)}$. ***La desigualdad es equivalente***

$\frac{1}{2R}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)\geq\sqrt{6(1+\cos A\cos B\cos C)}$. ***Luego:***

$$\frac{1}{2R}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)\geq\frac{1}{2R}\sqrt{3(a^2+b^2+c^2)}=\sqrt{3(\sin^2 A+\sin^2 B+\sin^2 C)}=$$

$$= \sqrt{6(1+\cos A\cos B\cos C)}$$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\cos A\cos B\cos C = \frac{\overbrace{\left(\sum a^2-2a^2\right)\left(\sum a^2-2b^2\right)\left(\sum a^2-2c^2\right)}^{Numerator}}{8a^2b^2c^2}$$

Numerator $= \left(\sum a^2\right)^3-\left(\sum a^2\right)^2\left(\sum a^2\right)+4\sum a^2\left(\sum a^2b^2\right)-8a^2b^2c^2$

$$= \left(\sum a^2\right)\left\{4\sum a^2b^2-4(s^2-4Rr-r^2)^2\right\}-128R^2r^2s^2$$

$$= 4\sum a^2\left\{\left(\sum ab\right)^2-2abc(2s)-(s^2-4Rr-r^2)^2\right\}-128R^2r^2s^2$$

$$= 4\sum a^2\left[\{(s^2+4Rr+r^2)^2-(s^2-4Rr-r^2)^2\}-16s^2Rr\right]-128R^2r^2s^2$$

$$= 4\sum a^2\{2s^2(8Rr+2r^2)-16s^2Rr\}-128R^2r^2s^2$$

$$= 4\sum a^2\{2s^2(8Rr+2r^2-8Rr)\}-128R^2r^2s^2$$

$$= 32s^2r^2(s^2-4Rr-r^2-4R^2)$$

$$\therefore \cos A\cos B\cos C = \frac{32s^2r^2(s^2-4Rr-r^2-4R^2)}{128R^2r^2s^2} \quad (*)$$

$$= \frac{s^2-4Rr-r^2-4R^2}{4R^2} \Rightarrow 1+\prod\cos A = \frac{s^2-4Rr-r^2}{4R^2}$$

$$\therefore \sqrt{6\left(1+\prod\cos A\right)} = \frac{1}{2R}\sqrt{3\cdot(s^2-4Rr-r^2)} = \frac{\sqrt{3\sum a^2}}{2R}$$

$\therefore RHS = \frac{\sqrt{3\sum a^2}}{2R}$ ***LHS*** $= \frac{1}{2R}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)$ $\therefore$ ***suffices to prove:***

$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \geq \sqrt{3\sum a^2} \Leftrightarrow \frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}+\frac{a^2b^2}{c^2}+2\sum a^2 \geq 3\sum a^2$ ***(squaring)***

$$\Leftrightarrow a^4b^4+b^4c^4+c^4a^4 \geq a^2b^2c^2(a^2+b^2+c^2)$$

$\rightarrow$ ***true*** $\because \sum x^2 \geq \sum xy$, ***where*** $x = a^2b^2, y = b^2c^2, z = c^2a^2$ ***(Proved)***

249. In $\Delta\ ABC$ the following relationship holds:

$$(m_a \sin A)^2 + (m_b \sin B)^2 + (m_c \sin C)^2 \le \frac{3}{4}(h_a^2 + h_b^2 + h_c^2)$$

Proposed by Daniel Sitaru – Romania

Solution by Rajsekhar Azaad-India

$$\sin A = \frac{a}{2R}, \dots m_a^2 = \frac{2(b^2 + c^2) - a^2}{4}, \dots h_a = \frac{bc}{2R}, \dots$$

$$LHS = \sum (m_a \sin A)^2 = \sum \frac{a^2}{4R^2} \cdot \frac{[2(b^2 + c^2) - a^2]}{4} = \frac{1}{16R^2}\left[4\sum a^2b^2 - \sum a^4\right]$$

$$\le \frac{1}{16R^2}\left[4\sum a^2b^2 - \sum a^2b^2\right] \quad \left\{\because \sum a^4 \ge \sum a^2b^2\right\}$$

$$= \frac{1}{16R^2}\left[3\sum a^2b^2\right] = \frac{3}{4}\left[\sum \frac{a^2b^2}{4R^2}\right] = \frac{3}{4}\left[h_a^2 + h_b^2 + h_c^2\right] \textit{ (proved)}$$

250. In acute ΔABC the following relationship holds:

$$\frac{1}{\cos^2 A(\cos B + \cos C)^2} + \frac{1}{\cos^2 B(\cos C + \cos A)^2} + \frac{1}{\cos^2 C(\cos A + \cos B)^2} \ge 12$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

In ΔABC, *let* $x = \cos A, y = \cos B, z = \cos C$ *we have* $x + y + z \le \frac{3}{2} \Rightarrow (x + y + z)^2 \le \frac{9}{4}$

$$\Rightarrow xy + yz + zx \le \frac{3}{2} \Rightarrow (xy + yz) + (yz + zx) + (zx + xy) \le \frac{3}{2}$$

$$\Rightarrow \frac{1}{x(y+z)} + \frac{1}{y(z+x)} + \frac{1}{z(x+y)} \ge 6 \Rightarrow$$

$$\Rightarrow \frac{1}{x^2(y+z)^2} + \frac{1}{y^2(z+x)^2} + \frac{1}{z^2(x+y)^2} \ge \frac{\left(\frac{1}{x(y+z)} + \frac{1}{y(z+x)} + \frac{1}{z(x+y)}\right)^2}{3}$$

$$\ge \frac{b^2}{3} = \frac{3b}{3} = 12. \textit{ Therefore it is to be true}$$

$$\frac{1}{\cos^2 A(\cos B + \cos C)^2} + \frac{1}{\cos^2 B(\cos C + \cos A)^2} + \frac{1}{\cos^2 C(\cos A + \cos B)^2} \ge 12$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\because \Delta ABC \text{ is acute – angled; } \therefore \cos A, \cos B, \cos C > 0$$

$$\Rightarrow \cos A\cos B + \cos C\cos A, \cos B\cos C + \cos A\cos B, \cos C\cos A + \cos B\cos C > 0$$

$$\therefore LHS = \frac{1^3}{(\cos A\cos B + \cos C\cos A)^2} + \frac{1^3}{(\cos B\cos C + \cos A\cos B)^2} +$$

$$+\frac{1^3}{(\cos C\cos A + \cos B\cos C)^2} \overset{Radon}{\geq}$$

$$\frac{27}{(\cos A\cos B + \cos C\cos A + \cos B\cos C + \cos A\cos B + \cos C\cos A + \cos B\cos C)^2}$$

$$= \frac{27}{4(\sum\cos A\cos B)^2} \overset{?}{\geq} 12 \Leftrightarrow 2\sum\cos A\cos B \overset{?}{\leq} \frac{3}{2} \Leftrightarrow \left(\sum\cos A\right)^2 - \sum\cos^2 A \overset{?}{\leq} \frac{3}{2}$$

$$\Leftrightarrow \left(1+\frac{r}{R}\right)^2 - \left(\frac{1}{2}\right)\sum(1+\cos 2A) \overset{?}{\leq} \frac{3}{2} \Leftrightarrow \frac{(R+r)^2}{R^2} - \left(\frac{1}{2}\right)\left(3-1-4\prod\cos A\right) \overset{?}{\leq} \frac{3}{2}$$

$$\Leftrightarrow \frac{(R+r)^2}{R^2} + 2\prod\cos A \overset{?}{\leq} \frac{5}{2} \Leftrightarrow \prod\cos A \overset{?}{\leq} \frac{3R^2-4Rr-2r^2}{4R^2} \rightarrow \textbf{(a)}$$

$$\textbf{Now, } \cos A\cos B\cos C = \frac{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}{2bc\cdot 2ca\cdot 2ab} \rightarrow \textbf{(1)}$$

$$\textbf{Numerator} = \left(\sum a^2 - 2a^2\right)\left(\sum a^2 - 2b^2\right)\left(\sum a^2 - 2c^2\right) =$$

$$= \left(\sum a^2\right)^3 - 2\left(\sum a^2\right)^2\left(\sum a^2\right) + 4\left(\sum a^2\right)\left(\sum a^2b^2\right) - 8a^2b^2c^2$$

$$= -\left(\sum a^2\right)^3 + 4\left(\sum a^2\right)\left\{\left(\sum ab\right)^2 - 2abc(2s)\right\} - 128R^2r^2s^2$$

$$= \left(\sum a^2\right)\left\{4\left(\sum ab\right)^2 - \left(\sum a^2\right)^2 - 16sabc\right\} - 128R^2r^2s^2$$

$$= 4\left(\sum a^2\right)\{(s^2+4Rr+r^2)^2 - (s^2-4Rr-r^2)^2 - 16Rrs^2\} - 128R^2r^2s^2$$

$$= 4\left(\sum a^2\right)\{2s^2(8Rr+2r^2) - 16Rrs^2\} - 128R^2r^2s^2$$

$$= 32r^2s^2(s^2-4Rr-r\text{^}2) - 128R^2r^2s^2 = 32r^2s^2(s^2-4R^2-4Rr-r^2) \rightarrow \textbf{(2)}$$

$$\textbf{(1), (2)} \Rightarrow \prod\cos A = \frac{s^2-4R^2-4Rr-r\text{^}2}{4R^2} \rightarrow \textbf{(3)}$$

$$\textbf{(3)} \Rightarrow \textbf{(a)} \Leftrightarrow s^2-4R^2-4Rr-r^2 \overset{?}{\leq} 3R^2-4Rr-2r^2 \Leftrightarrow s^2 \overset{?}{\leq} 7R^2-r^2 \rightarrow \textbf{(b)}$$

LHS of (b) $\overset{Gerretsen}{\leq} 4R^2 + 4Rr + 3r^2 \overset{?}{\leq} 7R^2 - r^2 \Leftrightarrow (R-2r)(3R+2r) \overset{?}{\geq} 0 \overset{Euler}{\rightarrow}$ **true**

$\Rightarrow$ **(b) is true (Proved)**

Solution 3 by Soumitra Mandal-Chandar Nagore-India

We know, $\sum_{cyc} \cos A \leq \frac{3}{2}$ **then** $\sum_{cyc} \frac{1}{\cos^2 A(\cos B + \cos C)^2}$

$$\overset{AM\geq GM}{\geq} \frac{3}{\sqrt[3]{\left(\prod_{cyc}\cos A\right)^2 \prod_{cyc}(\cos A + \cos B)^2}} = 3\left(\frac{1}{\sqrt[3]{\left(\prod_{cyc}\cos A\right)\prod_{cyc}(\cos A + \cos B)}}\right)^2$$

$$\overset{REVERSE\ A.M\ GM}{\geq} 3\left(\frac{1}{\left(\frac{\sum_{cyc}\cos A}{3}\right)\left(\frac{\sum_{cyc}(\cos A + \cos B)}{3}\right)}\right)^2 = \frac{243}{4}\left(\frac{1}{\sum_{cyc}\cos A}\right)^4 \geq \frac{243}{4}\left(\frac{2}{3}\right)^4 = 12 \ \textbf{(Proved)}$$

251. In $\Delta\ ABC$ the following relationship holds:

$$\frac{m_a^3 + m_b^3 + m_c^3}{m_a m_b m_c} \geq \frac{12r^2}{R^2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Daniel Sitaru-Romania

$$m_a + m_b + m_c \geq \frac{1}{2R}(a^2 + b^2 + c^2) \ \textbf{(1) – I. Cucurezeanu – Romania – 1989}$$

$$\frac{m_a^3 + m_b^3 + m_c^3}{m_a m_b m_c} = \sum \frac{m_a^2}{m_b m_c} \overset{BERGSTROM}{\geq} \frac{(\sum m_a)^2}{\sum m_b m_c} \overset{(1)}{\geq}$$

$$\geq \frac{1}{4R^2} \cdot \frac{(\sum a^2)^2}{\sum m_b m_c} \geq \frac{1}{4R^2} \cdot \frac{(\sum a^2)^2}{\sum m_a^2} = \frac{(\sum a^2)^2}{3R^2 \sum a^2} = \frac{1}{3R^2} \cdot \sum a^2 \geq$$

$$\overset{IONESCU-WEITZENBOCK}{\geq} \frac{1}{3R^2} \cdot 4\sqrt{3}S = \frac{4\sqrt{3}rs}{3R^2} \overset{MITRINOVIC}{\geq} \frac{4\sqrt{3}r \cdot 3\sqrt{3}r}{3R^2} = \frac{12r^2}{R^2}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$1 \geq \frac{4r^2}{R^2}(Euler) \Leftrightarrow 3 \geq \frac{12r^2}{R^2};$$

$$\frac{12r^2}{R^2} \leq 3 = \frac{3 \cdot m_a m_b m_c}{m_a m_b m_c} \overset{Mg\leq Ma}{\leq} \frac{m_a^3 + m_b^3 + m_c^3}{m_a m_b m_c}$$

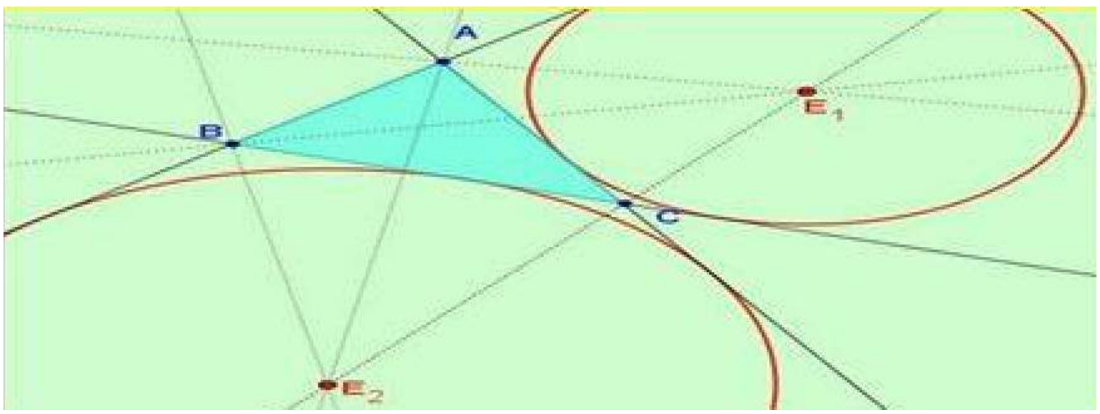
252. **Prove that in any triangle ABC**

$$6\sqrt{3}Rr \le S_{\Delta I_A I_B I_C} \le 3\sqrt{3}R^2$$

where I_A, I_B, I_C are the excenters correspond to A, B, C respectively.

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Daniel Sitaru – Romania



$$E_1 = \pi - \left(\frac{\pi}{2} + \frac{A}{2} + \frac{B}{2}\right) + \pi - \left(\frac{\pi}{2} + \frac{C}{2} + \frac{B}{2}\right) = \frac{\pi}{2} - \frac{B}{2}$$

$$S[\Delta\, E_1E_2E_3] = \frac{1}{2}E_1E_2 \cdot E_1E_3 \sin E = \frac{1}{2}4R\cos\frac{A}{2} \cdot 4R\cos\frac{C}{2} \cdot \sin\left(\frac{\pi}{2} - \frac{B}{2}\right) =$$

$$= 8R^2 \prod \cos\frac{A}{2} = 8R^2 \cdot \frac{s}{4R} = 2Rs$$

$$6\sqrt{3}Rr \le S[\Delta\, E_1E_2E_3] \le 3\sqrt{3}R^2 \leftrightarrow 6\sqrt{3}Rr \le 2Rs \le 3\sqrt{3}R^2 \leftrightarrow$$

$$\leftrightarrow 3\sqrt{3} \le s \le \frac{3\sqrt{3}R}{2} \quad \textit{(MITRINOVIC)}$$

Solution 2 by George Apostolopoulos – Messolonghi – Greece

It is well – known that $I_AI_B = 4R\cos\frac{C}{2}, I_BI_C = 4R\cos\frac{A}{2}$,

$I_CI_A = 4R\cos\frac{B}{2}$. So $I_AI_B \cdot I_BI_C \cdot I_CI_A = 64R^3\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right) =$

$$64R^3\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ca}}\cdot\sqrt{\frac{s(s-c)}{ab}}=$$

$$=64R^3\cdot s\cdot\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}=$$

$$64R^3\cdot s\cdot\frac{Area\ ABC}{abc}=64R^3\cdot s\cdot\frac{r\cdot s}{4R\cdot(rs)}=16R^2s$$

Also, we know that $3\sqrt{3}r\le\frac{3\sqrt{3}}{2}R$. *So*

$$48\sqrt{3}R^2\le I_AI_B\cdot I_BI_C\cdot I_CI_A\le 24\sqrt{3}R^3$$

Equality holds when the triangle ABC is equilateral.

253. In $\Delta\ ABC$:

$$am_a^2+bm_b^2+cm_c^2\ge 9rsR$$

Proposed by Abdilkadir Altintas - Afyonkarashisar-Turkey

Solution by Soumava Chakraborty-Kolkata-India

$$am_a^2+bm_b^2+cm_c^2\overset{(1)}{\ge}9rsR$$

$$(1)\Leftrightarrow a(2b^2+2c^2-a^2)+b(2c^2+2a^2-b^2)+c(2a^2+2b^2-c^2)\ge 36rsR$$

$$\Leftrightarrow 2\sum(a^2b+ab^2)-\sum a^3\ge 36rsR$$

$$\Leftrightarrow 2\sum\{ab(2s-c)\}-\left\{3abc+2s\left(\sum a^2-\sum ab\right)\right\}\ge 36rsR$$

$$\Leftrightarrow 2\{2s(s^2+4Rr+r^2)-12Rrs\}-\{12Rrs+2s(s^2-12Rr-3r^2)\}\ge 36rsR$$

$$\Leftrightarrow 4s(s^2+r^2-2Rr)-2s(s^2-6Rr-3r^2)\ge 36rsR$$

$$\Leftrightarrow 2(s^2+r^2-2Rr)-(s^2-6Rr-3r^2)\ge 18Rr$$

$$\Leftrightarrow s^2+2Rr+5r^2\ge 18Rr\Leftrightarrow s^2\ge 16Rr-5r^2$$

$\rightarrow$ *true by Gerretsen (Proved)*

254. In $\Delta\ ABC$

$$h_a^2\cdot\tan\frac{A}{2}+h_b^2\cdot\tan\frac{B}{2}+h_c^2\cdot\tan\frac{C}{2}\le\frac{9\sqrt{3}}{4}R^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Kevin Soto Palacios-Huarmey-Peru

Probar en un triàngulo ABC

$$h_a^2\cdot\tan\frac{A}{2}+h_b^2\cdot\tan\frac{B}{2}+h_c^2\cdot\tan\frac{C}{2}\le\frac{9\sqrt{3}}{4}R^2\quad (A)$$

Teniendo en cuenta la siguientes desigualdades en un triángulo ABC

$$2r\le R, 2p\le 3\sqrt{3}R,$$

$$\tan\frac{A}{2}=\frac{\big(a+(b-c)\big)\big(a-(b+c)\big)}{4S}\le\frac{a^2-(b-c)^2}{4S}=\frac{a^2}{4S}$$

Análogamente para los siguientes términos $\tan\frac{B}{2}\le\frac{b^2}{4S}, \tan\frac{C}{2}\le\frac{c^2}{4S}$

Utilizando las desigualdades previas en (A)

$$h_a^2\tan\frac{A}{2}+h_b^2\tan\frac{B}{2}+h_c^2\tan\frac{C}{2}=\frac{4S^2}{a^2}\cdot\frac{a^2}{4S}+\frac{4S^2}{b^2}\cdot\frac{b^2}{4S}+\frac{4S^2}{c^2}\cdot\frac{c^2}{4S}=3S=$$

$$=3pr\le 3\cdot\frac{3\sqrt{3}R}{2}\cdot\frac{R}{2}=\frac{9\sqrt{3}R^2}{4}\quad \textit{(LQQD)}$$

255. Prove that in any triangle:

$$m(A)\ge 90^\circ\to m_a\ge\sqrt{\frac{b^2+c^2}{2}}\cos\frac{A}{2}$$

Proposed by Adil Abdullayev-Azerbaidian, Marin Chirciu-Romania

Solution 1 by Daniel Sitaru – Romania

$$\cos A\le 0\to b^2+c^2-a^2\le 0\to -b^2-c^2+a^2\ge 0$$

$$m_a^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2 \ge \frac{s(s-a)(b^2+c^2)}{2bc} \leftrightarrow$$

$$\leftrightarrow bc(2(b^2+c^2) - a^2) \ge 2s(s-a)(b^2+c^2) \leftrightarrow$$

$$\leftrightarrow (b-c)^2(-b^2 - c^2 + a^2) \ge 0$$

Solution 2 by Soumava Chakraborty-Kolkata-India

In any triangle $\Delta\, ABC$ ***with*** $m(A) \ge 90°$***, we have:*** $m_a \overset{(1)}{\ge} \sqrt{\frac{b^2+c^2}{2}}\cos\frac{A}{2}$

(1) $\Leftrightarrow m_a^2 \ge \frac{b^2+c^2}{4}\left(2\cos^2\frac{A}{2}\right) \Leftrightarrow 2b^2 + 2c^2 - a^2 \ge (b^2+c^2)(1+\cos A)$

$$\Leftrightarrow b^2 + c^2 + 2bc\cos A \ge b^2 + c^2 + (b^2+c^2)\cos A$$

$\Leftrightarrow (2bc - b^2 - c^2)\cos A \ge 0 \Leftrightarrow (b-c)^2\cos A \le 0 \rightarrow$ ***true*** $\because \cos A \le 0$

(Proved). Equality when $\Delta\, ABC$ ***is either an isosceles right – angled triangle with*** $m(A) = 90°$***, or an isosceles triangle with*** $m(A) > 90°$***, or a right – angled with*** $m(A) = 90°$ ***and*** $b \ne c$

Solution 3 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC***, donde*** $\angle A \ge 90°;\ m_a \ge \sqrt{\frac{b^2+c^2}{2}}\cos\frac{A}{2}$

Dado que $\angle A \ge 90°$***, implica*** $\cos A \le 0 \Leftrightarrow -\cos A \ge 0$

En la desigualdad propuesta, elevando al cuadrado

$$\frac{b^2 + c^2 + 2bc\cos A}{4} \ge \frac{b^2+c^2}{2}\cdot\left(\frac{1+\cos A}{2}\right)$$

$$\Leftrightarrow b^2 + c^2 + 2bc\cos A \ge b^2 + c^2(1+\cos A)$$

$$\Leftrightarrow 2bc\cos A \ge (b^2+c^2)\cos A \Leftrightarrow -\cos A\,(b-c)^2 \ge 0,$$

(lo cual es cierto)

256. In $\Delta\ ABC$:

$$\frac{3r}{p}\le\tan^3\frac{A}{2}+\tan^3\frac{B}{2}+\tan^3\frac{C}{2}\le\frac{3R}{2p}\left[\left(\frac{3R}{2r}\right)^2-8\right]$$

Proposed by Marin Chirciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\frac{3r}{p}\le\tan^3\frac{A}{2}+\tan^3\frac{B}{2}+\tan^3\frac{C}{2}\le\frac{3R}{2p}\left[\left(\frac{3R}{2r}\right)^2-8\right]=\frac{3R}{2p}\left(\frac{3R}{2r}\right)^2-\frac{12R}{p}$$

Recordar las siguientes identidades

$$(x+y+z)^3-3(x+y)(y+z)(z+x)=x^3+y^3+z^3,$$

$$\frac{4R+r}{p}=\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}$$

Siendo $x=\tan\frac{A}{2}=\frac{r_a}{p}>0,\ y=\tan\frac{B}{2}=\frac{r_b}{p}>0,\ z=\tan\frac{C}{2}=\frac{r_c}{p}>0$

Además $\rightarrow(r_a+r_b)(r_b+r_c)(r_c+r_a)=4Rp^2,$

$R\ge 2r$ ***(Ineq. Euler)*** $\wedge\ p\ge 3\sqrt{3}r$ ***(Inequality Mitrinovic). En*** LHS ***es equivalente*** $\tan^3\frac{A}{2}+\tan^3\frac{B}{2}+\tan^3\frac{C}{2}\ge 3\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

$\Leftrightarrow x^3+y^3+z^3\ge 3xyz$ ***(Válido por MA ≥ MG). En RHS es equivalente***

$$x^3+y^3+z^3=\left(\frac{4R+r}{p}\right)^3-\frac{3}{p^3}\cdot(r_a+r_b)(r_b+r_c)(r_c+r_a)\le$$

$$\le\left(\frac{9R}{2p}\right)\left(\frac{9R}{2\cdot 3\sqrt{3}r}\right)^2-\frac{12R}{p}=\frac{3R}{2p}\left(\frac{3R}{2r}\right)^2-\frac{12R}{p}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

In any $\Delta\ ABC$, $\frac{3r}{s}\overset{(1)}{\lesssim}\tan^3\frac{A}{2}+\tan^3\frac{B}{2}+\tan^3\frac{C}{2}\overset{(2)}{\lesssim}\frac{3R}{2s}\left[\left(\frac{3R}{2r}\right)^2-8\right]$

$$s\tan\frac{A}{2}=s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}$$

$=\frac{\Delta}{s-a}=r_a\Rightarrow \tan\frac{A}{2}=\frac{r_a}{s}$. ***Similarly,*** $\tan\frac{B}{2}=\frac{r_b}{s},\tan\frac{C}{2}=\frac{r_c}{s}$

$$\therefore\sum\tan^3\frac{A}{2}=\frac{1}{s^3}\sum r_a^3\overset{A-G}{\geq}\frac{1}{s^3}\cdot 3r_ar_br_c=\frac{1}{s^3}\cdot 3\frac{r^2s^2}{r}=\frac{3r}{s}$$

$\Rightarrow$ ***(1) is true. Again,*** $\sum\tan^3\frac{A}{2}=\frac{1}{s^3}\sum r_a^3\overset{(i)}{=}\frac{1}{s^3}\left[\left(\sum r_a\right)^3-3\prod(r_a+r_b)\right]$

$$\prod(r_a+r_b)=(r_a+r_b)(r_b+r_c)(r_c+r_a)=2r_ar_br_c+\sum r_ar_b(r_a+r_b)$$

$$=\frac{2r^2s^2}{r}+\sum r_ar_b\left(\sum r_a-r_c\right)=2rs^2+\left(\sum r_a\right)\left(\sum r_ar_b\right)-3rs^2$$

$$=4(4R+r)s^2-rs^2=s^2(4R+r-r)\overset{(ii)}{=}4Rs^2$$

(i), (ii) $\Rightarrow\sum\tan^3\frac{A}{2}=\frac{1}{s^3}\{(4R+r)^3-12Rs^2\}$ ***(iii)***

$\therefore$ ***(2)*** $\Leftrightarrow\frac{1}{s^3}\{(4R+r)^3-12Rs^2\}\leq\frac{3R}{2s}\left\{\left(\frac{3R}{2r}\right)^2-8\right\}$ ***(from (iii))***

$$\Leftrightarrow(4R+r)^3-12Rs^2\leq\frac{27R^3s^2}{8r^2}-12Rs^2$$

$\Leftrightarrow 27R^3s^2\geq(4R+r)^3\cdot 8r^2$ ***(a)***

Gerretsen $\Rightarrow 27R^3s^2\geq 27R^3(16R-5r)r$ ***(b)***

it suffices to prove: (a), (b) $\Rightarrow 27R^3(16R-5r)\geq 8(4R+r)^3r$

$\Leftrightarrow 432t^4-647t^3-384t^2-96t-8\geq 0$ ***(where*** $\frac{R}{r}=t$***)***

$\Leftrightarrow(t-2)(432t^3+217t^2+50t+4)\geq 0\rightarrow$ ***true,*** $\because t=\frac{R}{r}\geq 2$ ***(Euler)***

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$\frac{A}{2};\frac{B}{2};\frac{C}{2}<\frac{\pi}{2}$; $y=\tan^2x$

$$y'' = \left(\frac{3\tan^2 x}{\cos^2 x}\right)' = \frac{6\tan x + 3\tan^2 x \cdot \sin 2x}{\cos^4 x} > 0$$

$$0 < x < \frac{\pi}{2} \quad f(x) = \tan^3 x \quad \textit{CONCAVE}$$

$$\sum \tan^3 \frac{A}{2} \geq 3 \cdot \tan^3 \frac{\pi}{6} = \frac{1}{\sqrt{3}} \geq \frac{3r}{p} \Rightarrow \textit{Mitrinovic}$$

2) RHS: $\tan\frac{A}{2} = x,\ \tan\frac{B}{2} = y,\ \tan\frac{C}{2} = z$

$$xy + yz + zx = 1$$

$$x^3 + y^3 + z^3 = \Big[(x^2 + y^2 + z^2) - \underbrace{(xy + yz + zx)}_{1}\Big] \cdot (x + y + z) + 3xyz =$$

$$= [x^2 + y^2 + z^2 - 1] \cdot (x + y + z) + 3xyz =$$

$$= \Big[(x + y + z)^2 - 2 \cdot \underbrace{(xy + yz + zx)}_{1} - 1\Big](x + y + z) + 3xyz =$$

$$= [(x + y + z)^3 - 3] \cdot (x + y + z) + 3xyz =$$

$$= (x + y + z)^3 - 3 \cdot (x + y + z) + 3xyz$$

$$x^3 + y^3 + z^3 = (x + y + z)^3 - 3 \cdot (x + y + z) + 3xyz \quad (*)$$

$$x + y + z = \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \sqrt{\frac{(p-a)(p-b)}{p \cdot (p-c)}} +$$

$$+ \sqrt{\frac{(p-b)(p-c)}{p(p-a)}} + \sqrt{\frac{(p-c)(p-a)}{p(p-b)}} =$$

$$\frac{1}{\sqrt{p \cdot (p-a) \cdot (p-b) \cdot (p-c)}} \cdot \big((p-a)(p-b) + (p-b)(p-c) + (p-c)(p-a)\big) =$$

$$= \frac{1}{S} \cdot (3p^2 - 4p^2 + ab + bc + ca) = \frac{4Rr + r^2}{S} = \frac{4R + r}{p}$$

$$x + y + z = \frac{4R + r}{p}$$

$$xyz=\tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}=\sqrt{\frac{\left((p-a)(p-b)(p-c)\right)^2}{p^3(p-a)(p-b)(p-c)}}=$$

$$=\sqrt{\frac{p(p-a)(p-b)(p-c)}{p^4}}=\frac{S}{p^2}=\frac{r}{p};\ xyz=\frac{r}{p}$$

$$(*)\Rightarrow x^3+y^3+z^3=\left(\frac{4R+r}{p}\right)^3-3\cdot\left(\frac{4R+r}{p}\right)+\frac{3r}{p}=$$

$$=\left(\frac{4R+r}{p}\right)^3-\frac{12R}{p}-\frac{3r}{p}+\frac{3r}{p}=\left(\frac{4R+r}{p}\right)^3-\frac{12R}{p}$$

$$\left(\frac{4R+r}{p}\right)^3-\frac{12R}{p}\le\frac{3R}{2p}\cdot\left[\left(\frac{3R}{2r}\right)^2-8\right]\ \textit{ASSURE}$$

$$\left(\frac{4R+r}{p}\right)^3-\frac{12R}{p}\le\frac{3R}{2p}\cdot\left(\frac{3R}{2r}\right)^2-\frac{12R}{p};\ \left(\frac{4R+r}{p}\right)^3\le\frac{3R}{2p}\cdot\left(\frac{3R}{2r}\right)^2$$

$$\frac{(4R+r)^3}{p^2}\le\frac{27R^3}{8p\cdot r^2};\ p^2\ge\frac{8r^2\cdot(4R+r)^3}{27R^3}$$

$$p^2\underbrace{\ge}_{Mitrinovic}27r^2\ge\frac{8r^2\cdot(4R+r)^3}{27R^3}$$

$$27^2\cdot R^3\ge\left(2\cdot(4R+r)\right)^3,(9R)^3\ge\left(2\cdot(4R+r)\right)^3;$$

$$9R\ge2\cdot(4R+r);\ R\ge2r\ \textit{Euler}$$

257. Let I be the incenter of triangle ABC. Prove that

$$3(IA^2+IB^2+IC^2)\ge a^2+b^2+c^2+k$$

$$k=\frac{r^2(R-2r)}{R-r}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Soumava Chakraborty-Kolkata-India

$$3\sum AI^2 \overset{(a)}{\geq} \sum a^2 + \frac{r^2(R-2r)}{R-r}, AI = \frac{r}{\sin\frac{A}{2}}$$

$$\therefore LHS = (3r^2)\sum\frac{bc}{(s-b)(s-c)} = \frac{3r^2\sum bc(s-a)}{\prod(s-a)}$$

$$= \frac{3r^2(s^2\sum ab - 12Rrs^2)}{r^2s^2} \overset{(1)}{=} 3(s^2-8Rr+r^2)$$

(a) $\Leftrightarrow 3(s^2-8Rr+r^2) \geq 2(s^2-4Rr-r^2) + \frac{r^2(R-2r)}{R-r}$

$\Leftrightarrow (s^2-16Rr+5r^2)(R-r) \geq r^2(R-2r)$ ***(b) (from (1))***

Rouche $\Rightarrow s^2 \geq 2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}$

$$\therefore (s^2-16Rr+5r^2)(R-r)$$

$$\geq \left\{2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right\}(R-r)$$

$$= (R-r)(R-2r)\left\{2(R-r)-2\sqrt{R^2-2Rr}\right\} \overset{(?)}{\geq} r^2(R-2r)$$

$$\Leftrightarrow (R-r)\left\{2(R-r)-2\sqrt{R^2-2Rr}\right\} \overset{(?)}{\geq} r^2(\because R-2r\geq 0)$$

$$\Leftrightarrow 2R^2+r^2-4Rr \overset{(?)}{\geq} 2\sqrt{R^2-2Rr}\,(R-r)$$

$$\Leftrightarrow (2R^2+r^2-4Rr)^2-4(R^2-2Rr)(R-r)^2 \overset{(?)}{\geq}$$

$\Leftrightarrow r^4 \overset{(?)}{\geq} 0 \rightarrow$ ***true*** $\Rightarrow$ ***(b) is true (Proved)***

258. In $\Delta\, ABC$:

$$\frac{9\sqrt{3}}{2}\cdot\frac{r^2}{R} \leq \sum_{cyc}\frac{r_ar_b}{a+b} \leq \frac{9\sqrt{3}}{8}\cdot R$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{9\sqrt{3}}{2}\cdot\frac{r^2}{R}\leq\sum\frac{r_ar_b}{a+b}\leq\frac{9\sqrt{3}}{8}R$

Tener en cuenta las siguientes identidades y desigualdades en un $\Delta\ ABC$

$$r_a=\frac{S}{s-a},\ r_b=\frac{S}{s-b},\ r_c=\frac{S}{s-c},\ 2s\leq 3\sqrt{3}R$$

$$r_ar_b\geq h_c^2,\ r_br_c\geq h_a^2,\ r_cr_a\geq h_b^2,\ h_a+h_b+h_c\geq 9r$$

$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$ ***(Desigualdad de Nesbitt)*** $\Leftrightarrow a,b,c>0$

Ahora bien en la desigualdad propuesta (LHS)

$$\sum\frac{r_ar_b}{a+b}\geq\frac{h_c^2}{a+b}+\frac{h_a^2}{b+c}+\frac{h_b^2}{c+a}\geq\frac{(h_a+h_b+h_c)^2}{4p}\geq\frac{81r^2}{6\sqrt{3}R}\geq\frac{9\sqrt{3}}{2}\cdot\frac{r^2}{R}$$

Finalmente en RHS, es equivalente $\sum\frac{r_ar_b}{a+b}=\frac{s(s-c)}{a+b}+\frac{s(s-a)}{b+c}+\frac{s(s-b)}{c+a}=$

$$=s\left(\frac{a+b-c}{2(a+b)}+\frac{b+c-a}{2(b+c)}+\frac{a+c-b}{2(c+a)}\right)$$

$$\sum\frac{r_ar_b}{a+b}=s\left(\frac{1}{2}-\frac{c}{2(a+b)}+\frac{1}{2}-\frac{a}{2(b+c)}+\frac{1}{2}-\frac{b}{2(c+a)}\right)\leq\frac{3s}{2}-\frac{3s}{4}=\frac{3s}{4}\leq\frac{9\sqrt{3}}{8}R$$

259. In $\Delta\ ABC$, O – circumcentre, I - incentre,

N – ninepoint center, I_a, I_b, I_c - excenters

$$ON^2+OI^2+OI_a^2+OI_b^2+OI_c^2>\frac{4s(R-2r)}{3}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$ON^2=\frac{1}{4}OH^2\overset{(1)}{=}\frac{9R^2-\sum a^2}{4}$ and $OI^2=R^2-2Rr$ **(2)**

$$\sum OI_a^2=\sum(R^2+2Rr_a)=3R^2+2R(4R+r)\overset{(3)}{=}11R^2+2Rr$$

$$\textbf{(1) + (2) + (3)} \Rightarrow LHS = \frac{9R^2-\sum a^2}{4} + 12R^2 = \frac{57R^2-\sum a^2}{4}$$

$$\therefore \text{ it suffices to prove: } \frac{57R^2-2(s^2-4Rr-r^2)}{4} > \frac{4s(R-2r)}{3}$$

$$\Leftrightarrow 171R^2 - 6s^2 + 24Rr + 6r^2 > 16s(R-2r)$$

$$\Leftrightarrow 171R^2 + 24Rr + 6r^2 > 6s^2 + 16s(R-2r) \quad (4)$$

$$\text{Now, } 6s^2 + 16s(R-2r) \underset{(5)}{\overset{Gerretsen}{\leq}} 24R^2 + 24Rr + 18r^2 + 16s(R-2r)$$

(4), (5)* ⇒ *it suffices to prove:

$$24R^2 + 24Rr + 18r^2 + 16s(R-2r) < 171R^2 + 24Rr + 6r^2$$

$$\Leftrightarrow 147R^2 - 12r^2 > 16s(R-2r)$$

$$\Leftrightarrow (147R^2 - 12r^2) > 256s^2(R-2r)^2 \quad (6)$$

$$256s^2(R-2r)^2 \underset{(7)}{\overset{Gerretsen}{\leq}} 256(4R^2 + 4Rr + 3r^2)(R-2r)^2$$

(6), (7)* ⇒ *it suffices to prove:

$$256(4R^2 + 4Rr + 3r^2)(R-2r)^2 < (147R^2 - 12r^2)^2$$

$$\Leftrightarrow 20585t^4 + 3072t^3 - 4296t^2 - 1024t - 2928 > 0 \text{ (where } t = \frac{R}{r}\text{)}$$

$$\Leftrightarrow (t-2)(20585t^3 + 44242t^2 + 84188t + 167352) + 331776 > 0 \rightarrow$$

$$\text{true} \because t = \frac{R}{r} \geq 2 \text{ (Euler) (Proved)}$$

260. Prove that in any acute triangle ABC,

$$m_a r_a + m_b r_b + m_c r_c \leq s^2.$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triàngulo acutángulo ABC

$$m_ar_a+m_br_b+m_cr_c\le s^2$$

Dado que es un triángulo acutángulo, tener en cuenta las siguientes desigualdades

$$m_a\le R(1+\cos A),\ m_b\le R(1+\cos B),\ m_c\le R(1+\cos C)$$

La desigualdad es equivalente

$$m_ar_a+m_br_b+m_cr_c\le R(1+\cos A)\cdot\frac{S}{s-a}+R(1+\cos B)\cdot\frac{S}{s-b}+R(1+\cos C)\cdot\frac{S}{s-c}$$

$$=R\frac{(b+c+a)(b+c-a)}{2bc}\cdot\frac{S}{s-a}+R\cdot\frac{(a+b+c)(a+c-b)}{2ca}\cdot\frac{S}{s-b}+$$

$$+R\cdot\frac{(a+b+c)(a+b-c)}{2ab}\cdot\frac{S}{s-c}$$

$$=R\cdot\frac{S}{bc}(a+b+c)+R\cdot\frac{S}{ca}(a+b+c)+R\cdot\frac{R}{ab}(a+b+c)=$$

$$=R\cdot S(a+b+c)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=R\cdot S\cdot\frac{(a+b+c)^2}{abc}=R\cdot S\cdot\frac{4s^2}{4RS}=s^2\quad \textbf{(LQQD)}$$

Solution 2 by Nirapada Pal-Jhargram-India

$$\sum m_ar_a\le\sum\left(2R\cos^2\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)$$

$$=s\sum R\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)=s\sum R\sin A=s\sum\frac{a}{2}=s^2$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS\le R(1+\cos A)r_a+R(1+\cos B)r_b+R(1+\cos C)r_c$$

$$=R\cdot 2\cos^2\frac{A}{2}r_a+R\cdot 2\cos^2\frac{B}{2}r_b+R\cdot 2\cos^2\frac{C}{2}r_c$$

$$=2R\frac{s(s-a)}{bc}\cdot\frac{\Delta}{s-a}+2R\frac{s(s-b)}{ca}\cdot\frac{\Delta}{s-b}+2r\frac{s(s-c)}{ab}\cdot\frac{\Delta}{s-c}$$

$$=2R\Delta s\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)=\frac{2R\Delta s(2s)}{4R\Delta}=s^2$$

(Proved)

261. In $\Delta\ ABC$:

$$\sum|\tan A| \leq \left|\prod \tan A\right| + \sum|\tan A - \tan B|$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum|\tan A| \overset{(1)}{\leq} \left|\prod \tan A\right| + \sum|\tan A - \tan B|$$

In a triangle, either (1) all $\angle s$ are acute, or (2) 2 angles are acute and 1 angle is obtuse. (N.B.: in this problem, an angle cannot be $= 90°$)

Case 1: All angles are acute. Then, (1)

$$\Leftrightarrow \tan A + \tan B + \tan C \leq \tan A \tan B \tan C + \sum|\tan A - \tan B|$$

($\because$ tangent of an acute angle is always > 0)

$$\Leftrightarrow \sum|\tan A - \tan B| \geq 0 \quad \textbf{(2)} \ (\because \sum \tan A = \prod \tan A)$$

But $\sum|\tan A - \tan B| = |\tan A - \tan B| + |\tan B - \tan C| +$

$+|\tan C - \tan A| \geq |\tan A - \tan B + \tan B - \tan C + \tan C - \tan A|$

$= |0| = 0 \Rightarrow$ ***(2) is true*** $\Rightarrow$ ***(1) is true***

Case 2 1 ***angle is obtuse WLOG, we may assume*** $m(\angle A) > 90°$

$\therefore \tan B, \tan C > 0$ ***and*** $\tan A < 0$

$\therefore$ ***(1)*** $\Leftrightarrow \tan B + \tan C - \tan A \leq -\tan A \tan B \tan C$

$+|\tan A - \tan B| + |\tan B - \tan C| + |\tan C - \tan A|$

$\Leftrightarrow \tan B + \tan C - \tan A \leq -(\tan A + \tan B + \tan C)$

$+ \tan B - \tan A + |\tan B - \tan C| + \tan C - \tan A$

$(\because \tan A < 0 < \tan B \ \ and \tan C > 0 > \tan A)$

$\Leftrightarrow |\tan B - \tan C| \geq 2\tan A + \tan B + \tan C$ ***(3)***

Case 2a $\tan B \geq \tan C$ ***and of course*** $m(\angle A) > 90°$

$\therefore$ *(3)* $\Leftrightarrow \tan B - \tan C \geq 2\tan A + \tan B + \tan C \Leftrightarrow \tan C + \tan A \leq 0$

$$\Leftrightarrow \frac{\sin C}{\cos C} + \frac{\sin A}{\cos A} \leq 0 \Leftrightarrow \frac{\sin(C+A)}{\cos C \cos A} \leq 0$$

$\Leftrightarrow \frac{\sin B}{\cos C \cos A} \leq 0 \rightarrow$ ***true*** $\because \sin B, \cos C > 0$ ***and*** $\cos A < 0$

$\Rightarrow$ ***(3) is true*** $\Rightarrow$ ***(1) is true***

Case 2b $\tan B < \tan C$ ***and of course*** $m(\angle A) > 90°$

(3) $\Leftrightarrow \tan C - \tan B \geq 2\tan A + \tan B + \tan C$

$$\Leftrightarrow \tan A + \tan B \leq 0 \Leftrightarrow \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} \leq 0$$

$\Leftrightarrow \frac{\sin(A+B)}{\cos A \cos B} \leq 0 \Leftrightarrow \frac{\sin C}{\cos A \cos B} \leq 0 \rightarrow$ ***true*** $\because \sin C, \cos B > 0$ ***and*** $\cos A < 0$

$\Rightarrow$ ***(3) is true*** $\Rightarrow$ ***(1) is true***

Thus, in case of acute-angled as well as obtuse-angled triangles, (1) is true. (Proved)

Solution 2 by Richdad Phuc-Hanoi-Vietnam

$A = \max\{A, B, C\}$ ***case*** $A < \frac{\pi}{2} \Rightarrow \tan A, \tan B, \tan C > 0$

we know $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$RHS - LHS = \sum_{cyc} |\tan A - \tan B| \geq 0 \Rightarrow$ ***Q.E.D***

equality holds if $A = B = C$ ***case*** $A > \frac{\pi}{2} \Rightarrow \tan A < 0$

we have $RHS - LHS = |\tan A \tan B \tan C| + |\tan B - \tan C| - \tan A > 0$

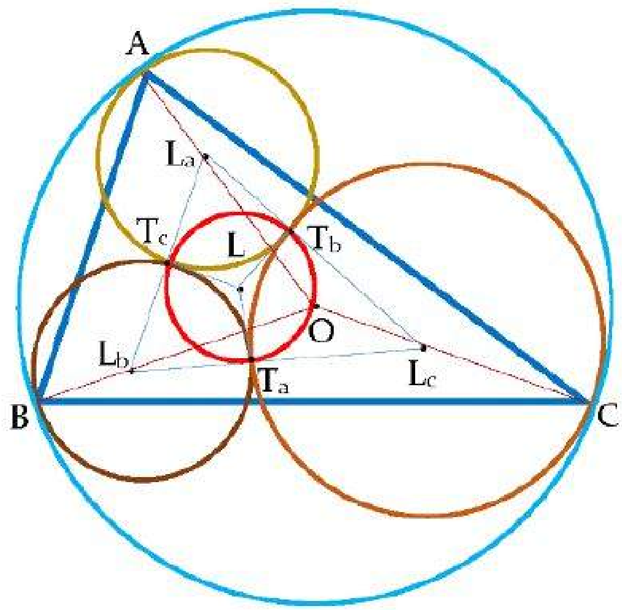
262. ABC be a triangle and R_a, R_b, R_c are radii of Lucas Circles of ABC.

Prove that: $R_a + R_b + R_c \geq \frac{8\cdot\Delta}{(1+\sqrt{3})^2 \cdot R}$

where Δ and R are area and circumradius of ABC respectively.

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Daniel Sitaru – Romania



$$\sum R_a = R\sum \frac{bc}{bc+2aR} = \sum R\cdot\frac{abc}{abc+2a^2R} =$$

$$= R\sum\frac{4RS}{4RS+2a^2R} = 2R\sum\frac{S}{S+a^2} \overset{BERGSTROM}{\geq}$$

$$\geq 2RS\cdot\frac{9}{3S+\sum a^2} \overset{HADWIGER}{\geq} 2RS\cdot\frac{9}{\left(\frac{\sqrt{3}}{4}+1\right)\sum a^2} \overset{LEIBNIZ}{\geq}$$

$$\geq 2RS\cdot\frac{9}{\left(\frac{\sqrt{3}}{4}+1\right)9R^2} = \frac{8S}{\left(1+\sqrt{3}\right)^2R}$$

263. **Prove that in any triangle**

$$3\left(r_a^2r_b+r_b^2r_c+r_c^2r_a\right)\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)\geq p^6$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

$$x = ra > 0, y = rb > 0, z = rc > 0 \to xy + yz + zx = p^2 \text{ by Holder's inequality} \to$$

$$(1 + 1 + 1)(x^2y + y^2z + z^2x)(y^{2x} + z^{2y} + x^{2z}) \geq (xy + yz + zx)^3 = p^6$$

Solution 2 by Daniel Sitaru – Romania

$$3\sum r_a^2 r_b \sum r_a r_b^2 \overset{HOLDER}{\geq} \left(\sum r_a r_b\right)^3 = (s^2)^3 = s^6$$

264. In ΔABC: O - circumcenter, I – incentre, I_a, I_b, I_c- excenters

$$(OI_a^2 - OI^2)(OI_b^2 - OI^2)(OI_c^2 - OI^2) > 16abcrR^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

Lemma 1 (Euler)

$$OI_a^2 = R^2 + 2Rr_a, OI^2 = R^2 - 2Rr$$

$$LHS \geq RHS \leftrightarrow (r_a + r)(r_b + r)(r_c + r) > 8pr^2$$

$$(r_a + r)(r_b + r)(r_c + r) > 8\sqrt{r_a r_b r_c r^3} = 8\sqrt{(pr^2)^2} = 8pr^2$$

Solution 2 by Marin Chirciu-Romania

We use the known identities in triangle $OI_a^2 = R^2 + 2Rr_a$ ***and*** $OI^2 = R^2 - 2Rr$ ***we obtain***

$$\prod(OI_a^2 - OI^2) = \prod(R^2 + 2Rr_a - R^2 - 2Rr) = 8R^3\prod(r + r_a) =$$

$$= 8R^3 \cdot 2r(p^2 + r^2 + 2Rr) = 16R^3r(p^2 + r^2 + 2Rr)$$

We use Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$ ***we obtain:***

$$M_s = \prod(OI_a^2 - OI^2) = 16R^3r(p^2 + r^2 + 2Rr) \geq$$

$$\geq 16R^3r(16Rr - 5r^2 + r^2 + 2Rr) = 16R^3r(18Rr - 4r^2) \geq$$

$$\geq 16R^3r(18Rr - 2Rr) = 256R^4r^2.$$

The equality holds if and only if the triangle is equilateral.

As $256R^4r^2 > 16abcrR^2 \Leftrightarrow 256R^4r^2 > 16 \cdot 4pRr \cdot rR^2 \Leftrightarrow 4R > p$,

obviously from Mitrinovic $p \leq \frac{3\sqrt{3}}{2} \cdot R$, ***it follows the conclusion.***

The inequality from the enunciation is strict.

265. Prove that in any triangle ABC,

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \geq \sqrt{\frac{6R}{r}}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \geq \sqrt{\frac{6R}{r}}$

Recordar lo siguiente: $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{4(p-a)(p-b)(p-c)}{abc}$,

$$\frac{2S}{bc} = \sin A, \frac{2S}{ac} = \sin B, \frac{2S}{ab} = \sin C$$

$$S = \sqrt{p(p-a)(p-b)(p-c)}, p = \frac{a+b+c}{2},$$

$ab + bc + ac \geq \sqrt{3abc(a+b+c)}$. ***Reemplazando en la desigualdad:***

$$\frac{bc}{2S} + \frac{ac}{2S} + \frac{ab}{2S} \geq \sqrt{\frac{6}{4}\csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2}} \to \sqrt{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left(\frac{bc}{2S} + \frac{ac}{2S} + \frac{ab}{2S}\right) \geq \frac{\sqrt{6}}{2}$$

$$\Rightarrow \sqrt{\frac{(p-a)(p-b)(p-c)}{abc}}\left(\sum \frac{bc}{2\sqrt{p(p-a)(p-b)(p-c)}}\right) \geq \frac{\sqrt{6}}{2}$$

$$\Rightarrow \frac{bc}{\sqrt{abcp}} + \frac{ac}{\sqrt{abcp}} + \frac{ab}{\sqrt{abcp}} = \frac{\sqrt{2}}{\sqrt{abc}\sqrt{(a+b+c)}}(bc + ac + ab) \geq$$

$$\geq \frac{\sqrt{2}\sqrt{3abc(a+b+c)}}{\sqrt{abc(a+b+c)}} = \sqrt{6} \ \textit{(LQQD)}$$

Solution 2 by Adil Abdullayev – Baku – Azerbaidjian

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} = \frac{2R}{a} + \frac{2R}{b} + \frac{2R}{c} =$$

$$= 2R \cdot \frac{ab + bc + ca}{abc} = \frac{ab + bc + ca}{2rp} \geq \sqrt{\frac{6R}{r}} \leftrightarrow$$

$$(ab + bc + ca)^2 \geq (2rp)^2 \cdot \frac{6R}{r} =$$

$$= (a + b + c)^2 \cdot 6Rr = (a + b + c)^2 \cdot 6 \cdot \frac{abc}{4S} \cdot \frac{2S}{a + b + c} =$$

$$= 3abc(a + b + c)^2 \leftrightarrow (ab + bc + ca)^2 \geq 3abc(a + b + c)\ldots \textit{(A)}$$

Let $ab = x; bc = y; ca = z;$ ***(A)*** $\leftrightarrow (x + y + z)^2 \geq 3(xy + yz + zx)$

Solution 3 by Martin Lukarevski – Skopje

$$\sum \frac{1}{\sin A} = 2R \sum \frac{1}{a} \geq 2R\sqrt{3\sum \frac{1}{ab}} = \sqrt{\frac{6R}{r}}$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \geq \sqrt{\frac{6R}{2}}; \ ab + bc + ca \geq \sqrt{3abc(a + b + c)} |: 2S$$

$$\frac{ab}{2S} + \frac{bc}{2S} + \frac{ca}{2S} \geq \sqrt{\frac{3abc(a + b + c)}{4S^2}}$$

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\geq\sqrt{\frac{6abc\cdot p}{4\cdot\frac{(abc)^2}{16R^2}}}=\sqrt{\frac{6\cdot R\cdot p}{\frac{abc}{4R}}}=\sqrt{\frac{6R\cdot p}{S}}=\sqrt{\frac{6R}{2}}$$

Solution 5 by George Apostolopoulos – Messolonghi – Greece

We have (Law of Sines) $\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}=\frac{2R}{a}+\frac{2R}{b}+\frac{2R}{c}=2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

It is well-known that $\frac{1}{4r^2}\geq\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq\frac{1}{2Rr}$

So $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{3}{2Rr}}$***. Now***

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}=2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 2R\sqrt{\frac{3}{2Rr}}=\sqrt{\frac{4R^2\cdot 3}{2Rr}}=\sqrt{\frac{6R}{r}}$$

Equality holds when the triangle ABC is equilateral.

Solution 6 by Marin Chirciu-Romania

We use the known identity in triangle $\sum\frac{1}{\sin A}=\frac{p^2+r^2+4Rr}{2pr}$***. We write the inequality***

$$\frac{p^2+r^2+4Rr}{2pr}\geq\sqrt{\frac{6R}{r}}\Leftrightarrow\left(\frac{p^2+r^2+4Rr}{2pr}\right)^2\geq\frac{6R}{r}\Leftrightarrow$$

$\Leftrightarrow p^2(p^2+2r^2-16Rr)+r^2(4R+r)^2\geq 0$. ***We distinguish the following cases:***

1) If $p^2+2r^2-16Rr\geq 0$***, the inequality is obvious.***

2) If $p^2+2r^2-16Rr<0$***, we rewrite the inequality:***

$p^2(16Rr-2r^2-p^2)\leq r^2(4R+r)^2$***, which follows from Gerretsen's inequality***

$16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2$***. It remains to prove that:***

$(4R^2+4Rr+3r^2)(16Rr-2r^2-16Rr+5r^2)\leq r^2(4R+r)^2\Leftrightarrow$

$\Leftrightarrow R^2-Rr-2r^2\geq 0\Leftrightarrow(R-2r)(R+r)\geq 0$***, obviously from Euler's inequality***

$R\geq 2r$***. The equality holds if and only if the triangle is equilateral.***

266. In $\Delta\ ABC$:

$$\sqrt{3}\le \tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\le \frac{R}{r}\sqrt{n\cdot\frac{R}{r}-2n+\frac{3}{4}},\ n\ge 0$$

Proposed by Marin Chirciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\sqrt{3}\le \tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\le \frac{R}{r}\sqrt{n\cdot\frac{R}{r}-2n+\frac{3}{4}},\ n\ge 0$$

Recordar las siguientes identidades y desigualdades

$\frac{4R+r}{p}=\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}$, $R\ge 2r$ ***(Euler)***, $p\ge 3\sqrt{3}r$ ***(Mitrinovic)***

$x+y+z\ge\sqrt{3(xy+yz+zx)}$, ***donde*** $x=\tan\frac{A}{2}>0$, $y=\tan\frac{B}{2}>0$, $z=\tan\frac{C}{2}>0$. ***Además*** $\rightarrow xy+yz+zx=1$. ***Por la tanto*** $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\ge\sqrt{3}$. ***Por ultimo***

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{4R+r}{p}\le\frac{9R}{2p}\le\frac{9R}{2\cdot 3\sqrt{3}r}=\frac{\sqrt{3}R}{2r}\le\frac{R}{r}\sqrt{n(R-2r)+\frac{3}{4}}$$

Lo cual es cierto ya que aplicando $R\ge 2r$; $\frac{R}{r}\sqrt{n(R-2r)+\frac{3}{4}}\ge\frac{R}{r}\sqrt{\frac{3}{4}}=\frac{\sqrt{3}R}{2r}$

Solution 2 Soumava Chakraborty-Kolkata-India

In any $\Delta\ ABC$, $\sqrt{3}\overset{(1)}{\le}\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\overset{(2)}{\le}\frac{R}{r}\sqrt{n\frac{R}{r}-2n+\frac{3}{4}}$; $n\ge 0$

$$s\tan\frac{A}{2}=\frac{s\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}=\frac{\Delta}{s-a}=r_a$$

$$\therefore \tan\frac{A}{2}=\frac{r_a}{s}.\ \textit{Similarly, } \tan\frac{B}{2}=\frac{r_b}{s}\ \textit{and}\ \tan\frac{C}{2}=\frac{r_c}{s}$$

$$\therefore \sum \tan\frac{A}{2} = \frac{\sum r_a}{s} = \frac{4R+r}{s}$$

$$\therefore \textit{(1)} \Leftrightarrow \frac{4R+r}{s} \geq \sqrt{3} \Leftrightarrow 3s^2 \leq (4R+r)^2 \quad \textit{(i)}$$

$$\textit{Now, Gerretsen} \Rightarrow 3s^2 \leq 3(4R^2+4Rr+3r^2) \quad \textit{(ii)}$$

$$\textit{(i), (ii)} \Rightarrow \textit{it suffices to prove: } 12R^2+12Rr+9r^2 \leq 16R^2+8Rr+r^2$$

$$\Leftrightarrow 4R^2-4Rr-8r^2 \geq 0 \Leftrightarrow R^2-Rr-2r^2 \geq 0$$

$$\Leftrightarrow (R+r)(R-2r) \geq 0 \to \textit{true} \because R \geq 2r \textit{ (Euler)} \Rightarrow \textit{(1) is true}$$

$$\textit{(2)} \Leftrightarrow \frac{R}{r}\sqrt{n\left(\frac{R}{r}-2\right)+\frac{3}{4}} \geq \frac{4R+r}{s} \Leftrightarrow \frac{R}{r}\sqrt{\frac{4n(R-2r)+3r}{4r}} \geq \frac{4R+r}{s} \quad \textit{(iii)}$$

$$\because n \geq 0, \therefore \frac{R}{r}\sqrt{\frac{4n(R-2r)+3r}{4r}} \geq \frac{R}{r}\sqrt{\frac{3}{4}} \quad \textit{(iv)}$$

(iii), (iv) ⇒ in order to prove (2), it suffices to prove:

$$\frac{R^2}{r^2}\cdot\frac{3}{4} \geq \frac{(4R+r)^2}{s^2} \Leftrightarrow 3s^2R^2 \geq 4r^2(4R+r)^2 \quad \textit{(v)}$$

$$\textit{(*) Gerretsen} \Rightarrow 3s^2R^2 \overset{(vi)}{\geq} 3R^2r(16R-5r)$$

(v), (vi) ⇒ in order to prove (2), it suffices to prove:

$$3R^2(16R-5r) \geq 4r(4R+r)^2$$

$$\Leftrightarrow 48t^3-79t^2-32t-4 \geq 0 \textit{ (where } t = \tfrac{R}{r}\textit{)}$$

$$\Leftrightarrow (t-2)(48t^2+17t+2) \geq 0 \to \textit{true} \because t \geq 2 \textit{ (Euler)} \Rightarrow \textit{(2) is true}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\textit{1) LHS: } \sum \tan\frac{A}{2} = \frac{4R+r}{p} \geq \sqrt{3}$$

$$(4R+r)^2 \geq \left(\sqrt{3}p\right)^2;\ 16R^2+8Rr+r^2 \geq 3p^2$$

$$16R^2+8Rr+r^2 \geq 3(R^2+4Rr+3r^2) \geq 3p^2;\ 4R^2 \geq 4Rr+8r^2$$

$$R^2 \geq Rr+2r^2;\ R^2 \geq 2Rr \geq Rr+2r^2;\ Rr \geq 2r^2 \textit{ (Euler)}$$

2) RHS: $R \ge 2r$ **(Euler);** $un \cdot R \ge 8n \cdot r \quad (n \ge 0)$

$$un \cdot R + (3-8n) \cdot r \ge 3r \Leftrightarrow 1 \ge \frac{3r}{4n \cdot R + (3-8n) \cdot r} \ (*)$$

$$p^2 > 27r^2 \overset{(*)}{\ge} 27r^2 \cdot \frac{3r}{4Rr + (3-8n)r} = \frac{81r^2 \cdot r}{un \cdot R + (3-8n) \cdot r}$$

$$4p^2 \ge \frac{81r^2 \cdot 4r}{un \cdot R + (3-8n)r} = \frac{81r^2}{n \cdot \frac{R}{r} - 2n + \frac{3}{4}} \Leftrightarrow \sqrt{n \cdot \frac{R}{r} - 2n + \frac{3}{4}} \ge \frac{9r}{4p} \Big| \cdot \frac{R}{r}$$

$$\frac{R}{r} \cdot \sqrt{n \cdot \frac{R}{r} - 2n + \frac{3}{4}} \ge \frac{9R}{4p} = \frac{8R + r}{4p} \ge \frac{8R + 2r}{4p} = \frac{4R + r}{p} = \sum \tan\frac{A}{2}$$

267. In $\Delta\, ABC, I$ - incentre, I_a, I_b, I_c – excenters

$$\frac{1}{II_a^2} + \frac{1}{II_b^2} + \frac{1}{II_c^2} + \frac{1}{I_bI_c^2} + \frac{1}{I_cI_a^2} + \frac{1}{I_aI_b^2} \le \frac{1}{4r^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{1}{II_a^2} + \frac{1}{II_b^2} + \frac{1}{II_c^2} + \frac{1}{I_b^2I_c^2} + \frac{1}{I_c^2I_a^2} + \frac{1}{I_a^2I_b^2} \le \frac{1}{4r^2}$

Siendo I ***- Incentro, tener en cuenta las siguientes notaciones y algunas desigualdades previas*** $II_a = 4R\sin\frac{A}{2}, II_b = 4R\sin\frac{B}{2}, II_C = 4R\sin\frac{C}{2}$,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \le \frac{1}{4r^2};\ I_aI_b = 4R\cos\frac{A}{2}, I_bI_c = 4R\cos\frac{A}{2}, I_cI_a = 4R\cos\frac{B}{2}$$

La desigualdad propuesta es equivalente

$$\frac{1}{16R^2}\left(\csc^2\frac{A}{2} + \csc^2\frac{B}{2} + \csc^2\frac{C}{2}\right) + \frac{1}{16R^2}\left(\sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2}\right) \le \frac{1}{4r^2}$$

$$\Leftrightarrow \frac{1}{16R^2}\left(\csc^2\frac{A}{2} + \sec^2\frac{A}{2}\right) + \frac{1}{16R^2}\left(\csc^2\frac{B}{2} + \sec^2\frac{B}{2}\right) + \frac{1}{16R^2}\left(\csc^2\frac{C}{2} + \sec^2\frac{C}{2}\right) \le \frac{1}{4r^2}$$

$$\Leftrightarrow \frac{1}{16R^2}\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)^2+\frac{1}{16R^2}\left(\tan\frac{B}{2}+\cot\frac{B}{2}\right)^2+\frac{1}{16R^2}\left(\tan\frac{C}{2}+\cot\frac{C}{2}\right)^2\le\frac{1}{4r^2}$$

$$\Leftrightarrow \frac{1}{16R^2}(4\csc^2 A+4\csc^2 B+4\csc^2 C)=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2}\quad \textit{(LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$II_a = a\sec\frac{A}{2} = 2R\sin A\sec\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{A}{2}\sec\frac{A}{2} = 4R\sin\frac{A}{2}$$

$$\Rightarrow \frac{1}{II_a^2}=\frac{1}{16R^2\sin^2\frac{A}{2}}\ \textit{(1)}.\ \textit{Similarly}, \frac{1}{II_b^2}=\frac{1}{16R^2\sin^2\frac{B}{2}}\ \textit{(2)};\ \frac{1}{II_c^2}=\frac{1}{16R^2\sin^2\frac{C}{2}}\ \textit{(3)}$$

$$I_bI_c = a\csc\frac{A}{2} = 2R\sin A\csc\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{A}{2}\csc\frac{A}{2} = 4R\cos\frac{A}{2}$$

$$\Rightarrow \frac{1}{I_bI_c^2}=\frac{1}{16R^2\cos^2\frac{A}{2}}\ \textit{(4)}.\ \textit{Similalry}, \frac{1}{I_cI_a^2}=\frac{1}{16R^2\cos^2\frac{B}{2}},\ \textit{(5)};\ \frac{1}{I_aI_b^2}=\frac{1}{16R^2\cos^2\frac{C}{2}}\ \textit{(6)}$$

(1)+(2)+(3)+(4)+(5)+(6)⇒

$$LHS = \frac{1}{16R^2}\sum\frac{1}{\sin^2\frac{A}{2}}+\frac{1}{\cos^2\frac{A}{2}}=\frac{1}{16R^2}\sum\frac{1}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}$$

$$=\sum\frac{1}{\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)^2}=\sum\frac{1}{(2R\sin A)^2}=\sum\frac{1}{a^2}=\frac{\sum a^2b^2}{a^2b^2c^2}\le\frac{4R^2S^2}{a^2b^2c^2}\ \textit{(Goldstone's inequality)}=\frac{4R^2S^2}{16R^2r^2s^2}=\frac{1}{4r^2}\ \textit{(Proved)}$$

268. In $\Delta\ ABC$:

$$\sum a^2\ge\sum(b-c)^2+4S\sqrt{\frac{2(2R-r)}{R}}$$

Proposed by Martin Lukarevski-Stip

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \sum a^2\ge\sum(b-c)^2+4S\sqrt{\frac{2(2R-r)}{R}}$

Tener en cuenta las siguientes identidades $ab = 2S\csc A$,

$bc = 2s\csc A$, $ca = 2S\csc B$, $a^2 + b^2 + c^2 = 4S(\cot A + \cot B + \cot C)$

Si $\to \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \leftrightarrow xy + yz + zx = 1 \leftrightarrow x = \tan\frac{A}{2}, y = \tan\frac{B}{2}, z = \tan\frac{C}{2}$

La desigualdad es equivalente

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 4S\sqrt{4 - \frac{2r}{R}}$$

$$4S\big((\csc A - \cot A) + (\csc B - \cot B) + (\csc C - \cot C)\big) \geq 4S\sqrt{4 - \frac{2r}{R}}$$

$$\Leftrightarrow \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{4 - \frac{2r}{R}}$$

$$\Leftrightarrow \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} + \frac{2r}{R} = \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} + 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \geq 2$$

1) Si: $m, n, p \in \mathbb{R}$ ***en un triángulo*** xyz***, se cumple la siguiente desigualdad:*** $m^2 + n^2 + p^2 \geq 2np\cos x + 2pm\cos y + 2mn\cos z$

Siendo: $m = \tan\frac{x}{2} > 0, y = \tan\frac{y}{2} > 0, z = \tan\frac{z}{2} > 0$***, resulta:***

$$\tan^2\frac{x}{2} + \tan^2\frac{y}{2} + \tan^2\frac{z}{2} \geq 2\tan\frac{y}{2}\tan\frac{z}{2}\cos x + 2\tan\frac{z}{2}\tan\frac{x}{2}\cos y + 2\tan\frac{x}{2}\tan\frac{y}{2}\cos z$$

$$\tan^2\frac{x}{2} + \tan^2\frac{y}{2} + \tan^2\frac{z}{2} \geq 2\tan\frac{y}{2}\tan\frac{z}{2}\cos x + 2\tan\frac{z}{2}\tan\frac{x}{2}\cos y + 2\tan\frac{x}{2}\tan\frac{y}{2}\cos z$$

Probaremos que

$$2\tan\frac{y}{2}\tan\frac{z}{2}\cos x + 2\tan\frac{z}{2}\tan\frac{x}{2}\cos y + 2\tan\frac{x}{2}\tan\frac{y}{2}\cos z = 2 - 8\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}$$

$$\Leftrightarrow 2\left(\tan\frac{y}{2}\tan\frac{z}{2} - 1\right)\cos x + 2\left(\tan\frac{z}{2}\tan\frac{x}{2} - 1\right)\cos y + 2\left(\tan\frac{x}{2}\tan\frac{y}{2} - 1\right)\cos z +$$

$$+2(\cos x + \cos y + \cos z)$$

$$\Leftrightarrow \frac{2\cos\left(\frac{y+z}{2}\right)}{\cos\frac{y}{2}\cos\frac{z}{2}}\cos x - \frac{2\cos\left(\frac{z+x}{2}\right)}{\cos\frac{z}{2}\cos\frac{x}{2}}\cos y - \frac{2\cos\left(\frac{x+y}{2}\right)}{\cos\frac{x}{2}\cos\frac{y}{2}}\cos z + 2\left(1 + 4\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\right)$$

$$\Leftrightarrow -\frac{(\sin 2x+\sin 2y+\sin 2z)}{2\cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{z}{2}}+2\left(1+4\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\right)=$$

$$=-16\sin\tfrac{x}{2}\sin\tfrac{y}{2}\sin\tfrac{z}{2}+2\left(1+4\sin\tfrac{x}{2}\sin\tfrac{y}{2}\sin\tfrac{z}{2}\right).$$ ***Por la tanto***

$$\tan^2\frac{x}{2}+\tan^2\frac{y}{2}+\tan^2\frac{z}{2}\ge 2-8\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}$$

$$\Leftrightarrow \tan^2\tfrac{x}{2}+\tan^2\tfrac{y}{2}+\tan^2\tfrac{z}{2}+8\sin\tfrac{x}{2}\sin\tfrac{y}{2}\sin\tfrac{z}{2}\ge 2 \quad \textit{(LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum a^2 \overset{(1)}{\underset{\smile}{\ge}} \sum (b-c)^2+4s\sqrt{\frac{2(2R-r)}{R}}$$

$$\textbf{(1)}\quad \Leftrightarrow 2\sum ab-\sum a^2\ge 4rs\sqrt{\frac{2(2R-r)}{R}}$$

$$\Leftrightarrow 2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)\ge 4rs\sqrt{\frac{2(2R-r)}{R}}$$

$$\Leftrightarrow 4r(4R+r)\ge 4rs\sqrt{\frac{2(2R-r)}{R}}$$

$$\Leftrightarrow R(4R+r)^2\ge 2s^2(2R-r)=s^2(4R-2r) \quad \textbf{(2)}$$

Using Rouche's inequality,

$$s^2(4R-2r)\le(4R-2r)(2R^2+10Rr-r^2)+$$

$$+2(4R-2r)(R-2r)\sqrt{R^2-2Rr}\overset{?}{\le} R(4R+r)^2\sqrt{R^2-2Rr}$$

$$\Leftrightarrow 8R^3-28R^2r+25Rr^2-2r^3\overset{?}{\ge} 4(R-2r)(2R-r)$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2)\overset{?}{\ge}4(R-2r)(2R-r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow 8R^2-12Rr+r^2\overset{?}{\ge}4(2R-r)\sqrt{R^2-2Rr}\ (\because R-2r\ge 0) \quad \textbf{(3)}$$

Now, $8R^2 - 12Rr + r^2 = r^2(8t^2 - 12t + 1)$ *(where* $t = \frac{R}{r}$*)*

$$= r^2\{8(t^2 - 4) - 12(t - 2) + 32 - 24 + 1\}$$

$$= r^2\{(t - 2)(8t + 16 - 12) + 9\} = r^2\{(t - 2)(8t + 4) + 9\}$$

$$> 0, \because t = \frac{R}{r} \geq 2 \text{ (Euler)}$$

$$\therefore \text{(3)} \Leftrightarrow (8R^2 - 12Rr + r^2)^2 - 16(2R - r)^2(R^2 - 2Rr) \geq 0$$

$$\Leftrightarrow 16R^2r^2 + 8Rr^3 + r^4 \overset{?}{\geq} 0 \rightarrow \text{true}$$

$\Rightarrow$ ***(3) is true*** $\Rightarrow$ ***(2) is true*** $\Rightarrow$ ***(1) is true (Proved)***

Solution 3 by Marin Chirciu-Romania

We use the known identities in triangle:

$\sum a^2 = 2(p^2 - r^2 - 4Rr)$ **and** $\sum(b - c)^2 = 2(p^2 - 3r^2 - 12Rr)$.

We write the inequality: $2(p^2 - r^2 - 4Rr) \geq$

$$\geq 2(p^2 - 3r^2 - 12Rr) + 4pr \cdot \sqrt{\frac{2(2R - r)}{R}} \Leftrightarrow 4R + r \geq p \cdot \sqrt{\frac{2(2R - r)}{R}}$$

$\Leftrightarrow p^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$ ***Blundon-Gerretsen's inequality, true from Gergonne's identity:*** $H\Gamma^2 = 4R^2\left[1 - \frac{2p^2(2R-r)}{R(4R+r)^2}\right] \geq 0$***, where*** Γ ***is Gergonne's point: the intersection of the lines:*** AA_1, BB_1, CC_1 ***where*** A_1, B_1, C_1 ***are the tangent point of the incentre with the sides*** BC, CA, AB ***of*** ΔABC***).***

The equality holds if and only if the triangle is equilateral.

269. In ΔABC**:**

$$(b + c - a)m_a^2 + (c + a - b)m_b^2 + (a + b - c)m_c^2 \geq 18rs(R - r)$$

Proposed by Abdilkadir Altintas - Afyonkarashisar-Turkey

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

KLAMKIN INERTIAL MOMENT

Siendo a, b, c los lados de un triángulo ABC y PA, PB, PC son las distancias de un punto P en plano ABC

Se cumple para todos los números reales x, y, z lo siguiente

$$(x+y+z)(xPA^2+yPB^2+zPC^2) \geq yza^2+zxb^2+xyc^2$$

Siendo $x = b+c-a, y = c+a-b, z = a+b-c$

Lo cual es equivalente

$$2s\left((b+c-a)PA^2+(c+a-b)PB^2+(a+b-c)PC^2\right) \geq$$

$$(a^2-(b-c)^2)a^2+\left(b^2-(c-a)\right)b^2+(c^2-(a-b)^2)c^2$$

$$2s\left((b+c-a)PA^2+(c+a-b)PB^2+(a+b-c)PC^2\right) \geq$$

$$\geq a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2+2abc(a+b+c)$$

$$2s\left((b+c-a)PA^2+(c+a-b)PB^2+(a+b-c)PC^2\right) \geq -16S^2+2(4sRr)(2s)$$

$$2s\left((b+c-a)PA^2+(c+a-b)PB^2+(a+b-c)PC^2\right) \geq -16s^2r^2+16s^2Rr$$

$$(b+c-a)PA^2+(c+a-b)PB^2+(a+b-c)PC^2 \geq 8sRr-8sr^2 = 8sr(R-r)$$

Un caso particular es cuando $P = G$ (Centroid) $\rightarrow GA = \frac{2}{3}m_a, GB = \frac{2}{3}m_b, GC = \frac{2}{3}m_c$

$$\Rightarrow aGA^2+bGB^2+cGC^2 \geq 8sr(R-r) \Leftrightarrow am_a^2+bm_b^2+cm_c^2 \geq \frac{9}{4}\cdot 8sr(R-r) =$$

$$= 18sr(R-r)$$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(b+c-a)m_a^2+(c+a-b)m_b^2+(a+b-c)m_c^2 \overset{(1)}{\geq} 18rs(R-r)$$

$$\textbf{(1)} \quad \Leftrightarrow (s-a)m_a^2+(s-b)m_b^2+(s-c)m_c^2 \geq 9rs(R-r)$$

$$\Leftrightarrow (s-a)(2b^2+2c^2-a^2)+(s-b)(2c^2+2a^2-b^2)+$$

$$+(s-c)(2a^2+2b^2-c^2)\geq 36rs(R-r)$$

$$\Leftrightarrow s\left(3\sum a^2\right)+\sum a^3-2\sum(a^2b+ab^2)\geq 36rs(R-r)$$

$$\Leftrightarrow 6s(s^2-4Rr-r^2)+3abc+2s\left(\sum a^2-\sum ab\right)-$$

$$-2\sum\{ab(2s-c)\}\geq 36rs(R-r)$$

$$\Leftrightarrow 6s(s^2-4Rr-r^2)+12Rrs+2s(s^2-12Rr-3r^2)-$$

$$-2\left(2s\sum ab-3abc\right)\geq 36rs(R-r)$$

$$\Leftrightarrow 6s(s^2-4Rr-r^2)+12Rrs+2s(s^2-12Rr-3r^2)-$$

$$-4s(s^2+4Rr+r^2)+24Rrs\geq 36Rrs-36r^2S$$

$$\Leftrightarrow 3(s^2-4Rr-r^2)+s^2-12Rr-3r^2-2(s^2+4Rr+r^2)+18r^2\geq 0$$

$$\Leftrightarrow 2s^2\geq 32Rr-10r^2\Leftrightarrow s^2\geq 16Rr-5r^2\rightarrow \textit{true, by Gerretsen}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$ab+bc+ca=p^2+r^2+4Rr$, $abc=4Rrp$ **and** $a=b+c=2p$

$$\sum_{cyc}(b+c-a)m_a^2=2\sum_{cyc}(p-a)m_a^2$$

$$=\frac{1}{2}\sum_{cyc}(p-a)\{2(a^2+b^2+c^2)-3a^2\}=\frac{1}{2}\sum_{cyc}(p-a)\{4(p^2-r^2-4Rr)-3a^2\}$$

$$=2(p^2-r^2-4Rr)\sum_{cyc}(p-a)-\frac{3}{2}\sum_{cyc}a^2(p-a)$$

$$=2p(p^2-r^2-4Rr)-\frac{3}{2}p(a^2+b^2+c^2)+\frac{3}{2}(a^3+b^3+c^3)$$

$$=2p(p^2-r^2-4Rr)-\frac{3}{2}p\cdot 2(p^2-r^2-4Rr)+\frac{3}{2}(a+b+c)^3-\frac{9}{2}(a+b+c)(ab+bc+ca)+\frac{9}{a}abc$$

$$=12p^3-9p(p^2+r^2+4Rr)+18Rrp-p(p^2-r^2-4Rr)$$

$$=12p^3-p(10p^2+8r^2+32Rr)+18Rrp=2p(p^2-4r^2-7Rr)$$

we need to prove, $2p(p^2-4r^2-7Rr)\ge 18rp(R-r)$

$\Leftrightarrow p^2-4r^2-7Rr\ge 9Rr-9r^2 \Leftrightarrow p^2\ge 16Rr-5r^2$***, which is true***

$$\therefore \sum_{cyc}(b+c-a)\,m_a^2\ge 18rp(R-r)$$

270. In $\Delta\ ABC$:

$$(3a-b-c)(3b-c-a)(3c-a-b)\le(b+c-a)(c+a-b)(a+b-c)$$

Proposed by Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC $(3a-b-c)(3b-c-a)(3c-a-b)\le$

$\le(b+c-a)(c+a-b)(a+b-c)$***. Como*** a,b,c ***son lados de un***

triángulo se cumple lo siguiente $x=b+c-a>0$,

$y=a+c-b>0, z=a+b-c>0$; $x+y-z=3c-a-b$,

$y+z-x=3a-b-c, z+x-y=3b-c-a$

La desigualdad es equivalente $(x+y-z)(z+x-y)(y+z-x)\le xyz$

Lo cual es equivalente

$\Leftrightarrow x^3+y^3+z^3+3xyz\ge xy(x+y)+yz(y+z)+zx(z+x)\to$

$\to$ ***(Válido por desigualdad de Schur)***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $x=a+b-c, y=b+c-a$ ***and*** $z=c+a-b$ ***now,***

$$xyz\ge\prod_{cyc}(x+y-z)$$

$$\Leftrightarrow\sum_{cyc}x^3+3xyz\ge\sum_{cyc}xy(x+y)\Leftrightarrow\sum_{cyc}x(x-y)(x-z)\ge 0,$$

which is true by Schurs Inequality so,

$$xyz \geq \prod_{cyc}(x+y-z) = \prod_{cyc}(3a-b-c)$$

$$\Rightarrow \prod_{cyc}(a+b-c) \geq \prod_{cyc}(3a-b-c)$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$RHS - LHS = 8\sum(a^2b+ab^2) - 4\sum a^3 - 36abc \geq 0$$

$$\Leftrightarrow 2\sum ab\,(2s-c) - \left[3abc + 2s\left(\sum a^2 - \sum ab\right)\right] - 9abc \geq 0$$

$$\Leftrightarrow 4s(s^2+4Rr+r^2) - 36Rrs - 2s(s^2-12Rr-3r^2) - 36Rrs \geq 0$$

$$\Leftrightarrow 2s^2 + 8Rr + 2r^2 - 36Rr - s^2 + 12Rr + 3r^2 \geq 0$$

$$\Leftrightarrow s^2 \geq 16Rr - 5r^2 \rightarrow \textit{ true by Gerretsen (Proved)}$$

Solution 4 by Sanong Hauerai-Nakonpathom-Thailand

$$(3a-(b+c))(3b-(c+a))(3c-(a+b)) \leq (a+b-c)(b+c-a)(c+a-b)$$

$$(3a+3b+3c-4(b+c))(3a+3b+3c-4(c+a))(3a+3b+3c-4(a+b))$$

$$\leq (3a+3b+3c-(2a+2b+4c))(3a+3b+3c-(4a+3b+2c))(3a+3b+3c-(2a+4b+2c))$$

$$\textit{If}\left((a+b+c)-\frac{4}{3}(b+c)\right)\left((a+b+c)-\frac{4}{3}(c+a)\right)\left((a+b+c)-\frac{4}{3}(a+b)\right)$$

$$\leq \left((a+b+c)-\frac{2}{3}(a+b+2c)\right)\left((a+b+c)-\frac{2}{3}(2a+b+c)\right)(a+b+c-2(a+2b+c))$$

$$\textit{From Expanding, Consider because } 8ab \leq 4a^2+4b^2$$

$$\textit{Hence}\left((a+b+c)-\frac{4}{3}(b+c)\right)\left((a+b+c)-\frac{4}{3}(c+a)\right) \leq$$

$$\leq \left((a+b+c)-\frac{2}{3}(a+b+2c)\right)^2. \textit{ Because } 8bc \leq 4b^2+4c^2$$

$$\textit{Hence}\left((a+b+c)-\frac{4}{3}(c+a)\right)\left((a+b+c)-\frac{2}{3}(a+b)\right) \leq$$

$$\leq \left((a+b+c)-\frac{2}{3}(2a+b+c)\right)^2 \textit{ and because } 8ac \leq 4a^2+4c^2$$

Hence $\left((a+b+c)-\frac{4}{3}(a+b)\right)\left((a+b+c)-\frac{4}{3}(b+c)\right)\le$

$$\le\left((a+b+c)-\frac{2}{3}(a+2b+c)\right)^2$$

Therefore $(3a-(b+c))(3b-(c+a))(3c-(a+b))\le$

$$\le(a+b-c)(b+c-a)(c+a-b)$$

271. In ΔABC, O – circumcenter, I_a, I_b, I_c – excenters

$$\frac{I_bI_c}{OI_a}+\frac{I_cI_a}{OI_b}+\frac{I_aI_b}{OI_c}\le\frac{27R^3}{abc}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS\overset{C-B-S}{\le}\sqrt{\sum I_bI_c^2}\sqrt{\sum\frac{1}{aI_a^2}}\quad (1)$$

$$I_bI_c=a\csc\frac{A}{2}=2R\sin A\csc\frac{A}{2}=4R\sin\frac{A}{2}\cos\frac{A}{2}\csc\frac{A}{2}=4R\cos\frac{A}{2}, \text{ etc.}$$

$$\therefore I_bI_c^2=16R^2\cos^2\frac{A}{2}$$

$$\therefore\sqrt{\sum I_bI_c^2}=4R\sqrt{\sum\cos^2\frac{A}{2}}=4R\sqrt{\frac{1}{2}\sum(1+\cos A)}$$

$$=4R\sqrt{\frac{1}{2}\left(3+1+\frac{r}{R}\right)}\overset{(2)}{=}4R\sqrt{\frac{4R+r}{2r}}. \text{ Now, } OI_a=\sqrt{R(R+2r_a)}, \text{ etc}$$

$$\therefore\sqrt{\sum\frac{1}{OI_a^2}}=\frac{1}{\sqrt{R}}\sqrt{\frac{1}{R+2r_a}+\frac{1}{R+2r_b}+\frac{1}{R+2r_c}}$$

$$= \frac{1}{\sqrt{R}}\sqrt{\frac{(R+2r_b)(R+2r_c)+(R+2r_c)(R+2r_a)+(R+2r_a)(R+2r_b)}{(R+2r_a)(R+2r_b)(R+2r_c)}}$$

$$\overset{(i)}{\triangleq} \frac{1}{\sqrt{2}}\sqrt{\frac{x}{y}} \textit{ (say)}$$

$$x = \sum(R+2r_b)(R+2r_c) = \sum\{R^2+2R(r_b+r_c)+4r_br_c\}$$

$$= 3R^2+4R(4R+r)+4s^2 \left(\because \sum r_a = 4R+r \text{ and} \sum r_ar_b = s^2\right)$$

$$\overset{(ii)}{=} 19R^2+4Rr+4s^2$$

$$y = (R+2r_a)(R+2r_b)(R+2r_c) = R^3+2R^2(4R+r)+4Rs^2+8r_ar_br_c$$

$$\overset{(iii)}{=} 9R^3+2R^2r+4Rs^2+8rs^2 \left(\because r\prod ra_a = \Delta^2\right)$$

(i), (ii), (iii) $\Rightarrow \sqrt{\sum\frac{1}{OI_a^2}}$

$$\overset{(3)}{=} \frac{1}{\sqrt{R}}\sqrt{\frac{19R^2+4Rr+4s^2}{9R^3+2R^2r+4Rs^2+8rs^2}}$$

(1), (2), (3) $\Rightarrow LHS$

$$\overset{(4)}{\leq} 4R\sqrt{\frac{4R+r}{2R}}\frac{1}{\sqrt{R}}\sqrt{\frac{19R^2+4Rr+4s^2}{9R^3+2R^2r+4Rs^2+8rs^2}}$$

$$= 4\sqrt{\frac{4R+r}{2}}\sqrt{\frac{19R^2+4Rr+4s^2}{9R^3+2R^2r+4Rs^2+8rs^2}}$$

(4) ⇒ it suffices to prove: $16\left(\frac{4R+r}{2}\right)\cdot\frac{19R^2+4Rr+4s^2}{9R^3+2R^2r+4Rs^2+8rs^2} \leq \frac{729R^6}{16R^2r^2s^2}$

$\Leftrightarrow 729R^4\{9R^3+2R^2r+(4R+8r)s^2\} \geq 128r^2(4R+r)s^2(19R^2+4Rr+4s^2)$ **(5)**

LHS of (5) $\overset{Gerretsen}{\geq} 729R^4\{9R^3+2R^2r+(16Rr-5r^2)(4R+8r)\}$

$$\text{RHS of (5)} \overset{Gerretsen}{\geq} 128r^2(4R+r)(4R^2+4Rr+3r^2)+$$

$$(16R^2+16Rr+12r^2+19R^2+4Rr) \therefore \text{ it suffices to prove:}$$

$$729R^4\{9R^3+2R^2r+(16Rr-5r^2)(4R+8r)\}$$

$$\geq 128r^2(4R+r)(4R^2+4Rr+3r^2)(35R^2+20Rr+12r^2)$$

$$\Leftrightarrow 6561t^7+48114t^6+7052t^5-159720t^4-147456t^3-$$

$$-84120t^2-32256t-4608 \geq 0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(6561t^6+61236t^5+129524t^4+99328t^3+51200t^2+17280t+2304) \geq 0 \rightarrow$$

$$\text{true} \because t=\frac{R}{r} \geq 2 \text{ (Euler) (Proved)}$$

272. In $\Delta\, ABC, O$ – circumcentre, I_a, I_b, I_c – excenters

$$OI_a \cdot OI_b \cdot OI_c + I_aI_b \cdot I_bI_c \cdot I_cI_a \geq 32rR(s+r)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Teniendo en cuenta las siguients notaciones y desigualdades

$$OI_a=\sqrt{R(R+2r_a)}, OI_b=\sqrt{R(R+2r_b)}, OI_c=\sqrt{R(R+2r_b)},$$

$$I_aI_b=4R\cos\frac{C}{2}, I_bI_c=4R\cos\frac{A}{2}, I_cI_a=4R\cos\frac{B}{2}$$

$r_ar_br_c=s^2r \geq 27r^3, R \geq 2r$. ***Luego aplicando la desigualdad de Holder***

$$OI_a \cdot OI_b \cdot OI_c = R\sqrt{R}\sqrt{(R+2r_a)(R+2r_b)(R+2r_c)} \geq$$

$$\geq R\sqrt{R}\sqrt{\left(R+2\sqrt[3]{r_ar_br_c}\right)^3} \geq R\sqrt{R(R+6r)^3} \geq R\sqrt{2r\cdot(8r)^3}=32Rr^2$$

$$\Leftrightarrow OI_a \cdot OI_b \cdot OI_c \geq 32Rr^2 \text{ (A). Además}$$

$$\rightarrow I_aI_b \cdot I_bI_c \cdot I_cI_a = 64R^3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=64R^3\cdot\frac{s}{4R}=16R^2s \geq 32Rrs \text{ (B)}$$

Sumando (A)+(B)→ $OI_a\cdot OI_b\cdot OI_c+I_aI_b\cdot I_bI_c\cdot I_cI_a\geq 32rR(s+r)$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$OI_a\cdot OI_b\cdot OI_c=\sqrt{(R^2+2Rr_a)(R^2+2Rr_b)(R^2+2Rr_c)}$$

$\overset{(1)}{=}\sqrt{R^3(R+2r_a)(R+2r_b)(R+2r_c)}$. **Now,** $(R+2r_a)(R+2r_b)(R+2r_c)$

$$=R^3+2R^2\left(\sum r_a\right)+4R\left(\sum r_ar_b\right)+8r_ar_br_c$$

$$=R^3+2R^2(4R+r)+4Rs^2+8rs^2$$

$$=9R^3+2R^2r+s^2(4R+8r)\overset{(2)}{\geq}9(2r)^3+2r(4r^2)+27r^2(8r+8r)$$

$$(\because R\geq 2r\ and\ s\geq 3\sqrt{3}r)=(72+8+27\times 16)r^3=512r^3$$

$$\therefore R^3\prod(R+2r_a)\geq R^2\cdot R(512r^3)\ \ \textbf{(using (2))}$$

$$\underset{(3)}{\overset{Euler}{\geq}}R^2\cdot 2r(512r^3)=1024R^2r^4$$

$$\therefore\sqrt{R^3\prod(R+2r_a)}\geq 32Rr^2\ \ \textbf{(using (3))}$$

$$\Rightarrow OI_a\cdot OI_b\cdot OI_c\geq 32Rr^2\ \ \textbf{(a) (from (1))}$$

$$I_aI_b\cdot I_bI_c\cdot I_cI_a=\left(c\csc\frac{C}{2}\right)\left(a\csc\frac{A}{2}\right)\left(b\csc\frac{B}{2}\right)$$

$$=abc\sqrt{\frac{ab}{(s-a)(s-b)}}\sqrt{\frac{bc}{(s-b)(s-c)}}\sqrt{\frac{ca}{(s-c)(s-a)}}$$

$$=\frac{16R^2r^2s^2\cdot s}{r^2s^2}=16R^2s\underset{(b)}{\overset{Euler}{\geq}}32Rrs\ \textbf{(a)+(b)}\Rightarrow LHS\geq 32Rr(r+s)$$

273. **In** $\Delta\ ABC$**:**

$$\sum\frac{a}{(b+c)(b+c-a)}\geq\frac{18r^2}{abc}$$

Proposed by Panagiote Ligouras-Florence-Italia

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\sum\frac{a}{(b+c)(b+c-a)}\geq\frac{18r^2}{abc}=\frac{18r^2}{4prR}=\frac{9r}{2Rp}$ ***(A)***

Además siendo a,b,c ***los lados de un triángulo se cumple la siguiente desigualdad*** $abc\geq(a+c-b)(b+c-a)(c+a-b)$, $R\geq 2r$

Lo equivalente en (A) se puede expresar como

$$\frac{R}{9r}\sum\frac{a}{(b+c)(b+c-a)}\geq\frac{1}{2p}$$

Aplicando la desigualdad de Euler y MA ≥ MG

$\frac{R}{9r}\sum\frac{a}{(b+c)(b+c-a)}\geq\frac{2}{9}\cdot 3\sqrt[3]{\frac{1}{(a+b)(b+c)(c+a)}}\geq\frac{1}{a+b+c}$***. Es suficiente probar lo siguiente*** $(a+b+c)^3\geq\frac{27}{8}(a+b)(b+c)(c+a)\Leftrightarrow$

$\Leftrightarrow\left((a+b)+(b+c)+(c+a)\right)^3\geq\frac{27}{8}(a+b)(b+c)(c+a)$ ***(MA ≥ MG)***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

We know, $abc\geq\prod_{cyc}(a+b-c)$

$$\therefore\sum_{cyc}\frac{a}{(b+c)(b+c-a)}\geq 3\sqrt[3]{\frac{1}{\prod_{cyc}(a+b)}\cdot\frac{abc}{\prod_{cyc}(a+b-c)}}$$

$$=3\sqrt[3]{\frac{1}{\prod_{cyc}(a+b)}}\geq\frac{9}{2(a+b+c)}=\frac{1}{2}\cdot\frac{9}{2p}\geq\frac{9r}{2Rp}\quad[\because R\geq 2r]$$

$=\frac{18r^2}{4Rrp}=\frac{18r^2}{abc}$ ***(Proved)***

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{a}{(b+c)(b+c-a)}\overset{(1)}{\geq}\frac{18r^2}{abc}$$

$$\textbf{(1)} \Leftrightarrow \sum \frac{a}{(b+c)(s-a)} \ge \frac{36r^2}{abc} \quad \textbf{(2)}$$

$$\text{Now, } \frac{a}{b+c} \ge \frac{b}{c+a} \Leftrightarrow ac + a^2 \ge b^2 + bc$$

$$\Leftrightarrow (a+b)(a-b) + c(a-b) \ge 0 \Leftrightarrow (a-b)(a+b+c) \ge 0$$

$$\overset{(i)}{\Leftrightarrow} a \ge b. \text{ Similarly, } \frac{b}{c+a} \ge \frac{c}{a+b} \Leftrightarrow b \ge c \text{ (ii)}$$

WLOG, we may assume $a \ge b \ge c$***. Then, from (i), (ii),*** $\frac{a}{b+c} \ge \frac{b}{c+a} \ge \frac{c}{a+b}$,

and also, $\frac{1}{s-a} \ge \frac{1}{s-b} \ge \frac{1}{s-c}$ $\therefore$ ***LHS of (2)*** $\ge \frac{1}{3}\sum \frac{a}{b+c} \cdot \sum \frac{1}{s-a}$ ***(Chebyshev)***

$$\overset{Nesbitt}{\ge} \frac{1}{3} \cdot \frac{3}{2} \cdot \sum \frac{1}{s-a} \overset{(3)}{\ge} \frac{1}{2} \cdot \frac{9}{\sum(s-a)} \text{ (Bergstrom)} = \frac{9}{2s}$$

(3), (2) $\Rightarrow$ ***it suffices to prove:*** $\frac{9}{2s} \ge \frac{36r^2}{4Rrs} \Leftrightarrow R \ge 2r \to$ ***true (Euler)***

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{a}{(b+c)(b+c-a)} = \sum \frac{(b+c)-(b+c-a)}{(b+c)\cdot(b+c-a)} =$$

$$= \sum \left(\frac{1}{b+c-a}\right) - \sum \frac{1}{b+c} = \frac{4R+r}{2S} - \frac{5p^2 + 4Rr + r^2}{2p(p^2 + 2Rr + r^2)}$$

$$\frac{4R+r}{2S} - \frac{5p^2 + 4Rr + r^2}{2p(p^2 + 2Rr + r^2)} \ge \frac{9r}{2p \cdot R} \Leftrightarrow$$

$$\left(\frac{4R+r}{2} - \frac{9r}{R}\right) \ge \frac{5p^2 + 4Rr + r^2}{2p(p^2 + 2Rr + r^2)} \Leftrightarrow$$

$$(4R^2 - 4Rr - 9r^2) \cdot p^2 \ge R \cdot r^2(4R + r) - r(2R + r)(4R^2 + Rr - 9r^2)$$

$$(4R^2 - 4Rr - 9r^2) \cdot (16Rr - 5r^2) \ge R \cdot r^2(4R + r) - r(2R + r)(4R^2 + Rr - 9r^2)$$

$$\frac{R}{2} = t;\ (4t^2 - 4t - 9)(16t - 5) \ge t \cdot (4t + 1) - (2t + 1)(4t^2 + t - 9)$$

$$72t^3 - 82t^2 - 142t + 36 \ge 0$$

$$\left.\begin{matrix}(t-2)\cdot(36t^2+31t-9)\geq 0\\ t\geq 2\end{matrix}\right\}\textit{TRUE}$$

274. In $\Delta\ ABC$:

$$\sqrt{h_a h_b h_c}\leq s\sqrt{r}\leq\sqrt{m_a m_b m_c}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \sqrt{h_a h_b h_c}\leq s\sqrt{r}\leq\sqrt{m_a m_b m_c}$

Utilizando las siguientes desigualdades e identidades en un $\Delta\ ABC$

$$S=sr=\sqrt{s(s-a)(s-b)(s-c)},\ h_a=\frac{bc}{2R},\ h_b=\frac{ca}{2R},\ h_c=\frac{ab}{2R}$$

$m_a\geq\sqrt{s(s-a)},\ m_b\geq\sqrt{s(s-b)},\ m_c\geq\sqrt{s(s-c)},\ R\geq 2r$. ***Por la tanto***

$$\sqrt{h_a h_b h_c}=\frac{abc}{2R\sqrt{2R}}=\frac{4Rsr}{2R\sqrt{2R}}=\frac{2sr}{\sqrt{2R}}\leq\frac{2sr}{\sqrt{4r}}=s\sqrt{r}\ \textit{(Válido por Euler)}$$

$$\sqrt{m_a m_b m_c}\geq\sqrt{\sqrt{s(s-a)(s-b)(s-c)\cdot s^2}}=\sqrt{s\cdot S}=\sqrt{s^2 r}=s\sqrt{r}$$

Solution 2 by George Apostolopoulos-Messolonghi-Greece

We have $h_a h_b h_c=\frac{2rs}{a}\cdot\frac{2rs}{b}\cdot\frac{2rs}{c}=\frac{8r^3s^3}{abc}=$

$\frac{8r^3s^3}{4R(rs)}=\frac{2r^2s^2}{R}\overset{Euler}{\leq}\frac{2r^2s^2}{2r}=r\cdot s^2$. ***So*** $\sqrt{h_a h_b h_c}\leq s\sqrt{r}$

Also, we have $m_a^2=\frac{2(b^2+c^2)-a^2}{4}\geq\frac{(b+c)^2-a^2}{4}=\frac{(b+c+a)(b+c-a)}{4}$

$=\frac{b+c+a}{2}\cdot\frac{b+c-a}{2}=s(s-a)=r_b r_c$. ***Namely*** $m_a^2\geq r_b r_c$

Similarly $m_b^2\geq r_c r_a$, ***and*** $m_c^2\geq r_a r_b$. ***So***

$$m_a^2\cdot m_b^2\cdot m_c^2\geq(r_a r_b r_c)^2\Leftrightarrow m_a m_b m_c\geq r_a r_b r_c=s^2 r$$

So $\sqrt{m_a m_b m_c}\geq s\sqrt{r}$. ***Namely*** $\sqrt{h_a h_b h_c}\leq s\sqrt{r}\leq\sqrt{m_a m_b m_c}$

275. In $\Delta\, ABC$, O – circumcentre, I_a, I_b, I_c - excenters:

$$\frac{216a^2b^2c^2R^3}{OI_a \cdot I_bI_c \cdot OI_b \cdot I_cI_a \cdot OI_c \cdot I_aI_b} \leq s^3$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{216a^2b^2c^2R^3}{OI_a \cdot I_bI_c \cdot OI_b \cdot I_cI_a \cdot OI_c \cdot I_aI_b} \overset{(1)}{\leq} s^3$$

$$I_aI_b \cdot I_bI_c \cdot I_cI_a = abc\sqrt{\frac{bc}{(s-b)(s-c)}} \cdot \sqrt{\frac{ca}{(s-c)(s-a)}} \cdot \sqrt{\frac{ab}{(s-a)(s-b)}}$$

$$= \frac{a^2b^2c^2s}{r^2s^2} \overset{(2)}{=} \frac{a^2b^2c^2}{r^2s}$$

using (2), (1) $\Leftrightarrow \left(\prod OI_a\right)a^2b^2c^2 \cdot s^3 \geq 216a^2b^2c^2R^3r^2S$

$$\Leftrightarrow s^2\left(\prod OI_a\right) \geq 216R^3r^2 \Leftrightarrow s^4\left(\prod OI_a^2\right) \geq 216 \times 216R^6r^4$$

$$\Leftrightarrow s^4R^3(R+2r_a)(R+2r_b)(R+2r_c) \geq 216^2R^6r^4$$

$$\Leftrightarrow s^4(R+2r_a)(R+2r_b)(R+2r_c) \geq 216^2R^3r^4 \quad \textit{(3)}$$

LHS of (3) $\overset{Gerretsen}{\geq} (16R-5r)^2r^2(R+2r_a)(R+2r_b)(R+2r_c)$

$$= r^2(16R-5r)^2\{R^3 + 2R^2(4R+r) + 4Rs^2 + 8rs^2\}$$

$$\left(\because \sum r_a = 4R + r, \sum r_ar_b = s^2, r_ar_br_c = rs^2\right)$$

$$\underset{(4)}{\overset{Gerretsen}{\geq}} r^2(16R-5r)^2\{9R^3 + 2R^2r + 4(16Rr - 5r^2)(R+2r)\}$$

(3), (4) $\Rightarrow$ *it suffices to prove:*

$$(16R-5r)^2\{9R^3 + 2R^2r + 4(16Rr - 5r^2)(R+2r)\} \geq 216^2R^3r^2$$

$$\Leftrightarrow 2304t^5 + 15456t^4 - 29343t^3 - 25870t^2 + 9100t - 1000 \geq 0,$$

(where, $t = \frac{R}{r}$***)***

$$\Leftrightarrow (t-2)\{(t-2)(2304t^3 + 24672t^2 + 60129t + 115958 + 232416\} \geq 0$$

$\rightarrow$ ***true,*** $\because t = \frac{R}{r} \geq 2$ ***(Euler) (Proved)***

276. In $\Delta\, ABC$:

$$\frac{s^2 + r^2}{2Rr} + \sqrt[3]{\frac{2r}{R}} \geq 8$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC***, la siguiente desigualdad*** $\frac{s^2+r^2}{2Rr} + \sqrt[3]{\frac{2r}{R}} \geq 8$

Aplicando la desigualdad de Euler $\rightarrow (R \geq 2r)$

$$\frac{s^2 + r^2}{2Rr} + \sqrt[3]{\frac{2r}{R}} = \frac{s^2 + r^2}{2Rr} + \sqrt[3]{\frac{R^2 \cdot 2r}{R^3}} \geq \frac{s^2 + r^2}{2Rr} + \frac{2r}{R} \geq 8$$

Lo cual es equivalente $s^2 + 5r^2 \geq 16Rr$ ***(Inequality Gerretsen)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS \underset{(1)}{\overset{Gerretsen}{\geq}} \frac{16Rr - 4r^2}{2Rr} + \sqrt[3]{\frac{2r}{R}} = 8 - \frac{2r}{R} + \sqrt[3]{\frac{2r}{R}}$$

$\therefore$ ***it suffices to prove:*** $\sqrt[3]{\frac{2r}{R}} \geq \frac{2r}{R}$ ***(from (1))***

$\Leftrightarrow \frac{2r}{R} \geq \left(\frac{2r}{R}\right)^3 \Leftrightarrow R^2 \geq (2r)^2 \Leftrightarrow R \geq 2r \rightarrow$ ***true by Euler (Proved)***

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\frac{2r}{R}=t^3\le 1\quad t\le 1;\ s^2\ge 16Rr-5r^2 \Rightarrow \textit{Gerretsen}$$

$$\frac{16Rr-4r^2}{2Rr}+\sqrt[3]{\frac{2r}{R}}\ge 8;\ 8-\frac{2r}{R}+\sqrt[3]{\frac{2r}{R}}\ge 8;\ t-t^3\ge 0$$

$$t(1-t^2)\ge 0\Rightarrow \textit{because } t\le 1$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\textit{In } \Delta\, ABC, p^2\ge 16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\therefore \frac{p^2+r^2}{2Rr}+\sqrt[3]{\frac{2r}{R}}\ge 8-\frac{2r}{R}+\frac{r}{2R}\left(\frac{R-2r}{R-r}\right)$$

$$\textit{We need to prove, } 8-\frac{2r}{R}+\frac{r}{2R}\left(\frac{R-2r}{R-r}\right)+\sqrt[3]{\frac{2r}{R}}\ge 8\Leftrightarrow -\frac{1}{t}+\frac{1}{4t}\left(\frac{t-1}{t-\frac{1}{2}}\right)+\frac{1}{\sqrt[3]{t}}\ge 0$$

$$\textit{where } t=\frac{R}{2r}\Leftrightarrow \frac{1}{t}\left(\frac{t-1}{4t-2}\right)-\frac{1}{t}+\frac{1}{\sqrt[3]{t}}\ge 0\Leftrightarrow \frac{1}{t}\left(\frac{1-3t}{4t-2}\right)+\frac{1}{\sqrt[3]{t}}\ge 0\Leftrightarrow \frac{1}{\sqrt[3]{t}}\ge \frac{1}{t}\left(\frac{3t-1}{4t-2}\right)$$

$$\Leftrightarrow 8t^2(2t-1)^3\ge (3t-1)^3\Leftrightarrow 64t^5-96t^4+21t^3-19t^2-9t+1\ge 0$$

$$\Leftrightarrow (t-1)(64t^4-32t^3-11t^2+8t-1)\ge 0$$

$$\Leftrightarrow (t-1)^2(64t^3+32t^2+21t+29)+(t-1)28\ge 0,$$

$$\textit{which is true} \because t\ge 1\therefore \frac{p^2+r^2}{2Rr}+\sqrt[3]{\frac{2r}{R}}\ge 8$$

277. In $\Delta\, ABC$:

$$\frac{R}{2r}+\frac{3s^2}{(4R+r)^2}\ge 2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{R}{2r}+\frac{3s^2}{(4R+r)^2}\ge 2$

Tener en cuenta las siguientes desigualdades en un triángulo ABC

$s^2\ge 16Rr-5r^2$ ***(Gerretsen's Inequality),*** $R\ge 2r$ ***(Euler's Inequality)***

La desigualdad es equivalente $\Leftrightarrow (4R+r)^2R+3s^2\cdot 2r\ge 4r(4R+r)^2$

Utilizando la desigualdad de Gerretsen

$$\Leftrightarrow (4R + r)^2R + 3s^2 \cdot 2r \ge (4R + r)^2R + 3(16Rr - 5r^2) \cdot 2r$$

Por último probaremos $(4R + r)^2R + 3(16Rr - 5r^2) \cdot 2r \ge 4r(4R + r)^2$

$$\Leftrightarrow 16R^3 + 8R^2r + 97r^2R - 30r^3 \ge 64R^2r + 32Rr^2 + 4r^3$$

$$\Leftrightarrow 16R^3 - 56R^2r + 65Rr^2 - 34r^3 = (R - 2r)(16R^2 - 24Rr + 17r^2) \ge 0$$

Lo cual es vàlido ya que $\rightarrow 16R^2 - 24Rr + 17r^2 = (4R - 3r)^2 + 8r^2 > 0$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{3s^2}{(4R + r)^2} + \frac{R}{2r} \overset{Gerretsen}{\ge} \frac{48Rr - 15r^2}{(4R + r)^2} + \frac{R}{2r} \overset{?}{\ge} 2$$

$$\Leftrightarrow 16R^3 - 56R^2r + 65Rr^2 - 34r^3 \overset{?}{\ge} 0 \Leftrightarrow (t - 2)(16t^2 - 24t + 17) \overset{?}{\ge} 0$$

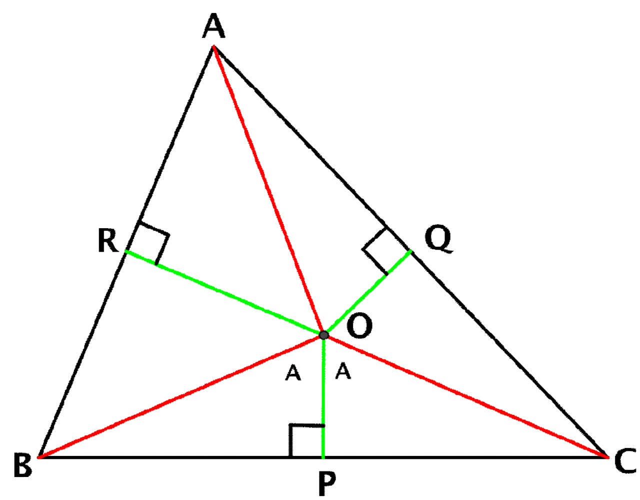
$\rightarrow$ ***true*** $\because t = \frac{R}{r} \ge 2$ ***and*** $16t^2 - 24t + 17 > 0$ ***as*** $\Delta = -512 < 0$ ***(Proved)***

278. In $\Delta\, ABC, O$ – circumcentre, R_a, R_b, R_c - circumradii of $\Delta\, BOC, \Delta\, COA, \Delta\, AOB$

$$\frac{9}{R_a^2 + R_b^2 + R_c^2} + \frac{R}{R_aR_bR_c} \le \frac{4}{R^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



$\because$ ***angle at centre = twice angle at circumference,*** $\therefore \angle BOC = 2\angle A$

$\therefore \angle BOP = \angle A$. Let $OP = t$; $\cos A = \frac{PO}{BO} = \frac{t}{R} \Rightarrow t = R\cos A$

Now, $Area\,(\Delta\, BOC) = \frac{1}{2}at = \frac{aR\cos A}{2}$. Also, $Area\,(\Delta\, BOC) = \frac{R\cdot R\cdot a}{4R_a} = \frac{aR^2}{4R_a}$

$\therefore \frac{aR\cos A}{2} = \frac{aR^2}{4R_a} \Rightarrow \cos A = \frac{R}{2R_a} \Rightarrow R_a = \frac{R}{2\cos A}$. Similarly, $R_b = \frac{R}{2\cos B}$ and

$R_c = \frac{R}{2\cos C} \therefore$ (1) $\Leftrightarrow \frac{9}{\frac{R^2}{4}\left(\frac{1}{\cos^2 A}+\frac{1}{\cos^2 B}+\frac{1}{\cos^2 C}\right)} + \frac{R}{\frac{R^3}{8\cos A\cos B\cos C}} \le \frac{4}{R^2}$

$$\Leftrightarrow \frac{9}{\frac{1}{\cos^2 A}+\frac{1}{\cos^2 B}+\frac{1}{\cos^2 C}} + 2\cos A\cos B\cos C \le 1 \quad (2)$$

Now, $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

$$\Rightarrow 2\sum \cos^2 A^{-3} = -1 - 4\cos A\cos B\cos C$$

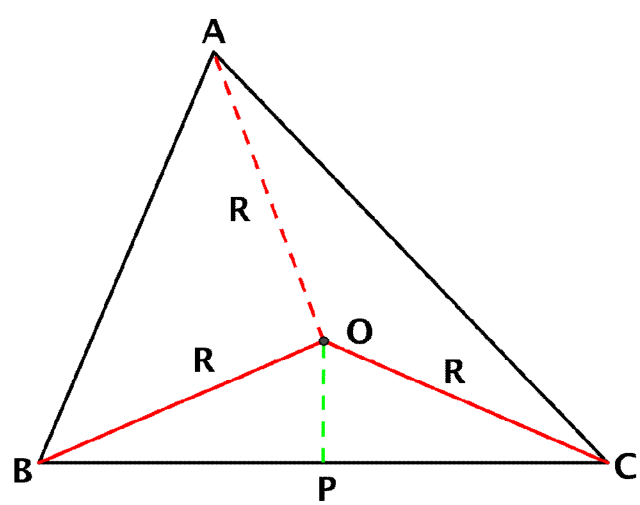
$$\Rightarrow \sum \cos^2 A = 1 - 2\cos A\cos B\cos C \quad (3)$$

Using (3), (2) $\Leftrightarrow \frac{9}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \le x + y + z.$

(where $x = \cos^2 A$, $y = \cos^2 B$, $z = \cos^2 C$)

→ which is true as HM ≤ AM (here, $x, y, z > 0$) (Proved)

Solution 2 by Geanina Tudose-Romania



We break the inequality into: $\frac{R}{R_aR_bR_c}\le\frac{1}{R^2}$ *and* $\frac{9}{R_a^2+R_b^2+R_c^2}\le\frac{3}{R^2}$

(1) *Let* $S_1=S_{BOC}=\frac{aR^2}{4R_a}\Rightarrow R_a=\frac{aR^2}{4S_1}$; $S_2=S_{AOC}=\frac{bR^2}{4R_b}\Rightarrow R_b=\frac{bR^2}{4S_2}$

$$S_3=S_{AOB}=\frac{cR^2}{4R_c}\Rightarrow R_c=\frac{cR^2}{4S_3};\ \frac{R}{R_aR_bR_c}=\frac{R}{\frac{abcR^6}{64S_1S_2S_3}}=\frac{64S_1S_2S_3}{abc\cdot R^5}$$

From GM $\le$ *AM* $\sqrt[3]{S_1S_2S_3}\le\frac{S_1+S_2+S_3}{3}=\frac{S}{3}=\frac{abc}{12R}\Rightarrow S_1S_2S_3=\frac{(abc)^3}{12^3R^3}$

Hence $\frac{R}{R_aR_bR_c}\le\frac{(abc)^2}{27R^8}$*; We show* $\frac{(abc)^2}{27R^8}\le\frac{1}{R^2}\Leftrightarrow abc\le3\sqrt{3}R^3\Leftrightarrow$

$\Leftrightarrow\sin A\sin B\sin C\le\frac{3\sqrt{3}}{8}$ *which is a true inequality*

(follows from $\sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}$ *and GM* $\le$ *IM)*

(2) *Is equivalent to* $R_a^2+R_b^2+R_c^2\ge3R^2$

In $\Delta\ BOC, m(BOC)=2m(A)\Rightarrow\frac{a}{\sin BOC}=2R_a\Rightarrow\frac{2R\sin A}{2\sin A\cos A}=2R_a\Rightarrow$

$\Rightarrow R_a=\frac{R}{2\cos A}$*. Thus:* $\frac{R^2}{4}\cdot\left(\frac{1}{\cos^2 A}+\frac{1}{\cos^2 B}+\frac{1}{\cos^2 C}\right)\ge3R^2$

$\Leftrightarrow\frac{1}{\cos^2 A}+\frac{1}{\cos^2 B}+\frac{1}{\cos^2 C}\ge12$ *(! $\Delta\ ABC$ not right – angled suppose acute)*

Let $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}\ f(x)=\frac{1}{\cos^2 x}, f'(x)=\frac{2\sin x}{\cos^3 x},$

$f''(x)=\frac{2\cdot(\cos^2 x+3\sin^2 x)}{\cos^4 x}>0\Rightarrow f''$ *convex fct* $\Rightarrow f\left(\frac{a+b+c}{3}\right)\le\frac{f(a)+f(b)+f(c)}{3}$

$$\Rightarrow\frac{1}{3}\cdot\left(\frac{1}{\cos^2 A}+\frac{1}{\cos^2 B}+\frac{1}{\cos^2 C}\right)\ge\frac{1}{\cos^2\left(\frac{A+B+C}{3}\right)}\Leftrightarrow$$

$\Leftrightarrow\frac{1}{\cos^2 A}+\frac{1}{\cos^2 B}+\frac{1}{\cos^2 C}\ge12$ *"true"*

Adding up the inequalities, we get the conclusion.

279. **In $\Delta\ ABC$:**

$$\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\ge\frac{1}{2}\left(3+\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\right)$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC la siguiente desigualdad

$$\frac{m_a}{l_a}+\frac{m_b}{l_b}+\frac{m_c}{l_c}\ge\frac{1}{2}\left(\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}+3\right)$$

Utilizando las siguientes desigualdades e identidades concidas en un triángulo ABC $m_a\ge\frac{b+c}{2}\cos\frac{A}{2}, m_b\ge\frac{c+a}{2}\cos\frac{B}{2}, m_c\ge\frac{a+b}{2}\cos\frac{C}{2}$

$$l_a=\frac{2bc}{b+c}\cos\frac{A}{2}, l_b=\frac{2ca}{c+a}\cos\frac{B}{2}, l_c=\frac{2ab}{a+b}\cos\frac{C}{2}$$

La desigualdad es equivalente

$$\frac{m_a}{l_a}+\frac{m_b}{l_b}+\frac{m_c}{l_c}\ge\frac{(b+c)^2}{4bc}+\frac{(c+a)^2}{4ca}+\frac{(a+b)^2}{4ab}\ge\frac{1}{2}\left(\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}+3\right)=$$

$=\frac{1}{2}\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}+\frac{3}{2}$. ***Ahora bien***

$$\frac{(b+c)^2}{4bc}+\frac{(c+a)^2}{4ca}+\frac{(a+b)^2}{4ab}=\frac{3}{2}+\frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\right)$$

Es suficiente demostrar lo siguiente

$$\frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}+3\right)=\frac{1}{4}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge$$

$\ge\frac{1}{2}\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}+\frac{3}{4}\Leftrightarrow\frac{1}{4}(x-3)(x+1)\ge 0$, ***lo cual es válido ya que***

$$x = \sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq 3 \rightarrow \textbf{(MA} \geq \textbf{MG)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_a \geq \sqrt{s(s-a)}\ \textbf{(1)}\ w_a = \frac{2bc}{b+c}\cos\frac{A}{2} \Rightarrow w_a = \frac{2bc}{b+c}\sqrt{\frac{s(s-a)}{bc}} = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}$$

$$\Rightarrow \frac{1}{w_a} = \frac{b+c}{2\sqrt{bc}} \cdot \frac{1}{\sqrt{s(s-a)}}\ \textbf{(2); (1), (2)} \Rightarrow \frac{m_a}{w_a} \geq \frac{b+c}{2\sqrt{bc}}\ \textbf{(3)}$$

$$\textbf{Similarly, } \frac{m_b}{w_b} \overset{(4)}{\geq} \frac{c+a}{2\sqrt{ca}} \textbf{ and, } \frac{m_c}{w_c} \overset{(5)}{\geq} \frac{a+b}{2\sqrt{ab}}$$

$$\textbf{(3) + (4) + (5)} \Rightarrow LHS \overset{(6)}{\geq} \frac{1}{2}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}+\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}\right)$$

$$\textbf{(6)} \Rightarrow \textbf{it suffices to prove: } \sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}+\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}$$

$$\overset{(7)}{\geq} 3 + \sqrt{3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}. \textbf{ Let } \sqrt{\frac{a}{b}} = x, \sqrt{\frac{b}{c}} = y, \sqrt{\frac{c}{a}} = z \therefore xyz = 1$$

$$\textbf{(7)} \Leftrightarrow x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 3 + \sqrt{3+x^2+y^2+z^2+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}$$

$$\Leftrightarrow \sum x + \sum xy - 3 \geq \sqrt{3+\sum x^2+\sum x^2y^2}$$

$$\Leftrightarrow \left(\sum x + \sum xy - 3\right)^2 \geq 3 + \sum x^2 + \sum x^2y^2$$

$$\left(\because \sum x + \sum xy \overset{A-G}{\geq} 3+3 = 6 \Rightarrow \sum x + \sum xy - 3 \geq 3 > 0\right)$$

$$\Leftrightarrow \left(\sum x\right)^2 + \left(\sum xy\right)^2 + 9 + 2\left(\sum x\right)\left(\sum xy\right) - 6\sum x - 6\sum xy$$

$$\geq 3 + \sum x^2 + \sum x^2y^2$$

$$\Leftrightarrow \sum x^2 + 2\sum xy + \sum x^2y^2 + 2\sum x + 9 + 2\left(\sum x\right)\left(\sum xy\right)$$

$$\geq 3+\sum x^2+\sum x^2y^2+6\sum x+6\sum xy \quad (\because xyz=1)$$

$$\Leftrightarrow 6+2\left(\sum x\right)\left(\sum xy\right)\geq 4\sum x+4\sum xy$$

$$\Leftrightarrow 3+uv\geq 2u+2v \text{ (where } \begin{matrix} u=\sum x\geq 3 \\ v=\sum xy\geq 3\end{matrix}\text{)} \Leftrightarrow u(v-2)\geq 2v-3$$

$$\Leftrightarrow u\geq \frac{2v-3}{v-2} \text{ } (\because (2v-3) \text{ and } (v-2)>0) \Leftrightarrow u^2\geq \frac{(2v-3)^2}{(v-2)^3} \text{ (8)}$$

Now, $\left(\sum x\right)^2\geq 3\sum xy\Rightarrow u^2\geq 3v$ ***(9); (8), (9)*** $\Rightarrow$ ***it suffices to prove:***

$$3v\geq \frac{(2v-3)^2}{(v-2)^2}\Leftrightarrow 3v(v-2)^2\geq (2v-3)^2\Leftrightarrow 3v^3-16v^2+24v-9\geq 0$$

$$\Leftrightarrow (v-3)(3v^2-7v+3)\geq 0\Leftrightarrow (v-3)\{(3v+2)(v-3)+9\}\geq 0$$

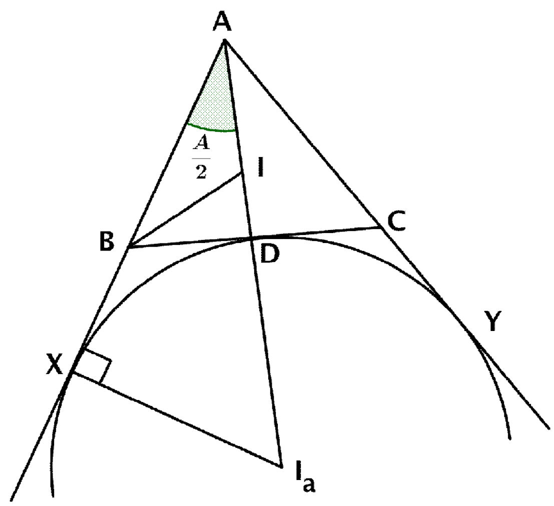
$$\rightarrow \textit{true} \because v=\sum xy=\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}\overset{A-G}{\geq} \text{(3) (Proved)}$$

280. In $\Delta\ ABC, I$ - incentre, I_a, I_b, I_c – excenters:

$$a\cdot AI\cdot AI_a+b\cdot BI\cdot BI_b+c\cdot CI\cdot CI_c\leq \frac{8s^2S}{9r}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Pal-Kolkata-India



In $\Delta\ ABD, \frac{AI}{ID} = \frac{AB}{BD}$ (angle bisector theorem)

$$= \frac{BC}{\frac{AB}{AB+AC}\cdot BC} \left(\because \frac{AB}{AC} = \frac{BD}{DC} \text{ by bisector theorem} \Rightarrow \frac{AB}{AB+AC} = \frac{BD}{BC}\right)$$

$$= \frac{c}{\frac{ca}{b+c}} = \frac{b+c}{a} \Rightarrow AI = \frac{b+c}{a+b+c}\ AD = \frac{2bc\cos\frac{A}{2}}{2S} \quad (*)$$

Also $AX = AB + BX = AB + BD$ (1); $AY = AC + CY = AC + CD$ (2)

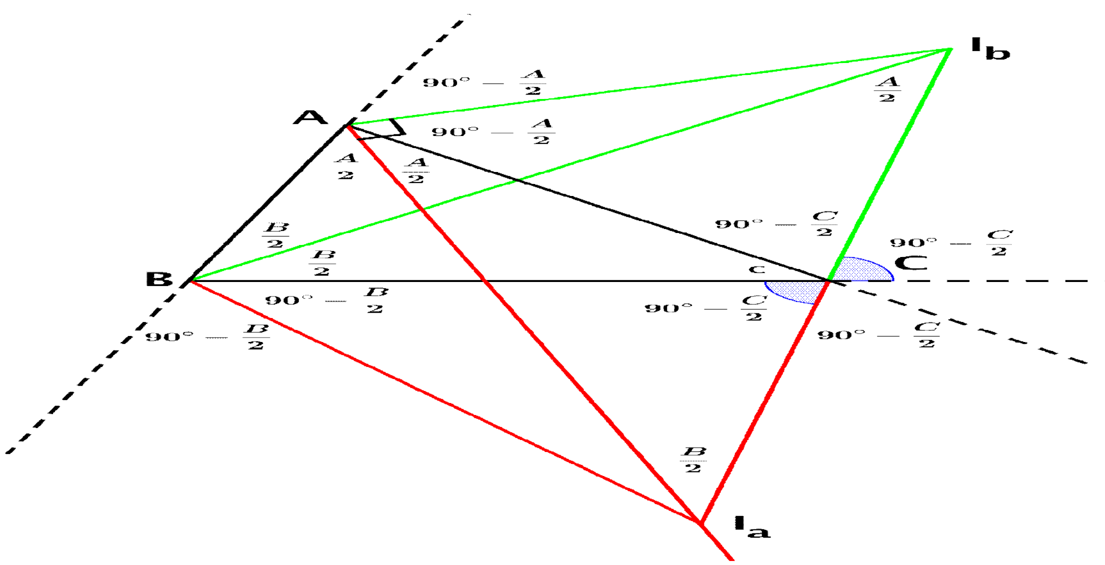
(1) + (2) = 2S and $AX = AY \Rightarrow AX = AY = s$

$AI_a = \frac{S}{\cos\frac{A}{2}}$. From (*) we get $AI = \frac{bc}{\frac{S}{\cos\frac{A}{2}}} = \frac{bc}{AI_a} \Rightarrow aAI \cdot AI_a = abc$

$\therefore$ the inequality stands $\sum a\ AI \cdot AI_a = 3abc \le \frac{8S^2\Delta}{9r} = \frac{8S^3\Delta}{9rs} \Leftrightarrow (a+b+$

$c3\ge 27abc \Leftrightarrow a+b+c\ge 33abc$ which is true by AM - GM

Solution 2 by Soumava Chakraborty-Kolkata-India



In $\Delta\ AI_aI_b, \angle I_aAI_b = 90^\circ \therefore \cos\frac{B}{2} = \frac{AI_a}{I_aI_b}$

$$\Rightarrow AI_a = \cos\frac{B}{2}\cdot C\csc\frac{C}{2} = \cos\frac{B}{2}\cdot 2R\sin C\cdot\csc\frac{C}{2}$$

$$= 4R\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}\csc\frac{C}{2} \stackrel{(1)}{=} 4R\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\therefore a\cdot AI\cdot AI_a = 2R\sin A\cdot\frac{R}{\sin\frac{A}{2}}\cdot 4R\cos\frac{B}{2}\cos\frac{C}{2}$$ ***(from (1))***

$$= 16R^2r\sin\frac{A}{2}\cos\frac{A}{2}\cdot\frac{1}{\sin\frac{A}{2}}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$$

$$= 16R^2r\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4R^2r(\sin A+\sin B+\sin C) = 4R^2r\left(\frac{s}{R}\right) \stackrel{(2)}{=} 4Rrs$$

Similarly, $b\cdot BI\cdot BI_b \stackrel{(3)}{=} 4Rrs$ ***and*** $c\cdot CI\cdot CI_c \stackrel{(4)}{=} 4Rrs$

$\therefore$ ***(2) + (3) + (4)*** $\Rightarrow LHS = 12Rrs \stackrel{?}{\le} \frac{8s^2S}{9r} = \frac{8S^2\cdot rs}{9r} = \frac{8S^3}{9}$

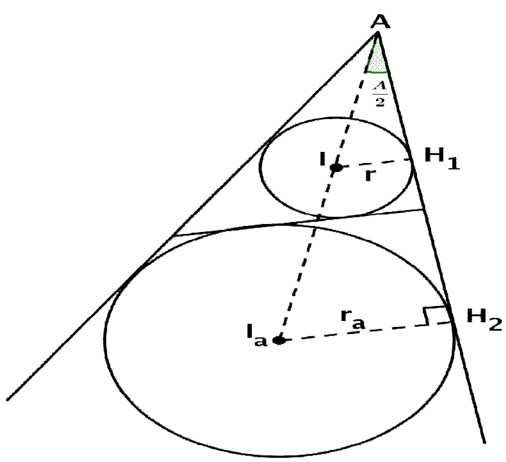
$$\Leftrightarrow 2s^2 \stackrel{?}{\ge} 27Rr$$

But $2s^2 \ge 32Rr - 10r^2 \rightarrow$ ***(Gerretsen)***

$\therefore$ ***suffices to prove:*** $32Rr - 10r^2 \ge 27Rr \Leftrightarrow R \ge 2r \rightarrow$ ***true (*)***

(Proved)

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia



$AI_a = \frac{r_a}{\sin\frac{A}{2}}$ $(\Delta\ AI_aH_2)$; $AI = \frac{r}{\sin\frac{A}{2}}(\Delta\ AIH_1)$; ***In*** $\Delta\ ABC, I -$ ***incentre***

$I_aI_bI_c$ ***– excenters;*** $a\cdot AI\cdot AI_a + b\cdot BI\cdot BI_b + C\cdot CI\cdot CI_c \le \frac{8s^2S}{9r}$

$$\sum \frac{a\cdot r\cdot r_a}{\sin^2\frac{A}{2}} = \sum \frac{a\cdot r\cdot \frac{S}{p-a}}{\left(\frac{(p-b)(p-c)}{bc}\right)} =$$

$$= abc\cdot r\cdot S\cdot \sum \frac{p}{p(p-a)(p-b)(p-c)} =$$

$$= (abc)\cdot r\cdot S\cdot \frac{3p}{S^2} = \frac{12R\cdot S^3}{S} = 12RS = LHS; LHS \le RHS \text{ (ASSURE)}$$

$$12RS \le \frac{9s^2\cdot S}{9r} \Rightarrow \underbrace{2p^2}_{Gerretsen} \ge 27pr; 2\cdot(16Rr-5r^2) \ge 27Rr$$

$$32Rr - 10r^2 \ge 27Rr; 5Rr \ge 10r^2; R \ge 2r$$

Solution 4 by Geanina Tudose – Romania

$\sum a\cdot AI\cdot AI_A \le \frac{8p^2S}{9r}$. **We use** $\begin{cases} AI_a = \frac{r_a}{\sin\frac{A}{2}} \text{ and } r_a = \frac{S}{p-a} \Rightarrow AI_a = \frac{S}{(p-a)\sin\frac{A}{2}} \\ AI = \frac{p-a}{\cos\frac{A}{2}} \end{cases}$

$$\textit{Thus } \sum a\cdot AI\cdot AI_A = \sum a\cdot \frac{p-a}{\cos\frac{A}{2}}\cdot\frac{1}{\sin\frac{A}{2}}\cdot\frac{S}{p-a} = \sum \frac{2aS}{\sin A} =$$

$$= \sum \frac{2\cdot 2R\sin AS}{\sin A} = \sum 4RS = 12RS. \textit{ We show } 12SR \le \frac{8p^2S}{9r} \Leftrightarrow 2p^2 \ge 27Rr$$

$$\Leftrightarrow 2\cdot\left(\frac{a+b+c}{2}\right)^2 \ge 27R\cdot\frac{S}{p} \Leftrightarrow 2p^3 \ge 27R\frac{abc}{4R} \Leftrightarrow 8p^3 = 27abc$$

$$\Leftrightarrow (a+b+c)^3 \ge 3^3abc \text{ (true } AM \ge GM)$$

281. In $\Delta\ ABC$:

$$a^2+b^2+c^2+\frac{1}{8}\sum(b-c)^2 \le 9R^2$$

Proposed by Nicolae and Cristina Nica – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ a^2+b^2+c^2+\frac{1}{8}\sum(b-c)^2\le 9R^2$

Tener en cuenta las siguientes desigualdades e identidades en un $\Delta\ ABC$

$s^2\le 4R^2+4Rr+3r^2$ ***(Inequality Gerretsen),*** $R\ge 2r$

$a^2+b^2+c^2=2(s^2-r^2-4Rr),\ ab+bc+ca=s^2+r^2+4Rr$

La desigualdad es equivalente

$$\Leftrightarrow 10(a^2+b^2+c^2)-2(ab+bc+ca)\le 72R^2$$

$$\Leftrightarrow 5(a^2+b^2+c^2)-(ab+bc+ca)=10(s^2-r^2-4Rr)-(s^2+r^2+4Rr)\le 36R^2$$

$$\Leftrightarrow 5(a^2+b^2+c^2)-(ab+bc+ca)=9s^2-11r^2-44Rr\le$$

$$\le 9(4R^2+4Rr+3r^2)-11r^2-44Rr$$

$$\Leftrightarrow 5(a^2+b^2+c^2)-(ab+bc+ca)\le 36R^2-8Rr+16r^2\le$$

$$\le 36R^2+8r(2r-R)\le 36R^2 \text{ (LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

To prove: $8\sum a^2+\sum(a-b)^2\le 72R^2\Leftrightarrow 10\sum a^2-2\sum ab\le 72R^2$

$$\Leftrightarrow 20(s^2-4Rr-r^2)-2(s^2+4Rr+r^2)\le 72R^2$$

$\Leftrightarrow 9s^2\le 36R^2+44Rr+11r^2$***. But*** $9s^2\le 36R^2+36Rr+27r^2$ ***(Gerretsen)***

$\therefore$ ***it suffices to prove:*** $36R^2+36Rr+27r^2\le 36R^2+44Rr+11r^2$

$\Leftrightarrow 8Rr\ge 16r^2\Leftrightarrow R\ge 2r\rightarrow$ ***true by Euler (Proved)***

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Lemma: In $\Delta\ ABC,\ 4R^2+4Rr+3r^2\ge p^2$

Where $p=a+b+c\cdot p^2+r^2+4Rr=ab+bc+ca$ **and** $abc=4Rrp$

$$9R^2\ge\frac{1}{8}\sum_{cyc}(a-b)^2+a^2+b^2+c^2$$

$$\Leftrightarrow 9R^2\ge\frac{(a+b+c)^2-3(ab+bc+ca)}{4}+a^2+b^2+c^2$$

$$\Leftrightarrow 9R^2 \geq \frac{1}{4}(p^2 - 3r^2 - 12Rr) + 2(p^2 - r^2 - 4Rr)$$

$$\Leftrightarrow 9R^2 \geq \frac{9p^2}{4} - \frac{11r^2}{4} - 11Rr \Leftrightarrow 4R^2 + \frac{44Rr}{9} + \frac{11r^2}{9} \geq p^2$$

we need to prove, $4R^2 + \frac{44Rr}{9} + \frac{11r^2}{9} \geq 4R^2 + 4Rr + 3r^2; \Leftrightarrow \frac{8r(R-2r)}{9} \geq 0$, ***which is true***

$$\because R \geq 2r; 9R^2 \geq \frac{1}{8}\sum_{cyc}(a-b)^2 + a^2 + b^2 + c^2 \text{ (Proved)}$$

282. In ΔABC:

$$(r + 4R)^3 + 5s^2r \geq 16s^2R$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en triángulo ABC $(4R + r)^3 + 5s^2r \geq 16s^2R$

Partimos de la desigualdad de Schur

$$r_a^3 + r_b^3 + r_c^3 + 3r_ar_br_c \geq r_ar_b(r_a + r_b) + r_br_c(r_b + r_c) + r_cr_a(r_cr_a)$$

$$\Leftrightarrow r_a^3 + r_b^3 + r_c^3 + 3r_ar_b(r_a + r_b) + 3r_br_c(r_b + r_c) + 3r_cr_a(r_cr_a) + 6r_ar_br_c \geq$$

$$\geq 4r_ar_b(r_a + r_b) + 4r_br_c(r_b + r_c) + 4r_cr_a(r_cr_a) - 3r_ar_br_c$$

$$\Leftrightarrow (r_a + r_b + r_c)^3 \geq 4(r_a + r_b + r_c)(r_ar_b + r_br_c + r_cr_a) - 9r_ar_br_c$$

$$\Leftrightarrow (4R + r)^3 \geq 4(4R + r)s^2 - 9s^2r \Leftrightarrow (4R + r)^3 + 5s^2r \geq 16s^2R$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow (4R + r)^3 \geq s^2(16R - 5r)$

Now, $s^2(16R - 5r) \leq (4R^2 + 4Rr + 3r^2)(16R - 5r)$

$\therefore$ ***it suffices to prove:*** $(4R + r)^3 \geq (4R^2 + 4Rr + 3r^2)(16R - 5r)$

$\Leftrightarrow r(R - 2r)^2 \geq 0 \to$ ***true (Proved)***

283. In $\Delta\ ABC$:

$$(b+c)m_a^2+(c+a)m_b^2+(a+b)m_c^2\ge 9rs(3R-2r)$$

Proposed by Abdilkadir Altintas-Afyonkarashisar-Turkey

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaidian

We have (Bertgstrom's Inequality)

$$\sum(b+c)m_a^2=\sum\frac{(b+c)^2m_a^2}{b+c}\ge\frac{(\sum(b+c)m_a)^2}{\sum(b+c)}=\frac{(\sum(b+c)m_a)^2}{2\sum a}=$$

$$=\frac{9s^2(s^2-3r^2-4Rr)}{4s}=\frac{9s(s^2-3r^2-4Rr)}{4}$$

We prove that $\frac{9s(s^2-3r^2-4Rr)}{4}\ge 9rs(3R-2r)\Leftrightarrow$

$$\Leftrightarrow s^2-3r^2-4Rr\ge 4r(3R-2r)\Rightarrow$$

$$\Rightarrow s^2-3r^2-4Rr-12Rr+8r^2\ge 0\Rightarrow s^2\ge 16Rr-5r^2$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(b+c)m_a^2+(c+a)m_b^2+(a+b)m_c^2\overset{(1)}{\ge} 9rs(3R-2r)$$

(1)$\Leftrightarrow (2s-a)(2b^2+2c^2-a^2)+(2s-b)(2c^2+2a^2-b^2)+(2s-c)(2a^2+2b^2-c^2)\ge$

$$\ge 36rs(3R-2r)$$

$$\Leftrightarrow 2s\cdot\left(3\sum a^2\right)+\sum a^3-2\left\{\sum(a^2b+ab^2)\right\}\ge 36rs(3R-2r)$$

$$\Leftrightarrow 12s(s^2-4Rr-r^2)+\left[3abc+2s\left(\sum a^2-\sum ab\right)\right]$$

$$-2\sum\{ab(2s-c)\}\ge 36rs(3R-2r)$$

$$\Leftrightarrow 12s(s^2-4Rr-r^2)+12Rrs+2s(s^2-12Rr-3r^2)$$

$$-4s(s^2+4Rr+r^2)+24Rrs\ge 36rs(3R-2r)$$

$$\Leftrightarrow 6s^2-24Rr-6r^2+18Rr+s^2-12Rr-3r^2-$$

$$-2s^2-8Rr-2r^2\ge 54Rr-36r^2$$

$$\Leftrightarrow 5s^2\ge 80Rr-25r^2\Leftrightarrow s^2\ge 16Rr-5r^2\to \textit{true, by Gerretsen}$$

284. In $\Delta\, ABC, I-$ incentre, I_a, I_b, I_c- excenters:

$$S[I_aI_bI_c]\cdot S[II_aI_b]\cdot S[II_bI_c]\cdot S[II_cI_a]\ge R^2\cdot\left(\frac{4S[ABC]}{\sqrt{3}}\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

$$\textit{Lemma 1. } S_{I_aI_bI_c}=2Rp, S_{II_aI_b}=8R^2\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

$$\textit{Lemma 2. } S\le\frac{3\sqrt{3}R^2}{4}.\ \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}, \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{p}{4R}.$$

$$LHS=2Rp\cdot 2^9R^6\cdot\frac{r^2}{16R^2}\cdot\frac{p}{4R}\ge R^2\cdot\frac{2^6\cdot S^3}{3\sqrt{3}}\leftrightarrow R^2\cdot S^2\ge\frac{4S^3}{3\sqrt{3}}\leftrightarrow S\le\frac{3\sqrt{3}R^2}{4}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$II_a=4R\sin\frac{A}{2}$ ***etc and*** $I_aI_b=4R\cos\frac{A}{2}$ ***etc*** $\because$ ***external angle bisector and internal angle bisector are mutually perpendicular,*** $\therefore CI\perp I_aI_b$

$$\therefore S[II_aI_b]=\frac{1}{2}\cdot CI\cdot I_aI_b=\frac{1}{2}\cdot\frac{r}{\sin\frac{C}{2}}\cdot 4R\cos\frac{C}{2}$$

$$=\frac{2Rr\cos\frac{C}{2}}{\sin\frac{C}{2}}=\frac{II_a\cdot II_b\cdot I_aI_b}{4R},$$ ***where R, is the circumradius of $\Delta\, II_aI_b$***

$$\Rightarrow\frac{2Rr\cos\frac{C}{2}}{\sin\frac{C}{2}}=\frac{4R\sin\frac{A}{2}\cdot 4R\sin\frac{B}{2}\cdot 4R\cos\frac{C}{2}}{4R}\quad (*)$$

$$\Rightarrow R_1=8\left(\prod\sin\frac{A}{2}\right)\cdot R\cdot\left(\frac{R}{r}\right)=\frac{8\left(\prod\sin\frac{A}{2}\right)R}{4\left(\prod\sin\frac{A}{2}\right)}=2R\Rightarrow R_1=2R\ (1)$$

$\because I, I_a, I_b, I_c$ ***form an orthocentric system,***

$\therefore \Delta\, I_aI_bI_c, \Delta\, II_aI_b, \Delta\, II_bI_c, \Delta\, II_cI_a$ ***all will have circumradius*** $= 2R$ ***(from (1))***

$$\therefore LHS = \frac{I_aI_b \cdot I_bI_c \cdot I_cI_a}{8R} \cdot \frac{II_a \cdot II_b \cdot I_aI_b}{8R} \cdot \frac{II_b \cdot II_c \cdot I_bI_c}{8R} \cdot \frac{II_c \cdot II_a \cdot I_cI_a}{8R}$$

$$= \frac{\left(4R\cos\frac{C}{2}\cdot 4R\cos\frac{A}{2}\cdot 4R\cos\frac{B}{2}\right)^2\left(64R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2}{64^2R^4}$$

$$= R^8\left(\prod\sin\frac{A}{2}\prod\cos\frac{A}{2}\right)^2 \times\ 64^2 = \frac{R^8(\sin A\sin B\sin C)^2 \times\ 64^2}{64}$$

$$= \frac{64\cdot R^8(abc)^2}{64R^6} = \frac{R^2(abc)^2 \times 64}{64} \overset{(2)}{=} R^2(abc)^2$$ $\therefore$ ***(2)*** $\Rightarrow$ ***it suffices to prove:***

$$R^2(abc)^2 \ge R^2\left(\frac{abc}{\sqrt{3}R}\right)^3 \Leftrightarrow 3\sqrt{3}R^3 \ge 4Rrs \Leftrightarrow 3\sqrt{3}R^2 \ge 4rs \text{ (3)}$$

$\because 3\sqrt{3}R \ge 2s$ ***and*** $R \ge 2r, \therefore 3\sqrt{3}R^2 \ge 4rs \Rightarrow$ ***(3) is true (Proved)***

285. In $\Delta\, ABC$:

$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} \ge 3\sqrt[3]{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} \ge 3\sqrt[3]{\frac{a^2+b^2+c^2}{ab+bc+ca}}$

Tener en cuenta las siguientes desigualdades

$a^2+b^2+c^2 \le 9R^2$ **(Leibniz),** $ab+bc+ca \ge 18Rr$

$$\Leftrightarrow \frac{R}{2r} = \frac{9R^2}{18Rr} \ge \frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1$$

LEMMA → Siendo x, y, z números R^+ se cumple la siguiente desigualdad conocida

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq\sqrt{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}$$ ***(demostrado anteriormente)***

Siendo $x=r_a, y=r_b, z=r_c\Leftrightarrow x+y+z=4R+r$ ***(Steiner)*** $\wedge\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{r}$

$\Leftrightarrow\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\sqrt{(4R+r)\left(\frac{1}{r}\right)}=\sqrt{1+\frac{4R}{r}}$. ***Luego aplicando MA ≥ MG y***

Euler $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\sqrt{1+\frac{4R}{r}}=\sqrt{1+\frac{R}{2r}(8)}\geq\sqrt{9\sqrt[9]{\left(\frac{R}{2r}\right)^6\cdot\left(\frac{R}{2r}\right)^2}}\geq$

$$\geq 3\sqrt[18]{\left(\frac{R}{2r}\right)^6\cdot 1}\geq 3\sqrt[3]{\frac{R}{2r}}\geq 3\sqrt[3]{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\overset{(1)}{\geq}3\sqrt[3]{\frac{a^2+b^2+c^2}{ab+bc+ca}}$. ***Using*** $r_a=\frac{\Delta}{s-a}, r_b=\frac{\Delta}{s-b}, r_c=\frac{\Delta}{s-c}$,

(1) $\Leftrightarrow\frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c}\overset{(2)}{\geq}3\sqrt[3]{\frac{\sum a^2}{\sum ab}}$

Let $s-a=x, s-b=y, s-c=z; s=x+y+z$

and, $a=y+z, b=z+x, c=x+y; x,y,z>0$

$\therefore\sum a^2=2\sum x^2+2\sum xy$ ***(3)***

$$\sum ab=\sum(y+z)(z+x)=\sum\left(\sum xy+z^2\right)\overset{(4)}{=}\sum x^2+3\sum xy$$

$\therefore$ ***(2)*** $\Leftrightarrow\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\overset{(5)}{\geq}3\sqrt[3]{\frac{2(\sum x^2+\sum xy)}{\sum x^2+3\sum xy}}$ ***(using (3), (4))***

$\frac{y}{x}+\frac{z}{y}+\frac{x}{z}=\frac{y^2}{xy}+\frac{z^2}{yz}+\frac{x^2}{zx}\overset{(6)}{\geq}\frac{(\sum x)^2}{\sum xy}$ ***(Bergstrom)***

$=\frac{\sum x^2+2\sum xy}{\sum xy}$. ***From (6), in order to prove (5), it suffices to prove:***

$$\frac{u+2v}{v}\ge 3\sqrt[3]{\frac{2(u+v)}{u+3v}}\ \left(\text{Where } \begin{matrix} u=\sum x^2 \\ v=\sum xy\end{matrix}\right) \Leftrightarrow \frac{(u+2v)^3}{v^3}\ge 27\cdot\frac{2(u+v)}{u+3v}$$

$$\Leftrightarrow (u+2v)^3-(u+3v)-54v^3(u+v)\ge 0$$

$$\Leftrightarrow u^4+9u^3v+30u^2v^2-10uv^3-30v^4\ge 0$$

$$\Leftrightarrow t^4+9t^3+30t^2-10t-30\ge 0\ \left(\text{where } t=\frac{u}{v}\right)$$

$$\Leftrightarrow (t-1)(t^3+10t^2+40t+30)\ge 0 \to \text{true}$$

$$\because t=\frac{u}{v}=\frac{\sum x^2}{\sum xy}\ge 1 \text{ as } x,y,z>0\ (\text{Proved})$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{r_a}{r_b}=\frac{\sum r_a^2r_c}{r_ar_br_c}\overset{Chebyshev}{\ge}$$

$$\ge\frac{\frac{1}{3}(r_a+r_b+r_c)\cdot\sum r_ar_c=p^2}{\underbrace{r_ar_br_c}_{r\cdot p^2}}=\frac{(4R+r)\cdot p^2}{3r\cdot p^2}=\frac{4R+r}{3r}$$

$$LHS\ge\frac{4R+r}{3r};\frac{4R+r}{3r}\ge 3\sqrt[3]{\frac{2p^2-8Rr-2r^2}{p^2+4Rr+r^2}}$$

$$(4R+r)^3\ge 729\cdot r^3\cdot\left(\frac{2p^2-8Rr-2r^2}{p^2+4Rr+r^2}\right)$$

$$(64R^3+48R^2r+12Rr^2-1457r^3)\cdot p^2+$$

$$+(64R^3+48R^2r+12Rr^2+r^3)(4Rr+r^2)+5832R\cdot r^4+1458r^5\ge 0$$

$$p^2\ge 16Rr-5r^2;\ \frac{p}{r}=t$$

$$(64t^3+48t^2+12t-1457)(16-1)+(64t^3+48t^2+12t+1)(t+1)+5832t+1458\ge$$

$$320t^4+176t^3+12t^2-4381t+2186\ge 0$$

$$\underbrace{(t-2)}_{\ge 0}\cdot\underbrace{(320t^3+816t^2+1644t-1093)}_{\substack{\ge 0\\ t\ge 2}}\ge 0$$

286. In any scalene acute – angled $\Delta\, ABC$:

$$\frac{\tan A}{\sin B + 5\sin C} + \frac{\tan B}{\sin C + 5\sin A} + \frac{\tan C}{\sin A + 5\sin B} > \frac{1}{2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\because 0 < A, B, C < \frac{\pi}{2}, \therefore \tan A > A \text{ etc and } \sin A < A \text{ etc}$$

$$\therefore LHS > \sum \frac{A}{B + 5C} = \sum \frac{A^2}{AB + 5CA}$$

$$\overset{Bergstrom}{\geq} \frac{(\sum A)^2}{\sum AB + 5\sum CA} = \frac{\sum A^2 + 2\sum AB}{6\sum AB} \therefore \text{ it suffices to show: } \frac{\sum A^2 + 2\sum AB}{6\sum AB} \geq \frac{1}{2}$$

$$\Leftrightarrow 2\sum A^2 + 4\sum AB \geq 6\sum AB \Leftrightarrow \sum A^2 \geq \sum AB \rightarrow \text{true (Proved)}$$

287. In $\Delta\, ABC$:

$$\sqrt{\frac{r_a + r_b}{r_c}} + \sqrt{\frac{r_b + r_c}{r_a}} + \sqrt{\frac{r_c + r_a}{r_b}} \leq \sqrt{\frac{2(4R + r)}{r}} \leq 3\sqrt{\frac{R}{r}}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC $\sqrt{\frac{r_a+r_b}{r_c}} + \sqrt{\frac{r_b+r_c}{r_a}} + \sqrt{\frac{r_c+r_a}{r_b}} \leq \sqrt{\frac{2(4R+r)}{r}} \leq 3\sqrt{\frac{R}{r}}$

Tener en cuenta lo siguiente $r_a + r_b + r_c = 4R + r, \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}, r \leq \frac{R}{2}$

Por la desigualdad de Cauchy

$$\sqrt{\frac{r_a+r_b}{r_c}} + \sqrt{\frac{r_b+r_c}{r_a}} + \sqrt{\frac{r_c+r_a}{r_b}} \leq$$

$$\sqrt{\left((r_a + r_b) + (r_b + r_c) + (r_c + r_a)\right)\left(\frac{1}{r_c} + \frac{1}{r_a} + \frac{1}{r_b}\right)}$$

$$\sqrt{\frac{r_a+r_b}{r_c}}+\sqrt{\frac{r_b+r_c}{r_a}}+\sqrt{\frac{r_c+r_a}{r_b}}\le\sqrt{2(4R+r)\cdot\frac{1}{r}}=\sqrt{\frac{2(4R+r)}{r}}\le$$

$$\le\sqrt{\frac{2\left(4R+\frac{R}{2}\right)}{r}}=3\sqrt{\frac{R}{r}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS\overset{C-B-S}{\le}\sqrt{\sum(r_a+r_b)}\cdot\sqrt{\sum\frac{1}{r_a}}$$

$$=\sqrt{2(4R+r)}\sqrt{\frac{\sum r_ar_b}{r_ar_br_c}}=\sqrt{2(4R+r)}\sqrt{\frac{s^2}{rs^2}}$$

$$\left(\because\sum r_ar_b=s^2,\ and\ \prod r_a=rs^2\right)$$

$$=\sqrt{\frac{2(4R+r)}{r}}=\sqrt{\frac{8R+2r}{r}}\overset{Euler}{\le}\sqrt{\frac{8R+R}{r}}=3\sqrt{\frac{R}{r}}$$

288. Let ABC be a triangle and r_a,r_b,r_c are exradii and h_a,h_b,h_c are altitudes of ABC. Prove that

$$\frac{r_a^2}{h_a}+\frac{r_b^2}{h_b}+\frac{r_c^2}{h_c}\ge 9r$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{r_a^2}{h_a}+\frac{r_a^2}{h_b}+\frac{r_c^2}{h_c}\overset{Bergstrom}{\ge}\frac{(r_a+r_b+r_c)^2}{h_a+h_b+h_c}=$$

$$= \frac{(4R+r)^2}{2S\cdot\frac{\sum ab}{abc}} = \frac{(4R+r)^2\cdot 2R}{\sum ab} = \frac{(4R+r)^2\cdot 2R}{p^2+4Rr+r^2} \geq$$

$$\overset{Gerretsen}{\geq} \frac{(4R+r)^2\cdot 2r}{4R^2+8Rr+4r^2} = \frac{R}{2}\cdot\left(\frac{4R+r}{R+r}\right)^2 =$$

$$= \frac{R}{2}\cdot\left(4-\frac{3r}{R+r}\right)^2 \geq r\cdot(4-1)^2 = 9r;\ R\geq 2r\text{ , Euler}$$

Solution 2 by Daniel Sitaru – Romania

$$a\leq b\leq c\to\frac{1}{h_a}\leq\frac{1}{h_b}\leq\frac{1}{h_c},\ r_a^2\leq r_b^2\leq r_c^2$$

$$\sum\frac{r_a^2}{h_a}\overset{Cebyshev}{\geq}\frac{1}{3}\sum r_a^2\sum\frac{1}{h_a} = \frac{1}{3}\cdot\frac{s}{S}\cdot\sum r_a^2 =$$

$$= \frac{1}{3R}\left((4R+r)^2-2s^2\right)\overset{Gerretsen}{\geq}\frac{1}{3r}\left((4R+r)^2-2(4R^2+4Rr+3r^2)\right) =$$

$$= \frac{1}{3R}(8R^2-5r^2)\overset{Euler}{\geq}\frac{1}{3r}\cdot 27r^2\geq 9r$$

Solution 3 by Adil Abdullayev-Baku-Azerbaidian

Lemma 1. $h_a+h_b+h_c\leq 4R+r.$

Lemma 2. $r_a+r_b+r_c = 4R+r.$

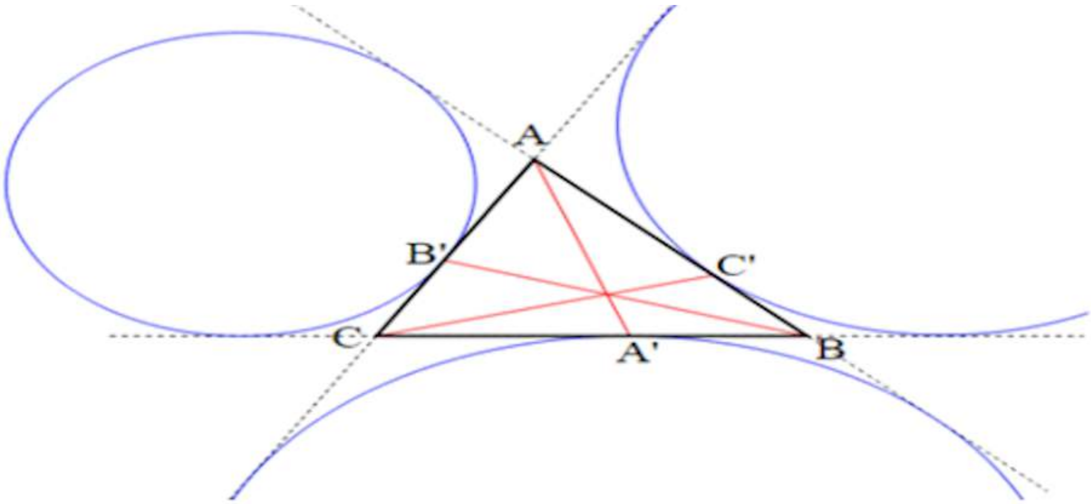
$$LHS\geq\frac{(r_a+r_b+r_c)^2}{h_a+h_b+h_c}\geq\frac{(4R+r)^2}{4R+r} = 4R+r\geq 9r\leftrightarrow R\geq 2r$$

(EULER)

289. In $\Delta\ ABC$:

$$a^3(AA')^2+b^3(BB')^2+c^3(CC')^2\geq 4s^3rR$$

Proposed by Abdilkadir Altintas-Afyonkarashisar-Turkey



Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo AA', BB', CC' Nagel's Cevians. Probar en un triángulo ABC

$$a^3(AA')^2 + b^3(BB')^2 + c^3(CC')^2 \ge 4s^3Rr$$

Teniendo en cuenta las siguientes notaciones y desigualdad

$$a^3(AA')^2 = -\frac{p}{2}a^2(a^2 - 2b^2 - 2c^2 - ab - ac + 4bc),$$

$$p^2 + 5r^2 \ge 16Rr \text{ (Gerretsen' Inequality)}$$

$$b^3(BB)^2 = -\frac{p}{2}b^2(b^2 - 2c^2 - 2a^2 - bc - ba + 4ca),$$

$$c^3(CC')^2 = -\frac{p}{2}c^3(c^2 - 2a^2 - 2b^2 - ca - cb + 4ab)$$

EL LHS es equivalente

$$\frac{p}{2}\left(\sum a^4 - 2\sum a^2(b^2 + c^2) - \sum a^3(b + c) + \sum 4a^2bc\right) =$$

$$= -\frac{p}{2}\left(\sum a^4 - 4\sum a^2b^2 - \sum ab(a^2 + b^2) + \sum 4a^2bc\right)$$

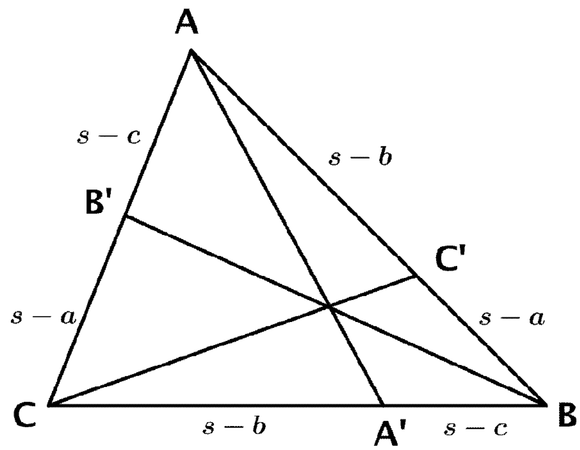
$$= -\frac{p}{2}\left(\left(\sum a^4 - 2\sum a^2b^2\right)\right) - 2\sum a^2b^2 - \left(\sum a^2\right)\left(\sum ab\right) + 5abc(a + b + c))$$

$$= -\frac{p}{2}(-16p^2r^2 - 2(p^2 + r^2 + 4Rr)^2 + 32p^2Rr - 2(p^2 - r^2 - 4Rr)(p^2 + r^2 + 4Rr) + 40p^2Rr)$$

$$= -\frac{p}{2}\left(-16p^2r^2 - 2p^4 - 2\left(r^2 + 4Rr\right)^2 - 4p^2\left(r^2 + 4Rr\right) - 2\left(p^4 - \left(r^2 + 4Rr\right)^2\right) + 72p^2Rr\right)$$

$$= -\frac{p}{2}\left(-4p^2 - 20p^2r^2 + 56p^2Rr\right) - 2p^3\left(p^2 + 5r^2 - 14Rr\right) \geq 2p^3 \cdot 2Rr - 4p^3Rr$$

Solution 2 by Soumava Chakraborty-Kolkata-India



Let $AA' = x, BB' = y, CC' = z$***. By Stewart's theorem,***

$$b^2(s-c) + c^2(s-b) = a\left(x^2 + (s-b)(s-c)\right) \text{ (*)}$$

$$\Rightarrow ax^2 = b^2(s-c) + c^2(s-b) - a(s-b)(s-c)$$

$$\Rightarrow a^3x^2 = a^2b^2(s-c) + a^2c^2(s-b) - a^3\{s^2 - s(2s-a) + bc\}$$

$$= s(a^2b^2 + a^2c^2) - abc(ab + ca) - a^3(-s^2 + as + bc)$$

$$\overset{(1)}{=} s(a^2b^2 + a^2c^2) - abc(ab + ca) + s^2a^3 - a^4s - a^3bc$$

Similarly, $b^3y^3 \overset{(2)}{=} s(b^2c^2 + b^2a^2) - abc(bc + ab) + s^2b^3 - b^4s - b^3ca$

and, $c^3z^2 \overset{(3)}{=} s(c^2a^2 + c^2b^2) - abc(ca + bc) + s^2c^3 - c^4s - c^3ab$

(1)+(2)+(3) $\Rightarrow LHS \overset{(4)}{=} 2s(\sum a^2b^2) - 2abc(\sum ab) +$

$$+s^2\left(\sum a^3\right) - s\left(\sum a^4\right) - abc\left(\sum a^2\right)$$

Now, $\sum a^4 = (\sum a^2)^2 - 2(\sum a^2b^2)$ ***(*)*** $\Rightarrow 2\sum a^2b^2 - (\sum a^2)^2 = -\sum a^4$

$$\Rightarrow -s\left(\sum a^4\right) = 2s\sum a^2b^2 - s\left(\sum a^2\right)^2$$

$$\Rightarrow -s\left(\sum a^4\right) + 2s\left(\sum a^2b^2\right) = 4s\left(\sum a^2b^2\right) - s\left(\sum a^2\right)^2$$

$$= 4s\left\{\left(\sum ab\right)^2 - 2abc(2s)\right\} - 4s(s^2 - 4Rr - r^2)^2$$

$$= 4s\{(s^2 + 4Rr + r^2)^2 - (s^2 - 4Rr - r^2)^2\} - 16s^2(4Rrs)$$

$$= 4s(2s^2)(8Rr + 2r^2) - 64s^3Rr$$

$$= 16s^3(4Rr + r^2) - 64s^3Rr = 16s^3(4Rr + r^2 - 4Rr)$$

$$= 16s^3r^2 \ \therefore 2s\sum a^2b^2 - s\left(\sum a^4\right) = 16s^3r^2 \ \textbf{(5)}$$

Again, $-2abc\left(\sum ab\right) - abc\left(\sum a^2\right)$

$$= -abc\left(\sum a^2 + 2\sum ab\right) = -abc(a + b + c)^2$$

$$= -abc(4s^2) = -4Rrs \cdot 4s^2 \overset{(6)}{=} -16s^3Rr$$

Also, $s^2\left(\sum a^3\right) = s^2\left\{3abc + 2s\left(\sum a^2 - \sum ab\right)\right\}$

$$= s^2\{12Rrs + 2s(s^2 - 12Rr - 3r^2)\}$$

$$= 2s^3(6Rr + s^2 - 12Rr - 3r^2) \overset{(7)}{=} 2s^3(s^2 - 6Rr - 3r^2)$$

(4) together with (5)+(6)+(7) gives

$$LHS = s^3\left(16r^2 - 16Rr + 2(s^2 - 6Rr - 3r^2)\right)$$

$$= 2s^3(8r^2 - 8Rr + s^2 - 6Rr - 3r^2)$$

$$= 2s^3(s^2 - 14R + 5r^2) \overset{?}{\geq} 4s^3Rr \Leftrightarrow s^2 - 14Rr + 5r^2 \overset{?}{\geq} 2Rr$$

$$\Leftrightarrow s^2 \overset{?}{\geq} 16Rr - 5r^2 \rightarrow \textit{true, by Gerretsen}$$

290. If in $\Delta\ ABC: c \leq b \leq a$ then:

$$\left(\frac{h_b}{m_a}+\frac{h_a}{m_c}+\frac{h_c}{m_b}\right)\left(\frac{h_b}{s_a}+\frac{h_a}{s_c}+\frac{h_c}{s_b}\right) \leq \left(\frac{b}{a}+\frac{a}{c}+\frac{c}{b}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$a \geq b \geq c \Leftrightarrow \frac{h_b}{m_a}+\frac{h_a}{m_c}+\frac{h_c}{m_b} \leq \frac{h_c}{m_a}+\frac{h_b}{m_c}+\frac{h_a}{m_b}$$

$$\frac{h_b}{s_a}+\frac{h_a}{s_c}+\frac{h_c}{s_b} \leq \frac{h_c}{s_a}+\frac{h_b}{s_c}+\frac{h_a}{s_b}$$

$$LHS \leq \sum\frac{h_c}{m_a}\cdot\sum\frac{h_c}{s_a} = 4\cdot S^2\left(\sum\frac{1}{c\cdot m_a}\right)\cdot\left(\sum\frac{1}{c\cdot s_a}\right) \leq$$

$$\underset{m_a \geq s_a \geq h_a}{\leq} 4\cdot S^2\cdot\sum\frac{1}{c\cdot h_a}\cdot\sum\frac{1}{c\cdot h_a} =$$

$$= 4S^2\cdot\left(\sum\frac{1}{c\cdot h_a}\right)^2 = \frac{4S^2}{4S^2}\left(\frac{a}{c}\right)^2 = \left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)^2$$

291. In $\Delta\ ABC$:

$$\frac{a^2+b^2}{ab}+\frac{b^2+c^2}{bc}+\frac{c^2+a^2}{ca} \geq 7-\frac{2r}{R}$$

Proposed by Rovsen Pirguliyev – Sumgait – Azerbaidian

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

$$\frac{a^2+b^2}{ab}+\frac{b^2+c^2}{bc}+\frac{c^2+a^2}{ca} = \frac{ab(a+b)+bc(b+c)+ca(c+a)}{abc} =$$

$$= \frac{p^2+r^2+4Rr-6Rr}{2Rr} \overset{?}{\geq} 7-\frac{2r}{R} p^2+r^2-2Rr \overset{?}{\geq} 14Rr-4r^2 \Leftrightarrow$$

$$p^2 \geq 16Rr-5r^2 \Rightarrow \textbf{Gerretsen}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{2r}{R}\geq 10$

Sea: $a=x+y, b=y+z, c=x+z$

Ademas: $\frac{2r}{r}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{8(p-a)(p-b)(p-c)}{abc}=\frac{(b+c-a)(a+c-b)(b+a-c)}{abc}$

$$(x+y)(y+z)(z+x)=xy(x+y)+yz(y+z)+xz(x+z)+2xyz,$$

$$(x+y+z)^3=x^3+y^3+z^3+3(x+y)(y+z)(x+z)$$

La desigualdad es equivalente:

$$(a+b+c)\left(\frac{bc+ac+ab}{abc}\right)+\frac{(b+c-a)(a+c-b)(b+a-c)}{abc}\geq 10$$

$$\Rightarrow (a+b+c)(bc+ac+ab)+(b+c-a)(a+c-b)(b+a-c)\geq 10abc$$

$$\Rightarrow 2(x+y+z)\big((y+z)(x+z)+(x+y)(x+z)+(x+y)(y+z)\big)+8xyz\geq$$

$$\geq 10(x+y)(x+z)(y+z)$$

$$\Rightarrow 2(x+y+z)((x+y+z)^2+xy+yz+xz)+8xyz\geq$$

$$\geq 10(x+y)(y+z)(z+x)$$

$$\Rightarrow 2(x+y+z)^3+2(x+y+z)(xy+yz)+8xyz\geq$$

$$\geq 10(x+y)(y+z)(x+z)$$

$$\Rightarrow 2x^3+2y^3+2z^3+6(x+y)(y+z)(z+x)+2xy(x+y)+$$

$$+2yz(y+z)+2xz(x+z)+6xyz+8xyz\geq 10(x+y)(y+z)(z+x)$$

$$\Rightarrow 2x^3+2y^3+2z^3+2xy(x+y)+2yz(y+z)+2xz(x+z)+14xyz\geq$$

$$\geq 4(x+y)(y+z)(x+z)$$

$$\Rightarrow 2x^3+2y^3+2z^3+2xy(x+y)+2yz(y+z)+2xz(x+z)+14xyz\geq$$

$$\geq 4xy(x+y)+4yz(y+z)+4xz(x+z)+8xyz\rightarrow$$

$\rightarrow$ ***(Simplificando se tiene ...)***

$$\Rightarrow x^3+y^3+z^3-xy(x+y)-yz(y+z)-zx(z+x)+3xyz\geq 0$$

→ ***A lo que es equivalente:***

$$x(x-y)(x-z)+y(y-x)(y-z)+z(z-x)(z-y)\geq 0\rightarrow$$

→ ***(Válido por desigualdad Schur)***

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS=\frac{\sum a^2-c^2}{ab}+\frac{\sum a^2-a^2}{bc}+\frac{\sum a^2-b^2}{ca}$$

$$=\sum a^2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)-\left(\frac{c^2}{ab}+\frac{a^2}{bc}+\frac{b^2}{ca}\right)$$

$$=\frac{\sum a^2\,(a+b+c)}{abc}-\frac{3abc+2s(\sum a^2-\sum ab)}{abc}$$

$$=\frac{2(s^2-4Rr-r^2)(2s)}{4Rrs}-\frac{12Rrs+2s(s^2-12Rr-3r^2)}{4Rrs}$$

$$=\frac{2(s^2-4Rr-r^2)}{2Rr}-\frac{6Rr+s^2-12Rr-3r^2}{2Rr}=\frac{s^2-2Rr+r^2}{2Rr}$$

$$\therefore LHS-7=\frac{s^2-16Rr+r^2}{2Rr}\therefore LHS-7+\frac{2r}{R}=\frac{s^2-16Rr+r^2}{2Rr}+\frac{2r}{R}$$

$$=\frac{s^2-16Rr+r^2+4r^2}{2Rr}=\frac{s^2-16Rr+5r^2}{2Rr}$$

$$\Rightarrow LHS-RHS=\frac{s^2-(16Rr-5r^2)}{2Rr}\geq 0,\because s^2\geq 16Rr-5r^2$$ ***(Gerretsen)***

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{(a^2+b^2)\cdot\frac{\sin C}{2}}{\frac{ab\cdot\sin C}{2}}=\frac{1}{2S}\cdot\sum(a^2+b^2+c^2-c^2)\cdot\sin C=$$

$$=\frac{1}{2S}\cdot\left((a^2+b^2+c^2)\cdot(\sin A+\sin B+\sin C)-\sum a^2\cdot\sin A\right)$$

$$=\frac{1}{2S}\cdot\left(\sum a^2\cdot\frac{\sum a}{2R}-\frac{\sum a^3}{2R}\right)=$$

$$=\frac{1}{4SR}\cdot\left(\sum a^2\cdot\sum a-\sum a^3\right)=\frac{1}{4SR}\cdot\sum(a^2b+ab^2)=$$

$$=\frac{1}{4SR}\cdot\left(\sum ab\cdot\sum a-3abc\right)=\frac{1}{4SR}\cdot 2p\cdot\left(\sum ab-6pr\right)=$$

$$=\frac{1}{2Rr}\cdot(p^2-2Rr+r^2)\overset{Gerrestsen}{\geq}\frac{1}{2Rr}(16Rr-5r^2-2Rr+r^2)=$$

$$=\frac{1}{2Rr}\cdot(14Rr-4r^2)=7-\frac{2r}{R}$$

Solution 5 by Soumitra Mandal-Chandar Nagore-India

$$ab+bc+ca=p^2+r^2+4Rr,\ a+b+c=2p$$

$$\sum_{cyc}\frac{a^2+b^2}{ab}=\frac{1}{abc}\sum_{cyc}c(a^2+b^2)=\frac{(a+b+c)(ab+bc+ca)}{abc}-3$$

$=\frac{p^2+r^2+4Rr}{2Rr}-3=\frac{p^2}{2Rr}+\frac{r}{2R}-1$. ***We need to prove,*** $\frac{p^2}{2Rr}+\frac{r}{2R}-1\geq 7-\frac{2r}{R}$

$\Leftrightarrow p^2\geq 16Rr-5r^2$, ***which is true.*** $\therefore \sum_{cyc}\frac{a^2+b^2}{ab}\geq 7-\frac{2r}{R}$ ***(Proved)***

292. In $\Delta\, ABC$:

$$\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}\geq\sqrt{1+8\cdot\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

LEMMA: Siendo x,y,z números R^+ se cumple la siguiente desigualdad

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}+6\geq(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$\Leftrightarrow\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}+3\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}$. ***Realizando los siguientes***

cambios de variables $\frac{x}{y}=\frac{a^2}{bc}>0,\frac{y}{z}=\frac{b^2}{ca}>0,\frac{z}{x}=\frac{c^2}{ab}>0\Leftrightarrow a,b,c>0$

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La desigualdad es equivalente

$$\frac{a^4}{b^2c^2}+\frac{b^4}{c^2a^2}+\frac{c^4}{a^2b^2}+3\ge\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}$$

$$\Leftrightarrow a^6+b^6+c^6+3a^2b^2c^2\ge a^4bc+b^4ac+c^4ab+b^3c^3+c^3a^3+a^3b^3$$

Por la desigualdad de Schur y MA ≥ MG

$$a^6+b^6+c^6+3a^2b^2c^2\ge a^4(b^2+c^2)+b^4(c^2+a^2)+c^4(a^2+b^2)\ge$$

$$\ge 2a^4bc+2b^4ac+2c^4ab\quad (1)$$

$$a^6+b^6+c^6+3a^2b^2c^2\ge b^2c^2(b^2+c^2)+c^2a^2(c^2+a^2)+a^2b^2(a^2+b^2)\ge$$

$$\ge 2b^3c^3+2c^3a^3+2a^3b^3\quad (2)$$

$$(1)+(2)\Rightarrow a^6+b^6+c^6+3a^2b^2c^2\ge a^4bc+b^4ac+c^4ab+b^3c^3+c^3a^3+a^3b^3$$

(LQQD). Ahora bien $x=r_a,y=r_b,z=r_c$

$$\Leftrightarrow\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}+6\ge(r_a+r_b+r_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)$$

$$\Leftrightarrow\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}+6\ge\frac{4R+r}{r}=1+\frac{4R}{r}\Leftrightarrow\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}\ge\frac{4R}{r}-5$$

Es necesario demostrar lo siguiente $\frac{4R}{r}-5\ge\sqrt{1+8\cdot\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2}$

Es muy conocido lo siguiente

$$\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2\le\left(\frac{R}{2r}\right)^2\Leftrightarrow\sqrt{1+8\cdot\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2}\le$$

$$\le\sqrt{9\cdot\left(\frac{R}{2r}\right)^2}\le\frac{3R}{2r}\le\frac{4R}{r}-5\quad \textit{(Válido por Euler)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}\overset{(1)}{\gtrsim}\sqrt{1+8\cdot\left(\frac{\sum a^2}{\sum ab}\right)^2}$$

Using $r_a = \frac{\Delta}{s-a}, r_b = \frac{\Delta}{s-b}, r_c = \frac{\Delta}{s-c}$,

$$\textit{(1)} \Leftrightarrow \frac{(s-b)^2}{(s-a)^2} + \frac{(s-c)^2}{(s-b)^2} + \frac{(s-a)^2}{(s-c)^2} \overset{(2)}{\geq} \sqrt{1 + 8\left(\frac{\sum a^2}{\sum ab}\right)^2}$$

Let $s - a = x, s - b = y, s - c = z$

Then, $s = x + y + z \Rightarrow a = y + z, b = z + x, c = x + y$

$\therefore \sum a^2 = 2\sum x^2 + 2\sum xy$, *and* $\sum ab = \sum x^2 + 3\sum xy$

$$\therefore \textit{(2)} \Leftrightarrow \frac{y^2}{x^2} + \frac{z^2}{y^2} + \frac{x^2}{z^2} \geq \sqrt{1 + \frac{32(\sum x^2 + \sum xy)^2}{(\sum x^2 + 3\sum xy)^2}}$$

Now, $\frac{y^2}{x^2} + \frac{z^2}{y^2} + \frac{x^2}{z^2} \geq \frac{1}{3}\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right)^2$ *(Chebyshev)*

$$= \frac{1}{3}\left(\frac{y^2}{xy} + \frac{z^2}{yz} + \frac{x^2}{zx}\right)^2 \overset{Bergstrom}{\geq} \frac{1}{3}\left[\frac{(\sum x)^2}{\sum xy}\right]^2$$

$$= \frac{(\sum x^2 + 2\sum xy)^2}{3(\sum xy)^2} \overset{?}{\geq} \sqrt{1 + \frac{32(\sum x^2 + \sum xy)^2}{(\sum x^2 + 3\sum xy)^2}}$$

$\Leftrightarrow \frac{(u+2v)^4}{9v^3} \overset{?}{\geq} 1 + \frac{32(u+v)^2}{(u+3v)^2}$, *where* $u = \sum x^2, v = \sum xy$

$$\Leftrightarrow (u + 3v)^2\{(u + 2v)^4 - 9v^4\} - 288v^4(u + v)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow u^6 + 14u^5v + 81u^4v^2 + 248u^3v^3 + 127u^2v^4 - 246uv^5 - 225v^6 \overset{?}{\geq} 0$$

$$\Leftrightarrow t^6 + 14t^5 + 81t^4 + 248t^3 + 127t^2 - 246t - 225 \overset{?}{\geq} 0 \left(t = \frac{u}{v}\right)$$

$$\Leftrightarrow (t - 1)\left(t^5 + 15t^4 + 96t^3 + 344t^2 + 471t + 225\right) \overset{?}{\geq} 0$$

$\rightarrow$ *true* $\because t = \frac{u}{v} = \frac{\sum x^2}{\sum xy} \geq 1$ *(Proved)*

293. In acute $\Delta\ ABC$:

$$\sum \sqrt{w_a s_a r_a} \leq s\sqrt{s\sqrt{3}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo $ABC \sum \sqrt{w_a s_a r_a} \leq s\sqrt{s\sqrt{3}}$

1) Tener en cuenta las siguiente desigualdades conocidas en un

$\Delta\ ABC\ s_a \leq m_a, s_b \leq m_b, s_c \leq m_c$

$w_a \leq \sqrt{s(s-a)}, w_a \leq \sqrt{s(s-b)}, w_c \leq \sqrt{s(s-c)}$. ***Además***

$$(w_a + w_b + w_c)^2 \leq \left(\sqrt{s(s-a)} + \sqrt{s(s-b)} + \sqrt{s(s-c)}\right)^2 \leq$$

$$\leq (s+s+s)(s-a+s-b+s-c) \leq 3s^2$$

$$\Leftrightarrow w_a + w_b + w_c \leq s\sqrt{3}$$

2) En un triángulo se cumple la siguiente desigualdad

$$m_a r_a + m_b r_b + m_c r_c \leq s^2$$

Dado que es un triángulo acutángulo, tener en cuenta las siguientes desigualdades

$$m_a \leq R(1+\cos A), m_b \leq R(1+\cos B), m_c \leq R(1+\cos C)$$

La desigualdad es equivalente

$$m_a r_a + m_b r_b + m_c r_c \leq R(1+\cos A)\cdot\frac{S}{s-a} + R(1+\cos B)\cdot\frac{S}{s-b} + R(1+\cos C)\cdot\frac{S}{s-c}$$

$$= R\frac{(b+c+a)(b+c-a)}{2bc}\cdot\frac{S}{s-a} + R\cdot\frac{(a+c+b)(a+c-b)}{2ca}\cdot\frac{S}{s-b} +$$

$$+R\cdot\frac{(a+b+c)(a+b-c)}{2ab}\cdot\frac{S}{s-c}$$

$$= R\cdot\frac{S}{bc}(a+b+c) + R\cdot\frac{S}{ca}(a+b+c) + R\cdot\frac{S}{ab}(a+b+c) =$$

$$= R\cdot S(a+b+c)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) = R\cdot S\cdot\frac{(a+b+c)^2}{abc} = R\cdot S\cdot\frac{4s^2}{4RS} = s^2$$ ***(LQQD)***

Luego $\sum\sqrt{w_a s_a r_a} \le \sum\sqrt{w_a m_a r_a} \le s\sqrt{s\sqrt{3}}$***. Por la desigualdad de Cauchy*** $\left(\sum\sqrt{w_a s_a r_a}\right)^2 \le \left(\sum\sqrt{w_a m_a r_a}\right)^2 \le \left(\sum w_a\right)\left(\sum m_a r_a\right) \le \left(s\sqrt{3}\right)(s^2)$

Por transitividad $\to \sum\sqrt{w_a s_a r_a} \le s\sqrt{s\sqrt{3}}$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt{w_a}\sqrt{s_a r_a} + \sqrt{w_b}\sqrt{s_b r_b} + \sqrt{w_c}\sqrt{s_c r_c} \overset{C-B-S}{\underset{(1)}{\le}} \sqrt{\sum w_a}\sqrt{\sum s_a r_a}$$

$$\sum w_a \overset{C-B-S}{\le} \sqrt{3}\sqrt{w_a^2+w_b^2+w_c^2}$$

$$\le \sqrt{3}\sqrt{s(s-a)+s(s-b)+s(s-c)}\ \left(\because w_a \le \sqrt{s(s-a)}, etc\right)$$

$$= \sqrt{3}\sqrt{3s^2 - s(2s)} = s\sqrt{3} \Rightarrow \sqrt{\sum w_a} \le \sqrt{s\sqrt{3}}\ \ (2)$$

Again, $\sum s_a r_a \overset{s_a\le m_a}{\le} \sum m_a r_a \overset{m_a\le R(1+\cos A)}{\le} \sum R(1+\cos A) r_a$

$$= 2R\sum\left(\cos^2\frac{A}{2}\right)r_a = 2R\sum\left\{\frac{s(s-a)}{bc}\cdot\frac{\Delta}{s-a}\right\}$$

$$= 2Rrs^2\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right) = \frac{2Rrs^2}{4Rrs}(2s) = s^2$$

$$\Rightarrow \sqrt{\sum s_a r_a} \le s\ \ (3)$$

(1), (2), (3) $\Rightarrow LHS \le \sqrt{s\sqrt{3}}s = s\sqrt{s\sqrt{3}} = RHS$ ***(Proved)***

294. In $\Delta\, ABC, AA' = w_a, BB' = w_b, CC' = w_c, A' \in (BC), B' \in (CA), C' \in (AB)$

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 2 - \frac{4S[A'B'C']}{S[ABC]}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo $ABC, AA' = w_a, BB' = w_b, CC' = w_c,$

$A' \in (BC), B' \in (CA), C' \in (AB)$. **Probar que** $\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 4 - \frac{4S[A'B'C']}{S[ABC]}$

Sea $\rightarrow BA' = yr, A'C = r, BC' = q, C'A = qz, AB' = p, B'C = px$

Ahora bien $S[C'BA'] = \frac{1}{2}(BC')(BA')\sin B = \frac{pqz}{2}\sin B$ **… (1)**

$S[ABC] = \frac{1}{2}(AB)(BC)\sin B = \frac{1}{2}q(1+z)p(1+x)\sin B$ **… (2)**

Dividiendo (1) y (2) $\frac{S[C'BA']}{S[ABC]} = \frac{z}{(1+z)(1+x)}$ **… (3)**

Análogamente se cumplira lo siguiente

$\frac{S[A'CB']}{S[ABC]} = \frac{x}{(x+1)(y+1)}$ **… (4)** $\wedge$ $\frac{S[B'AC']}{S[ABC]} = \frac{y}{(y+1)(z+1)}$ **… (5)**

Además $y = \frac{a}{b}, z = \frac{b}{c}, x = \frac{c}{a} \Leftrightarrow xyz = 1$. **Por la tanto**

$$1 = \frac{S[C'BA']}{S[ABC]} + \frac{S[A'CB']}{S[ABC]} + \frac{S[B'AC']}{S[ABC]} + \frac{S[A'BC']}{S[ABC]}$$

$$1 = \frac{z}{(1+z)(1+x)} + \frac{x}{(x+1)(y+1)} + \frac{y}{(y+1)(z+1)} + \frac{S[A'BC']}{S[ABC]}$$

$$\frac{(1+x)(1+y)(1+z) - z(1+x) - x(1+z) - y(1+x)}{(1+x)(1+y)(1+z)} = \frac{S[A'BC']}{S[ABC]}$$

$$\frac{S[A'BC']}{S[ABC]} = \frac{1+xyz}{(1+x)(1+y)(1+z)} = \frac{2}{\left(1+\frac{c}{a}\right)\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)} = \frac{2abc}{(a+b)(b+c)(c+a)}$$

Por último demostraremos $\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$

Supongamos sin **pérdida de generalidad** $\rightarrow a \geq b \geq c > 0$

LEMMA $\rightarrow$ **Siendo** $m \geq n > 0$ **y** $p \geq 0$ **se cumple la siguiente desigualdad**

$\frac{m}{n} \geq \frac{m+p}{n+p} \Leftrightarrow \frac{m}{n} - \frac{m+p}{n+p} = \frac{p(m-n)}{n(n+p)} \geq 0$ **, donde** $m = a^2 + b^2 + c^2$,

$$n = ab + bc + ca, p = c^2$$

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq \frac{a^2+b^2+c^2}{ab+bc+ca+c^2} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

$$\Leftrightarrow \frac{a^2+b^2+c^2}{(c+b)(c+a)} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

$$\Leftrightarrow (a^2+b^2+2c^2)(a+b) + 8abc \geq 2(a+b)(b+c)(c+a)$$

$$\Leftrightarrow a^3 + a^2b + b^2a + b^3 + 2c^2a + 2c^2b + 8abc \geq$$

$$\geq 2a^2b + 2b^2a + 2c^2a + 2b^2c + 2c^2b + 4abc$$

$$\Leftrightarrow a^3 + b^3 - a^2b - b^2a - 2a^2c + 4abc - 2b^2c \geq 0$$

$$\Leftrightarrow (a+b)(a^2-ab+b^2) - ab(a+b) - 2c(a^2-2ab+b^2) \geq 0$$

$$\Leftrightarrow (a+b)(a-b)^2 - 2c(a+b)^2 = \big((a-c)+(b-c)\big)(a-b)^2 \geq 0$$

(LQQD)

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaidian

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 2 - \frac{4S[A'B'C']}{S[ABC]}$$ ***(*). It is known that***

$$S_{A'B'C'} = \frac{2S_{ABC} \cdot abc}{(a+b)(b+c)(a+c)}, a^2+b^2+c^2 = 2(p^2-r^2-4Rr)$$

$$abc = 4Rrp, (a+b)(b+c)(a+c) = 2p(p^2+2Rr+r^2)$$

$ab+bc+ca = p^2+r^2+4Rr$, ***we have:*** $\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 2 - \frac{4\cdot\frac{2S\cdot abc}{(a+b)(b+c)(a+c)}}{S} \Rightarrow$

$$\Rightarrow \frac{2(p^2-r^2-4Rr)}{p^2+r^2+4Rr} \geq 2 - \frac{32Rrp}{2p(p^2+2Rr+r^2)} \Rightarrow$$

$$\Rightarrow \frac{2(p^2-r^2-4Rr)}{p^2+r^2+4Rr} \geq \frac{2p^2-12Rr+2r^2}{p^2+2Rr+r^2} \Rightarrow$$

$$\Rightarrow \frac{p^2-r^2-4Rr}{p^2+r^2+4Rr} \geq \frac{p^2-6Rr+r^2}{p^2+2Rr+r^2} \Rightarrow$$

$$\Rightarrow p^2 + 2Rr + r^2 \le 9R^2 \Rightarrow \frac{27}{4}R^2 \le 10R^2 - (R + r) \Rightarrow \textit{true!}$$

295. In $\Delta\ ABC$:

$$\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \ge 27\left(\frac{a^2 + b^2 + c^2}{ab + bc + ca}\right)^2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \ge 27\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2$

Utilizando las siguientes desigualdades y notaciones en un $\Delta\ ABC$

$$m_a \ge \frac{b^2 + c^2}{4R}, m_b \ge \frac{c^2 + a^2}{4R}, m_c \ge \frac{a^2 + b^2}{4R}, h_a = \frac{bc}{2R}, h_b = \frac{ca}{2R}, h_c = \frac{ab}{2R}$$

Por la tanto $\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \ge \left(\frac{b^2+c^2}{2bc} + \frac{c^2+a^2}{2ca} + \frac{a^2+b^2}{2ab}\right)^3$

Aplicando la desigualdad de Cauchy

$$\frac{b^2 + c^2}{2bc} + \frac{c^2 + a^2}{2ca} + \frac{a^2 + b^2}{2ab} \ge \frac{\left(\sqrt{b^2 + c^2} + \sqrt{c^2 + a^2} + \sqrt{a^2 + b^2}\right)^2}{2(ab + bc + ca)} =$$

$$= \frac{2(a^2 + b^2 + c^2) + 2\sum\sqrt{(c^2 + a^2)(c^2 + b^2)}}{2(ab + bc + ca)} \ge$$

$$\ge \frac{2(a^2+b^2+c^2)+2\sum(c^2+ab)}{2(ab+bc+ca)} = \frac{4(a^2+b^2+c^2)+2(ab+bc+ca)}{2(ab+bc+ca)} =$$

$= \frac{2(a^2+b^2+c^2)}{ab+bc+ca} + 1$*. Luego por transitividad y aplicando MA* $\ge$ *MG*

$$\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \ge \left(\frac{b^2 + c^2}{2bc} + \frac{c^2 + a^2}{2ca} + \frac{a^2 + b^2}{2ab}\right) \ge$$

$$\geq\left(\frac{(a^2+b^2+c^2)}{ab+bc+ca}+\frac{(a^2+b^2+c^2)}{ab+bc+ca}+1\right)^3\geq 27\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^3$$

296. In $\Delta\ ABC$:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{2r}{R}\geq 4$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\geq 4-\frac{2r}{R}$

$$\Rightarrow\frac{\tan^2\frac{A}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}}+\frac{\tan^2\frac{B}{2}}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\tan^2\frac{C}{2}}{\tan\frac{C}{2}\tan\frac{A}{2}}\geq 4-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Por la desigualdad de Cauchy:

$$\Rightarrow\frac{\tan^2\frac{A}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}}+\frac{\tan^2\frac{B}{2}}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\tan^2\frac{C}{2}}{\tan\frac{C}{2}\tan\frac{A}{2}}\geq\frac{\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}}$$

$$\Rightarrow\frac{\tan^2\frac{A}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}+\frac{\tan^2\frac{B}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}+\frac{\tan^2\frac{C}{2}}{\tan\frac{C}{2}+\tan\frac{A}{2}}\geq$$

$$\geq\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+2\geq 4-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Es necesario demostrar lo siguiente

$$\Leftrightarrow\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+2\geq 4-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Leftrightarrow\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq 2$$

(lo cual ya se demostro anteriormente)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{2r}{R}\overset{(1)}{\geq}4$$

(1) $\Leftrightarrow \frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\overset{(2)}{\geq}\frac{2(2R-r)}{R}$. **Now,** $\sum\frac{r_a}{r_b}=\sum\frac{r_a^2}{r_ar_b}\underset{(3)}{\overset{Bergstrom}{\geq}}\frac{(\sum r_a)^2}{\sum r_ar_b}=\frac{(4R+r)^2}{s^2}$

It suffices to prove: (2),(3) $\Rightarrow \frac{(4R+r)^2}{s^2}\geq\frac{2(2R-r)}{R}$

$\Leftrightarrow (4R+r)^2\cdot R\geq 2(2R-r)s^2$ ***(4). Now, Rouche*** $\Rightarrow 2(2R-r)s^2$

$\overset{(5)}{\leq} 2(2R-r)(2R^2+10Rr-r^2)+2^2(2R-r)(R-2r)\sqrt{R^2-2Rr}$

(4), (5) $\Rightarrow$ ***it suffices to prove:***

$$R(4R+r)^2-2(2R-r)(2R^2+10Rr-r^2)\geq$$

$$\geq 4(2R-r)(R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2)\geq 4(R-2r)(2R-r)\sqrt{R^2-2Rr}$$

$\therefore$ ***it suffices to prove:*** $8R^2-12Rr+r^2>4(2R-r)\times\sqrt{R^2-2Rr}$

$$\Leftrightarrow (8R^2-12Rr+r^2)^2-16(2R-r)^2(R62-2Rr)>0$$

$\Leftrightarrow 16R^2r^2+8Rr^3+r^4>0\rightarrow$ ***true***

297. In $\Delta\ ABC$:

$$\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}+\frac{\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}}+\frac{\cos^2\frac{C}{2}}{\cos^2\frac{A}{2}}+\frac{3s^2}{(r+4R)^2}\geq 4$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Rovsen Pirguliyev-Sumgait-Azerbaidian

$$\sum \frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}} + \frac{3s^2}{(r+4R)^2} \ge 4$$ ***(*). It is known that if*** $x, y, z > 0$***, then***

$$\sum \frac{x}{y} \ge \frac{9\sum x^2}{(\sum x)^2}$$ ***(1) if*** $x, y, z \in \left\{\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right\}$,

then (1) $\Rightarrow \sum \frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}} \ge \frac{9\left((4R+r)^2-s^2\right)}{2(4R+r)^2}$ ***(2)***

(*) $\overset{(2)}{\Rightarrow} \frac{9\left((4R+r)^2-s^2\right)}{2(4R+r)^2} + \frac{3s^2}{(4R+r)^2} = \frac{9}{2} - \frac{3s^2}{2(4R+r)^2}$***. Now we prove that***

$\frac{9}{2} - \frac{3s^2}{2(4R+r)^2} \ge 4 \Leftrightarrow \frac{1}{2} \ge \frac{3s^2}{2(4R+r)^2} \Leftrightarrow (4R+r)^2 \ge 3s^2$***. True***

298. In $\Delta\, ABC$:

$$m_a \cos\frac{A}{2} + m_b \cos\frac{B}{2} + m_c \cos\frac{C}{2} \ge \frac{9\sqrt{3}}{2} r$$

Proposed by Kevin Soto Palacios – Huarmey – Peru

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS \overset{A-G}{\ge} 3\sqrt[3]{m_a m_b m_c \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$\underbrace{\overset{m_a \ge \sqrt{s(s-a)}etc}{\ge}}_{(1)} 3\sqrt[3]{\sqrt{s(s-a)\cdot s(s-b)\cdot s(s-c)\cdot \frac{s(s-a)}{bc}\cdot\frac{s(s-b)}{ca}\cdot\frac{s(s-c)}{ab}}}$$

$$= 3\sqrt[3]{\frac{s^3(s-a)(s-b)(s-c)}{abc}} = 3\sqrt[3]{\frac{s^2\cdot r^2 s^2}{4Rrs}}$$

$= 3\sqrt[3]{\frac{s^3 r}{4R}} = 3s\sqrt[3]{\frac{r}{4R}}$; ***(1)*** $\Rightarrow$ ***it suffices to prove:*** $3s\sqrt[3]{\frac{r}{4R}} \ge \frac{9\sqrt{3}}{2} r$

$$\Leftrightarrow s^3\cdot\frac{r}{4R}\geq\frac{27}{8}\cdot 3\sqrt{3}r^3\Leftrightarrow 2s^3\geq 27\cdot 3\sqrt{3}r^2R\ \ (2)$$

Now, $s\geq 3\sqrt{3}r\Rightarrow 2s^3\geq 2s^2\cdot 3\sqrt{3}r$ **(3)**

(2), (3) ⇒ it suffices to prove:

$$2s^2\cdot 3\sqrt{3}r\geq 27\cdot 3\sqrt{3}Rr^2\Leftrightarrow 2s^2\geq 27Rr\ \ (4)$$

Gerretsen ⇒ $2s^2\overset{(5)}{\geq}32Rr-10r^2$**; (4), (5) ⇒ it suffices to prove:**

$32Rr-10r^2\geq 27Rr\Leftrightarrow 5r\geq 10r^2\Leftrightarrow R\geq 2r\rightarrow$ **true (Euler) (Proved)**

Solution 2 by proposer

Probar en triángulo ABC $P=m_a\cos\frac{A}{2}+m_b\cos\frac{B}{2}+m_c\cos\frac{C}{2}\geq\frac{9\sqrt{3}r}{2}$

Utilizando las siguientes desigualdades e identidades conocidas en un triángulo

ABC $m_a\geq\frac{b+c}{2}\cos\frac{A}{2}, m_b\geq\frac{c+a}{2}\cos\frac{B}{2}, m_c\geq\frac{a+b}{2}\cos\frac{C}{2}, p\geq 3\sqrt{3}r$

$a=b\cos C+c\cos B$, $b=c\cos A+a\cos C$, $c=a\cos B+b\cos B$

La desigualdad propuesta es equivalente

$$P\geq\left(\frac{b+c}{4}\right)(1+\cos A)+\left(\frac{c+a}{4}\right)(1+\cos B)+\left(\frac{a+b}{4}\right)(1+\cos C)$$

$$\Leftrightarrow P\geq\frac{a+b+c}{2}+\frac{b\cos C+c\cos B}{4}+\frac{c\cos A+a\cos C}{4}+\frac{a\cos B+b\cos A}{4}=$$

$$=\frac{a+b+c}{2}+\frac{a+b+c}{4}\Leftrightarrow P\geq\frac{3}{4}(a+b+c)=\frac{3p}{2}\geq\frac{9\sqrt{3}r}{2}\ \ \textbf{(LQQD)}$$

299. In $\Delta\ ABC$:

$$\frac{b^2c^2}{s-a}\leq 4R^2s$$

Proposed by Colin Springer-Canada

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC $\frac{b^2c^2}{s-a}\leq 4R^2s$**. Tener en cuenta lo siguiente**

$\frac{abc}{4R}=\sqrt{s(s-a)(s-b)(s-c)}$. ***La desigualdad es equivalente***

$$\frac{a^2b^2c^2}{16R^2}\le\frac{a^2s(s-a)}{4}\Leftrightarrow s(s-a)(s-b)(s-c)\le\frac{a^2s(s-a)}{4}$$

Es suficiente probar lo siguiente $(s-b)(s-c)\le\frac{a^2}{4}$. ***Aplicando MA*** $\ge$ ***MG***

$$(s-b)(s-c)\le\left(\frac{(s-b)+(s-c)}{2}\right)^2=\frac{a^2}{4}\ \textit{(LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{b^2c^2}{s-a}\overset{(1)}{\le}4R^2s;\ (1)\Leftrightarrow s(s-a)\ge\frac{b^2c^2}{4R^2}\quad (2)$$

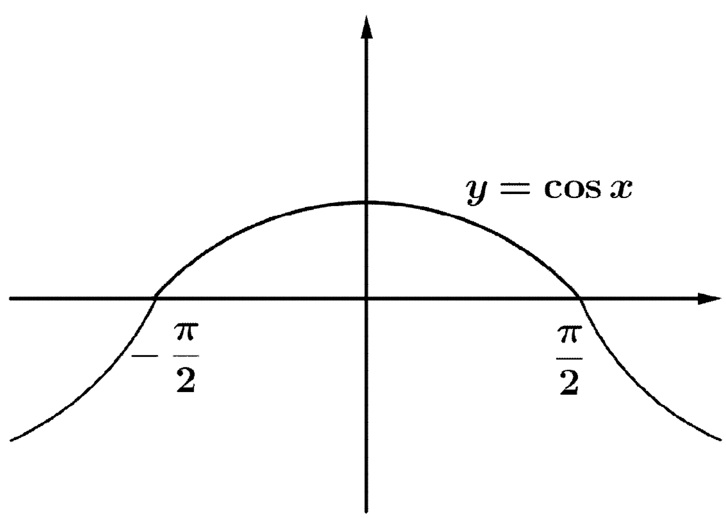
We know that, $s(s-a)\ge w_a^2$ ***(3); (2), (3)*** $\Rightarrow$ ***it suffices to prove:*** $w_a\ge\frac{bc}{2R}$

$$\Leftrightarrow\frac{2bc}{b+c}\cos\frac{A}{2}\ge\frac{bc}{2R}\Leftrightarrow 4R\cos\frac{A}{2}\ge b+c$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\ge 2R(\sin B+\sin C)=4R\cdot\cos\frac{A}{2}\cos\frac{B-C}{2}$$

$$\Leftrightarrow\cos\frac{A}{2}\left(1-\cos\frac{B-C}{2}\right)\ge 0\quad \textit{(4). Now, } 0<B<\pi \textit{ and } -\pi<-C<0$$

$$\Rightarrow -\frac{\pi}{2}<\frac{B-C}{2}<\frac{\pi}{2}$$



From the graph, it is clear that $0 < \cos\frac{B-C}{2} \le 1 \Rightarrow 1 - \cos\frac{B-C}{2} \ge 0$

$\Rightarrow \cos\frac{A}{2}\left(1 - \cos\frac{B-C}{2}\right) \ge 0 \ \left(\because \cos\frac{A}{2} > 0 \ as \ 0 < \frac{A}{2} < \frac{\pi}{2}\right) \Rightarrow$ ***(4) is true***

Solution 3 by Rovsen Pirguliyev-Sumgait-Azerbaidian

$\frac{b^2c^2}{s-a} \le 4R^2 s$ ***(*). Prove:*** $a = y+z, b = x+z, c = x+y, s = x+y+z$

$R = \frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}}$***, then we have:***

$$\textbf{(*)} \Rightarrow \frac{(x+z)^2(x+y)^2}{x+y+z-y-z} \le 4 \cdot \frac{(y+z)^2(z+x)^2(x+y)^2}{16xyz(x+y+z)} \cdot (x+y+z)$$

$$\Rightarrow (y+z)^2 \ge 4yz \Rightarrow (y-z)^2 \ge 0$$

Solution 4 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\therefore 4R^2S(s-a) \ge b^2c^2; 4R^2s(s-a)(s-b)(s-c) \ge b^2c^2(s-b)(s-c)$$

$$4R^2s^2 \ge b^2c^2(s-b)(s-c); a^2b^2c^2 \ge 4b^2c^2\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)$$

$$a^2 \ge (a+c-b)(a+b-c)$$

$$a^2 \ge a^2 + ab - ac + ac + bc - c^2 - ab - b^2 + bc$$

$$b^2 + c^2 \ge 2bc \Rightarrow (b-c)^2 \ge 0$$

Solution 5 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$\left(\frac{b}{2R}\right) \cdot \left(\frac{c}{2R}\right) \le \frac{p(p-a)}{bc}; \sin B \cdot \sin C \le \cos^2\frac{A}{2}$ ***(ASSURE)***

$$\cos^2\frac{A}{2} = \left(\cos\left(\frac{\pi}{2} - \left(\frac{B}{2} + \frac{C}{2}\right)\right)\right)^2 = \left(\sin\left(\frac{B}{2} + \frac{C}{2}\right)\right)^2 =$$

$$= \left(\sin\frac{B}{2} \cdot \cos\frac{C}{2} + \cos\frac{B}{2} \cdot \sin\frac{C}{2}\right)^2 \overset{AM\ge GM}{\ge}$$

$$\Rightarrow 0 < \left(\frac{B}{2}; \frac{C}{2}; \frac{A}{2} < \frac{\pi}{2}\right)$$

$$\geq 4\cdot\sin\frac{B}{2}\cdot\cos\frac{B}{2}\cdot\sin\frac{C}{2}\cdot\cos\frac{C}{2}=\sin B\cdot\sin C$$

Solution 6 by Geanina Tudose – Romania

$$\frac{b^2c^2}{p-a}\leq 4R^2p;\ p-a=\frac{b+c-a}{2}=R(\sin B+\sin C-\sin A)$$

$$=2R\left(\sin\frac{B+C}{2}\cos\frac{B-C}{2}-2\sin\frac{A}{2}\cos\frac{A}{2}\right)=2R\cos\frac{A}{2}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)$$

$$=2R\cos\frac{A}{2}\left(\sin\left(\frac{\pi}{2}-\frac{B-C}{2}\right)-\sin\frac{A}{2}\right)=4R\cos\frac{A}{2}\sin\frac{\pi-B+C-A}{4}\cos\frac{\pi-B+C+A}{4}$$

$$=4R\cos\frac{A}{2}\sin\frac{C}{2}\sin\frac{B}{2}$$

Hence $LHS=\frac{4R^2\sin^2 B\cdot 4R^2\sin^2 C}{4R\cos\frac{A}{2}\sin\frac{C}{2}\sin\frac{B}{2}}=\frac{4R^3\cdot 4\sin^2\frac{B}{2}\cos^2\frac{B}{2}\cdot 4\sin^2\frac{C}{2}\cos^2\frac{C}{2}}{\cos\frac{A}{2}\sin\frac{C}{2}\sin\frac{B}{2}}$

On the other hand $p=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

The inequality becomes $4\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}\leq\cos^2\frac{A}{2}$

$$\Leftrightarrow\sin B\sin C\leq\cos^2\frac{A}{2}\Leftrightarrow\frac{\cos(B-C)-\cos(B+C)}{2}\leq$$

$\leq\frac{\cos A+1}{2}\Leftrightarrow\cos(B-C)\leq 1$, ***which is true***

300. Prove that in any triangle ABC:

$$\frac{bc\cdot m_a}{h_a}+\frac{ca\cdot m_b}{h_b}+\frac{ab\cdot m_c}{h_c}\geq a^2+b^2+c^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{bc\cdot m_a}{h_a}+\frac{ca\cdot m_b}{h_b}+\frac{ab\cdot m_c}{h_c}\geq a^2+b^2+c^2$

Recordar las siguientes desigualdades e identidades en un $\Delta\ ABC$

$$m_a \geq \frac{b^2+c^2}{4R}, m_b \geq \frac{c^2+a^2}{4R}, m_c \geq \frac{a^2+b^2}{4R}$$

$h_a = \frac{bc}{2R}, h_b = \frac{ca}{2R}, h_c = \frac{ab}{2R}$. ***Luego, utilizando en la desigualdad propuesta***

$$\frac{bc \cdot m_a}{h_a} + \frac{ca \cdot m_b}{h_b} + \frac{ab \cdot m_c}{h_c} \geq \frac{bc(b^2+c^2)}{2bc} + \frac{ca(c^2+a^2)}{2ca} + \frac{ab(a^2+b^2)}{2ab}$$

$$\frac{bc \cdot m_a}{h_a} + \frac{ca \cdot m_b}{h_b} + \frac{ab \cdot m_c}{h_c} \geq \frac{b^2+c^2}{2} + \frac{c^2+a^2}{2} + \frac{a^2+b^2}{2} = a^2 + b^2 + c^2 \quad \textbf{(LQQD)}$$

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Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru