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INEQUALITY IN TRIANGLE - 449
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1. In ΔABC

$$\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)}\ge\frac{1}{2Rr}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

We prove the following lemma:

Lemma
2. In ΔABC

$$\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)}=\frac{p^2+(4R+r)^2}{4Rrp^2}$$

Proof.

We have $\sum\frac{1}{a(p-a)}=\frac{\sum bc(p-b)(p-c)}{abc(p-a)(p-b)(p-c)}=\frac{r^2[p^2+(4R+r)^2]}{4Rrp\cdot r^2p}=\frac{p^2+(4R+r)^2}{4Rrp^2}$

□

Let's pass to solving the inequality from enuntiation.

*Using the **Lemma** we write the inequality:*

$$\frac{p^2+(4R+r)^2}{4Rrp^2}\ge\frac{1}{2Rr}\Leftrightarrow 3p^2\ge(4R+r)^2$$

(Doucet's inequality, which follows from Gerretsen's inequality $p^2\ge 16Rr-5r^2$

and Euler's inequality $R\ge 2r$).

The equality holds if and only if the triangle is equilateral.

Remark 1.

The inequality can be strengthened

3. In ΔABC

$$\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)}\ge\frac{5R-2r}{4R^2r}$$

Proof.

*Using the **Lemma** we write the inequality:*

$$\frac{p^2+(4R+r)^2}{4Rrp^2} \geq \frac{5R-2r}{4R^2r} \Leftrightarrow p^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$$

(Blundon-Gerretsen's inequality, which follows from Gergonne's identity

$$H\Gamma^2 = 4R^2\left[1-\frac{2p^2(2R-r)}{R(4R+r)^2}\right], \textit{ where } \Gamma \textit{is Gergonne's point).}$$

The equality holds if and only if the triangle is equilateral.

□

Remark 2.

*Inequality **3.** is stronger then inequality **1.**:*

4. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)} \geq \frac{5R-2r}{4R^2r} \geq \frac{1}{2Rr}}$$

Proof.

*The first inequality is inequality **3.***

The second inequality is equivalent with $R \geq 2r$ (Euler's inequality).

The equality holds if and only if the triangle is equilateral.

□

Remark 3.

Let's find an inequality having an opposite sense.

5. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)} \leq \frac{1}{2r^2}}.$$

Proof.

*Using the **Lemma** we write the inequality:*

$$\frac{p^2+(4R+r)^2}{4Rrp^2} \leq \frac{1}{2r^2} \Leftrightarrow p^2(2R-r) \geq r(4R+r)^2$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)(2R-r) \geq r(4R+r)^2 \Leftrightarrow 16R^2-17Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(8R-r) \geq 0,$$

obviously form Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark 6.

The double inequality take place:

6. In ΔABC

$$\frac{5R-2r}{4R^2r} \leq \frac{1}{a(p-a)} + \frac{1}{b(p-b)} + \frac{1}{c(p-c)} \leq \frac{1}{2r^2}$$

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Proof.

*See inequalities **3.** and **5.***

The equality holds if and only if the triangle is equilateral.

□

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