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# TRIGONOMETRIC SUBSTITUTIONS IN PROBLEM SOLVING PART 3

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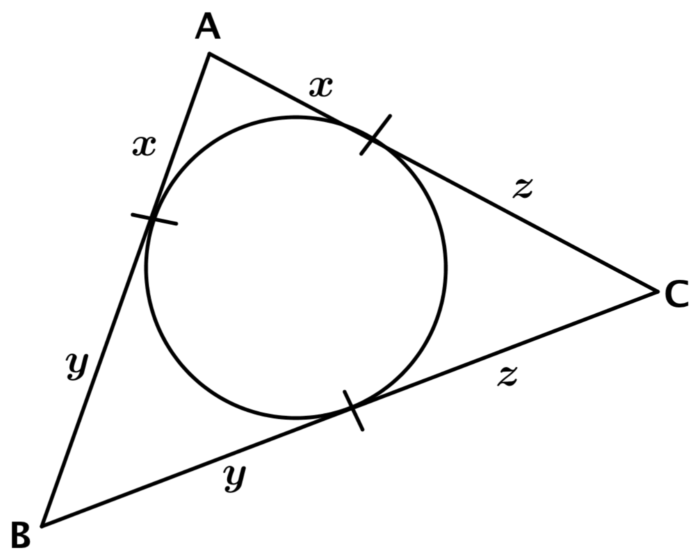
Abstract. In this paper are indicated a few useful trigonometric substitutions for solving problems. Solved problems are also a part of this article.




$$a = y + z, b = z + x, c = x + y$$
$$s = x + y + z, S = \sqrt{xyz(x + y + z)}$$
$$R = \frac{(x + y)(y + z)(z + x)}{4\sqrt{xyz(x + y + z)}}, r = \sqrt{\frac{xyz}{x + y + z}}$$
$$r_a = \sqrt{\frac{(x + y + z)yz}{x}}; r_b = \sqrt{\frac{(x + y + z)zx}{y}}; r_c = \sqrt{\frac{(x + y + z)xy}{z}}$$
$$\sin\frac{A}{2} = \sqrt{\frac{yz}{(x + y)(x + z)}}; \sin\frac{B}{2} = \sqrt{\frac{zx}{(y + z)(y + x)}}; \sin\frac{C}{2} = \sqrt{\frac{xy}{(z + x)(z + y)}}$$
$$\cos\frac{A}{2} = \sqrt{\frac{(x + y + z)x}{(x + y)(x + z)}}; \cos\frac{B}{2} = \sqrt{\frac{(x + y + z)y}{(y + x)(y + z)}}; \cos\frac{C}{2} = \sqrt{\frac{(x + y + z)z}{(z + x)(z + y)}}$$
$$\tan\frac{A}{2} = \sqrt{\frac{yz}{x}}; \tan\frac{B}{2} = \sqrt{\frac{zx}{y}}; \tan\frac{C}{2} = \sqrt{\frac{xy}{z}}$$

*Key words and phrases.* Trigonometric substitutions.

$$\cot\frac{A}{2}=\sqrt{\frac{x}{yz}};\cot\frac{B}{2}=\sqrt{\frac{y}{xz}};\cot\frac{C}{2}=\sqrt{\frac{z}{zx}}$$
$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\sqrt{xyz};\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\frac{1}{\sqrt{xyz}}$$
$$\cos A=\frac{x(x+y+z)-yz}{(x+y)(x+z)};\cos B=\frac{y(x+y+z)-xz}{(y+z)(y+x)}$$
$$\cos C=\frac{z(x+y+z)-xy}{(z+x)(z+y)}$$
$$\sin A=\frac{2\sqrt{xyz(x+y+z)}}{(x+y)(x+z)},\sin B=\frac{2\sqrt{xyz(x+y+z)}}{(y+z)(y+x)}$$
$$\sin C=\frac{2\sqrt{xyz(x+y+z)}}{(z+x)(z+y)}$$

**Problem 43.**
**If $x,y,z>0$ then:**
$$\frac{2xy}{(z+x)(z+y)}+\frac{2yz}{(x+y)(x+z)}+\frac{3zx}{(y+z)(y+x)}\geq\frac{5}{3}$$

*Proof.*
$$\cos A=1-2\sin^2\frac{A}{2}=1-\frac{2yz}{(x+y)(x+z)}$$
$$\frac{2yz}{(x+y)(x+z)}=1-\cos A$$
$$\frac{yz}{(x+y)(x+z)}=\frac{1}{2}-\frac{1}{2}\cdot\cos A;$$
$$\frac{xy}{(z+x)(z+y)}=\frac{1}{2}-\frac{1}{2}\cos C;\ \frac{xz}{(y+x)(y+z)}=\frac{1}{2}-\frac{1}{2}\cos B$$
*Inequality to prove can be written:*
$$2\left(\frac{1}{2}-\frac{1}{2}\cos C\right)+2\left(\frac{1}{2}-\frac{1}{2}\cos A\right)+3\left(\frac{1}{2}-\frac{1}{2}\cos B\right)\geq\frac{5}{3}$$
$$1-\cos C+1-\cos A+\frac{3}{2}-\frac{3}{2}\cos B\geq\frac{5}{3}$$
$$\frac{7}{2}-\left(\cos A+\cos C+\frac{3}{2}\cos B\right)\geq\frac{5}{3}\tag{1}$$
$$\cos A+\cos C+\frac{3}{2}\cos B=2\cos\frac{A+C}{2}\cos\frac{A-C}{2}+\frac{3}{2}\cos B=$$
$$=2\sin\frac{B}{2}\cos\frac{A-C}{2}+\frac{3}{2}\left(1-2\sin^2\frac{B}{2}\right)$$
*By* (1) :
$$\frac{7}{2}-2\sin\frac{B}{2}\cos\frac{A-C}{2}-\frac{3}{2}+3\sin^2\frac{B}{2}\geq\frac{5}{3}\ \textit{(to prove)}$$
$$3\sin^2\frac{B}{2}-2\sin\frac{B}{2}\cos\frac{A-C}{2}+\frac{1}{3}\geq 0$$
$$3\left(\sin\frac{B}{2}-\frac{1}{3}\cos\frac{A-C}{2}\right)^2-\frac{1}{9}\cos^2\frac{A-C}{2}+\frac{1}{3}\geq 0$$
$$3\left(\sin\frac{B}{2}-\frac{1}{3}\cos\frac{A-C}{2}\right)^2+\frac{1}{9}-\frac{1}{9}\cos^2\frac{A-C}{2}+\frac{2}{9}\geq 0$$

$$3\Big(\sin\frac{B}{2}-\frac{1}{3}\cos\frac{A-C}{2}\Big)^2+\frac{1}{9}\sin^2\frac{A-C}{2}+\frac{2}{9}\geq 0$$

□

**Problem 44.**
**If $a,b,c>0; a+b+c=1$ then:**

$$\sqrt{\frac{ab}{c+ab}}+\sqrt{\frac{bc}{a+bc}}+\sqrt{\frac{ca}{b+ca}}\leq\frac{3}{2}$$

*Proof.*

$$\sum\sqrt{\frac{ab}{c+ab}}\leq\frac{3}{2}\Leftrightarrow\sum\sqrt{\frac{ab}{(c+ab)\cdot 1}}\leq\frac{3}{2}\Leftrightarrow$$

$$\Leftrightarrow\sum\sqrt{\frac{ab}{(c+ab)(a+b+c)}}\leq\frac{3}{2}$$

$$\sum\sqrt{\frac{ab}{ca+cb+c^2+ab(a+b+c)}}\leq\frac{3}{2}$$

$$\sum\sqrt{\frac{ab}{ca+cb+c^2+ab}}\leq\frac{3}{2}$$

(2)
$$\sum\sqrt{\frac{ab}{(c+a)(c+b)}}\leq\frac{3}{2}$$

*Donde:* $x=a+b; y=b+c; z=c+a; s=a+b+c$

*By* (2) :

$$\sum\sqrt{\frac{(s-y)(s-z)}{yz}}\leq\frac{3}{2}\ \textit{(to prove)}$$

$$f:(0,\pi)\to\mathbb{R}; f(x)=\sin\frac{x}{2};$$

$$f'(x)=\frac{1}{2}\cos\frac{x}{2}; f''(x)=-\frac{1}{4}\sin\frac{x}{2}\leq 0$$

*By Jensen:*

$$f(A)+f(B)+f(C)\leq 3f\Big(\frac{A+B+C}{3}\Big)$$

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq 3\sin\frac{\pi}{3}=\frac{3}{2}$$

$$\sin\frac{A}{2}=\sqrt{\frac{(s-y)(s-z)}{yz}};\sin\frac{B}{2}=\sqrt{\frac{(s-x)(s-z)}{xz}};\sin\frac{C}{2}=\sqrt{\frac{(s-x)(s-y)}{xy}}$$

□

**Problem 45.**
**If $x,y,z>0$ then:**

$$\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}\geq 2\sqrt{\frac{(x+y)(y+x)(z+x)}{x+y+z}}$$

***(D. Grinberg)***

*Proof.*

*Inequality to prove becomes:*

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}+\sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}}+\sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}}\geq 2$$

$$a=x+y;b=y+z;c=z+x;s=x+y+z$$

$$\sqrt{\frac{s(s-b)}{ac}}+\sqrt{\frac{s(s-c)}{ab}}+\sqrt{\frac{s(s-a)}{bc}}\geq 2$$

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\geq 2$$

$$A=\pi-2A',B=\pi-2B',C=\pi-2C'$$

$$\cos\frac{\pi-2A'}{2}+\cos\frac{\pi-2B'}{2}+\cos\frac{\pi-2C'}{2}\geq 2$$

$$\sin A'+\sin B'+\sin C'\geq 2$$

*By Jordan's inequality:*

$$\sin A'\geq\frac{2A'}{\pi};\sin B'\geq\frac{2B'}{\pi};\sin C'\geq\frac{2C'}{\pi}$$

*By adding:*

$$\sin A'+\sin B'+\sin C'\geq\frac{2(A'+B'+C')}{\pi}=\frac{2\pi}{\pi}=2$$

□

**Problem 46.**
**If $a,b,c>0;a+b+c=abc$ then:**

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}}\geq 2\sqrt{3}$$

***(A. Nicolaescu; C. Pătraşcu)***

*Proof.*

$$a=\tan A;b=\tan B;c=\tan C;A,B,C\in\left(0,\frac{\pi}{2}\right)$$

*Inequality can be written:*

$$\sqrt{1+\frac{1}{\tan^2 A}}+\sqrt{1+\frac{1}{\tan^2 B}}+\sqrt{1+\frac{1}{\tan^2 C}}\geq 2\sqrt{3}$$

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\geq 2\sqrt{3}$$

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\overbrace{\geq}^{CBS}\frac{(1+1+1)^2}{\sin A+\sin B+\sin C}\geq\frac{9}{\frac{3\sqrt{3}}{2}}=2\sqrt{3}$$

□

**Problem 47. If $x,y,z>0;x+y+z=xyz$ then:**

$$\sqrt{\frac{x^4}{3}}+\sqrt{\frac{y^4}{3}+1}+\sqrt{\frac{z^4}{3}+1}\geq 6$$

***(George Apostolopoulos)***

*Proof.*

$$\textit{Denote: } a^2=\sqrt{3}\tan A, b^2=\sqrt{3}\tan B, c^2=\sqrt{3}\tan C$$

$$\sqrt{\frac{3\tan^2 A}{3}+1}+\sqrt{\frac{3\tan^2 B}{3}+1}+\sqrt{\frac{3\tan^2 C}{3}+1}\geq 6$$

$$\sqrt{1+\tan^2 A}+\sqrt{1+\tan^2 B}+\sqrt{1+\tan^2 C}\geq 6$$

$$\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\geq 6$$

$$\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\overbrace{\geq}^{CBS}\frac{(1+1+1)^2}{\cos A+\cos B+\cos C}\geq\frac{9}{\frac{3}{2}}=6$$

□

**Problem 48. If $x, y, z > 0; x+y+z=xyz$ then:**

$$\boldsymbol{xy+yz+zx\geq 3+\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}}$$

*Proof.*

$$\textit{Denote } x=\tan A, y=\tan B, z=\tan C$$

*Inequality can be written:*

$$\tan A\tan B+\tan B\tan C+\tan C\tan A\geq 3+\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}$$

$$(\tan A\tan B-1)+(\tan B\tan C-1)+(\tan C\tan A-1)\geq\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}$$

$$\frac{\sin A\sin B-\cos A\cos B}{\cos A\cos B}+\frac{\sin B\sin C-\cos B\cos C}{\cos B\cos C}+\frac{\sin C\sin A-\cos C\cos A}{\cos C\cos A}\geq$$

$$\geq\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}$$

$$-\frac{\cos(A+B)}{\cos A\cos B}-\frac{\cos(B+C)}{\cos B\cos C}-\frac{\cos(C+A)}{\cos C\cos A}\geq\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}$$

$$\frac{\cos C}{\cos A\cos B}+\frac{\cos A}{\cos B\cos C}+\frac{\cos B}{\cos C\cos A}\geq\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}$$

$$\cos^2 A+\cos^2 B+\cos^2 C\geq\cos B\cos C+\cos A\cos C+\cos A\cos B$$

$$(\cos A-\cos B)^2+(\cos B-\cos C)^2+(\cos C-\cos A)^2\geq 0$$

□

**Problem 49. If $x, y, z > 0, xy+yz+zx=1$ then:**

$$\boldsymbol{\frac{1-x^2}{1+x^2}+\frac{1-y^2}{1+y^2}+\frac{1-z^2}{1+z^2}\leq\frac{3}{2}}$$

***(C. Popescu)***

*Proof.*

$$\textit{Denote } x=\tan\frac{A}{2}; y=\tan\frac{B}{2}; z=\tan\frac{C}{2}$$
$$\textit{Inequality can be written:}$$
$$\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}+\frac{1-\tan^2\frac{B}{2}}{1+\tan^2\frac{B}{2}}+\frac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}}\le\frac{3}{2}$$
$$\cos A+\cos B+\cos C\le\frac{3}{2}$$

□

**Problem 50. If $a,b,c>0; a+b+c=1$ then:**

$$\boldsymbol{a^2+b^2+c^2+2\sqrt{2abc}\le 1}$$

*Proof.*

$$\textit{Denote } a=xy; b=yz; c=zx;$$
$$a+b+c=1\Leftrightarrow xy+yz+zx=1$$
$$\textit{For } x=\tan\frac{A}{2}; y=\tan\frac{B}{2}; z=\tan\frac{C}{2}$$
$$A,B,C\in\left(0,\frac{\pi}{2}\right)$$
$$\textit{Inequality can be written:}$$
$$x^2y^2+y^2z^2+z^2x^2+2\sqrt{3}xyz\le 1$$
$$(xy+yz+zx)^2-2xyz(x+y+z)+2\sqrt{3}+xyz\le 1$$
$$1-2xyz(x+y+z)+2\sqrt{3}xyz\le 1$$
$$x+y+z\ge\sqrt{3}$$
$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\ge\sqrt{3}$$

□

**Problem 51. If $x,y,z>1; \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$ then:**

$$\boldsymbol{\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\le\sqrt{x+y+z}}$$

$$\textit{Denote } x=a+1, y=b+1, z=c+1, a,b,c>0$$
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2\Leftrightarrow ab+bc+ca+2abc=1$$
$$\textit{Inequality can be written:}$$
$$\sqrt{a}+\sqrt{b}+\sqrt{c}\le\sqrt{a+b+c}$$
$$\textit{For } ab=\sin^2\frac{A}{2}; bc=\sin^2\frac{B}{2}; ca=\sin^2\frac{C}{2};$$
$$A,B,C\in\left(0,\frac{\pi}{2}\right)\textit{ the constraint can be written:}$$
$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1$$
$$\textit{By squaring inequality to prove becomes:}$$
$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\le\frac{3}{2}$$

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}$$

**Proposed problems**

**52. If $a,b\in\mathbb{R};15a+12b+7|=13$ then:**

$$a^2+b^2+2(b-a)\ge -1$$

$$(\text{Use: } a-1=R\sin p; b+1=R\cos p)$$

**53. If $a,b\in\mathbb{R};|a|\ge 1;|b|\ge 1$ then:**

$$\left|\frac{\sqrt{a^2-1}+\sqrt{b^2-1}}{ab}\right|\le 1$$

$$\left(\text{Use: } a=\frac{1}{\cos p}; b=\frac{1}{\cos q}; p,q\in\left[0,\frac{\pi}{2}\right]\cup\left[\pi,\frac{3\pi}{2}\right)\right)$$

**54. If $|x|\ge 1;|y|\ge 1$ then:**

$$y\sqrt{x^2-1}+4\sqrt{y^2-1}+3\le xy\sqrt{26}$$

$$\left(\text{Use: } x=\frac{1}{\cos p}; y=\frac{1}{\cos q}; p,q\in\left(0,\frac{\pi}{2}\right)\right)$$

**55. If $x,y,y,v\in\mathbb{R}\quad x^2+y^2=u^2+v^2=1$ then:**

**a.** $|xu+yv|\le 1$

**b.** $|xv+yu|\le 1$

**c.** $-2\le(x-y)(u+v)+(x+y)(u-v)\le 2$

**d.** $-2\le(x+y)(u+v)-(x-y)(u-v)\le 2$

$$\left(\text{Use: } x=\cos a; y=\sin a; u=\cos b; v=\sin b; a,b\in(0,2\pi)\right)$$

**56. If $a,b\in\mathbb{R}$ then: $(a+b)^4\le 8(a^4+b^4)$**

**a.** $(a+b)^4\le 8(a^4+b^4)$

**b.** $(a+b)^6\le 32(a^6+b^6)$

$$\left(\text{Use: } \tan x=\frac{b}{a}; x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

**57. If $x,y\in\mathbb{R};xy\ne 0$ then:**

$$-2\sqrt{2}-2\le\frac{x^2-(x-4y)^2}{x^2+4y^2}\le 2\sqrt{2}-2$$

$$\left(\text{Use: } x=2y\tan p; p\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

**58. If $x,y\in\mathbb{R};36x^2+16y^2=9$ then:**

$$\frac{15}{4}\le y-2x+5\le\frac{25}{4}$$

$$\left(\text{Use: } x=\frac{1}{2}\cos p; y=\frac{3}{4}\sin p; p\in[0,2\pi]\right)$$

**59.** If $x, y > 0; 3x + 4y = 5$ then $x^2 + y^2 \geq 1$

$$\left(\text{Use: } \sin p = \frac{3}{5}; \cos p = \frac{4}{5}\right)$$

**60.** If $a, b \in \mathbb{R}; 4a^2 + 9b^2 = 25$ then:

$$|6a + 12b| \leq 25$$

$$\left(\text{Use: } \frac{2}{5}a = \sin p; \frac{3}{5}b = \cos p; p \in [0, 2\pi]\right)$$

**61.** If $x, y, a, b, c > 0, ax + by = 0, a^2 + b^2 = c^2$ then: $x^2 + y^2 \geq 1$

$$\left(\text{Use: } \frac{a}{c} = \cos p; \frac{b}{c} = \sin p; p \in [0, 2\pi)\right)$$

**62.** If $a > b > c > 0$ then:

$$\sqrt{c(a-c)} + \sqrt{c(b-c)} \leq \sqrt{ab}$$

$$\Bigg(\text{Use: } \sqrt{\frac{c}{a}} = \sin p; \sqrt{\frac{a-c}{a}} = \cos p; \sqrt{\frac{c}{b}} = \sin v;$$

$$\sqrt{\frac{b-c}{b}} = \cos v; u, v \in \left[0, \frac{\pi}{2}\right]\Bigg)$$

# TO BE CONTINUED!

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