Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

SOLUTION
INEQUALITY IN TRIANGLE - 413
ROMANIAN MATHEMATICAL MAGAZINE 2017

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1. In ΔABC

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{27(a+b)(b+c)(c+a)}{8abc}$$

Proposed by Abdullayev - Baku - Azerbaidian

Remark.

The inequality can be strengthened:

2. In ΔABC

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{2p^2}{Rr}$$

Marin Chirciu - Romania

Proof.

We prove that following Lemma.

Lemma 1.
3. In ΔABC

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \geq \frac{p^2+r^2-2Rr}{4Rr}.$$

Proof.

Using Tereşin's inequality $m_a \geq \frac{b^2+c^2}{4R}$, *formula* $h_a = \frac{bc}{2R}$ *and the known inequality in triangle* $\sum \frac{b^2+c^2}{bc} = \frac{p^2+r^2-2Rr}{2Rr}$, *we obtain:*

$$\sum \frac{m_a}{h_a} \geq \sum \frac{\frac{b^2+c^2}{4R}}{\frac{bc}{2R}} = \frac{1}{2}\sum \frac{b^2+c^2}{bc} = \frac{p^2+r^2-2Rr}{4Rr}$$

Equality holds if and only if the triangle is equilateral.

□

□

Remark.

We can write the inequalities:

4. In ΔABC

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\geq\frac{p^2+r^2-2Rr}{4Rr}\geq\frac{7R-2r}{2R}\geq 3.$$

Proof.
*The first inequality is **Lemma 1**, the second inequality follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$, and the third inequality follows from Euler's inequality $R\geq 2r$.*

Equality holds if and only if the triangle is equilateral.

□

Lemma 2.
5. In ΔABC

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^2\geq\frac{2p^2}{3Rr}$$

Proof.

*Using **Lemma 1**, is enough to prove that:* $\left(\frac{p^2+r^2-2Rr}{4Rr}\right)^2\geq\frac{2p^2}{3Rr}\Leftrightarrow$

$$3p^4+p^2(6r^2-44Rr)+12R^2r^2-12Rr^3+3r^4\geq 0\Leftrightarrow p^2(3p^2+6r^2-44Rr)+3r^2(2R-r)^2\geq 0$$

□

We distinguish the cases:
1) If $\mathbf{3p^2+6r^2-44Rr\geq 0}$*, the inequality is obvious.*
2) If $\mathbf{3p^2+6r^2-44Rr<0}$*, inequality we can rewrite:*

$p^2(44Rr-6r^2-3p^2)\leq 3r^2(2R-r)^2$*, true from Gerretsen's inequality:*

$$(4R^2+4Rr+3r^2)\Big[44Rr-6r^2-3(16Rr-5r^2)\Big]\leq 3r^2(2R-r)^2\Leftrightarrow$$

$$\Leftrightarrow 4R^3-2R^2r-9Rr^2-6r^3\geq 0\Leftrightarrow(R-2r)(4R^2+6Rr+3r^2)\geq 0$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark 2.

We can rewrite the inequalities:

6. In ΔABC

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^2\geq\frac{2p^2}{3Rr}\geq\frac{9(a+b)(b+c)(c+a)}{8abc}\geq 9.$$

Proof.

First inequality is ***Lemma 2.***

Let's prove the second inequality.

Using the known identities in triangle: $(a+b)(b+c)(c+a) = 2p(p^2+r^2+2Rr)$

and $abc = 4Rrp$*, the second inequality:*

$$\frac{2p^2}{3Rr} \geq \frac{9 \cdot 2p(p^2+r^2+2Rr)}{8 \cdot 4Rrp} \Leftrightarrow 32p^2 \geq 27(p^2+r^2+2Rr) \Leftrightarrow 5p^2 \geq 27(r^2+2Rr)$$

which follows from Gerretsen's inequality: $p^2 \geq 16Rr - 5r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

The third inequality is the well known inequality $(a+b)(b+c)(c+a) \geq 8abc$ *(Cesaro)*

We've obtained a strengthened inequality in triangle $\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \geq 3$.

Let's pass to solving inquality ***2****:* $\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \geq \frac{2p^2}{Rr}$

Base on ***Lemma 2*** *and the the last inequality from* ***Remark 1*** *we obtain:*

$$\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 = \left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^2 \cdot \left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right) \geq \frac{2p^2}{3Rr} \cdot 3 = \frac{2p^2}{Rr}$$

Equality holds if and only if the triangle is equilateral.

Remark 3.

Inequality ***2*** *is stronger then inequality* ***1****:*

7. In ΔABC

$$\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \geq \frac{2p^2}{Rr} \geq \frac{27(a+b)(b+c)(c+a)}{8abc}$$

Proof.

The first inequality is ***6****.*

Let's prove the second inequality.

Using the known identities in triangle: $(a+b)(b+c)(c+a) = 2p(p^2+r^2+2Rr)$

and $abc = 4Rrp$*, the second inequality:*

$$\frac{2p^2}{Rr} \geq \frac{27 \cdot 2p(p^2+r^2+2Rr)}{8 \cdot 4Rrp} \Leftrightarrow 32p^2 \geq 27(p^2+r^2+2Rr) \Leftrightarrow 5p^2 \geq 27(r^2+2Rr)$$

which follows from Gerretsen's inequality: $p^2 \geq 16Rr - 5r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark 4.

We can write the inequalities:

8. In ΔABC

$$\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \geq \frac{2p^2}{Rr} \geq \frac{27(a+b)(b+c)(c+a)}{8abc} \geq 27$$

Proof.

*See **7** and Cesaro's inequality* $(a+b)(b+c)(c+a) \geq 8abc$

□

Remark 5.

*Inequality **2** can also be strengthened:*

9. In ΔABC

$$\boldsymbol{\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{p^2}{3Rr}\left(7-\frac{2r}{R}\right)}$$

Proof.

*Base on **Lemma 2** and on the second inequality from **Remark 1** we obtain:*

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3=\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^2\cdot\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right) \geq \frac{2p^2}{3Rr}\cdot\frac{7R-2r}{2R}=\frac{p^2}{3Rr}\left(7-\frac{2r}{R}\right)$$

Equality holds if and only if the triangle is equilateral.

□

Remark 6.

*Inequality **9.** is stronger then inequality **2.**:*

10. In ΔABC

$$\boldsymbol{\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{p^2}{3Rr}\left(7-\frac{2r}{R}\right) \geq \frac{2p^2}{Rr}}$$

Proof.

*See inequality **9.** and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark 7.

We can write the inequalities:

11. In ΔABC

$$\boldsymbol{\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{p^2}{3Rr}\left(7-\frac{2r}{R}\right) \geq \frac{2p^2}{Rr} \geq \frac{27(a+b)(b+c)(c+a)}{8abc} \geq 27.}$$

Proof.

*See **10.** and **8.***

Equality holds if and only if the triangle is equilateral.

□

We've obtained again a strengthening of the well known inequality in triangle

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\geq 3$$

*Finally we can propose a development of inequality **2.**:*

12. In ΔABC

$$\boldsymbol{\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^n\geq 3^{n-3}\cdot\frac{2p^2}{Rr},\ \text{where } n\geq 2.}$$

Proof.

*Base on **Lemma 2** and the last inequality from **Remark 1** we obtain:*

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^n=\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^2\cdot\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^{n-2}\geq\frac{2p^2}{3Rr}\cdot 3^{n-2}=3^{n-3}\cdot\frac{2p^2}{Rr}.$$

Equality holds if and only if the triangle is equilateral.

□

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